ON THE MOMENTS OF SOME ONE-SIDED STOPPING RULES[1]

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1. Introduction. The moments of stopping rules (or stopping times) have been discussed in [1], [3], and [4], and the following results have been proved. Let x_n be independent random variables with $Ex_n = 0$, $Ex_n^2 = 1$, and $S_n = x_1 + \cdots + x_n$. For $c > 0$ and $m = 1, 2, \cdots$, define t_m to be the first $n \geqq m$ such that $|S_n| > cn^{\frac{1}{2}}$. If $c \geqq 1$, then $Et_1 = \infty$. If $P[|x_n| \leqq K] = 1$ for some $K < \infty$ and $n = 1, 2, \cdots$, then $Et_m < \infty$ for every m if $c < 1$, $Et_m^2 < \infty$ for every m if $c < 3 - 6^{\frac{1}{2}}$, and $Et_m^2 = \infty$ for all large m if $c \geqq 3 - 6^{\frac{1}{2}}$.

In this note, we are interested in the following one-sided stopping rules, instead of the above stated two-sided stopping rules. For $c > 0$ and $1 > p \geqq 0$, define

$$s = \text{first} \quad n \geqq 1 \quad \text{such that} \quad S_n \geqq cn^p.$$

One of the results states that, if x_n are independent, $Ex_n = \mu > 0$, and $Ex_n^2 - \mu^2 = \sigma^2 < \infty$, then $Es^2 < \infty$ and

$$\lim_{c\to\infty} \mu^2 Es^2/(c^2 Es^{2p}) = \lim \mu Es^2/(cEs^{1+p}) = 1. \tag{1}$$

When $p = 0$, $Es^2 < \infty$ implies that $P[S_1 < c, \cdots, S_n < c] = P[s > n] = o(n^{-2})$ as $n \to \infty$, which completes a result of Morimura [9]. Also (1) extends the elementary renewal theorem from first moments to second moments and generalizes some results due to Chow and Robbins [2], Hatori [6], and Heyde [7].

2. The first moment. Let $(\Omega, \mathfrak{F}, P)$ be a probability space and x_n be a sequence of integrable random variables. Let $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots \subset \mathfrak{F}$ be Borel fields such that x_n is $\mathfrak{F}_n$-measurable and $\mathfrak{F}_0 = \{\varnothing, \Omega\}$. Put $S_n = x_1 + \cdots + x_n$, $S_0 = 0$, $m_n = E(x_n \mid \mathfrak{F}_{n-1})$ and $T_n = \sum_1^n m_j$. Assume that for some constant $\infty > \mu > 0$ and for some null set N,

$$\lim_{n\to\infty} T_n/n = \mu, \quad \text{uniformly on } \Omega - N. \tag{2}$$

For $c > 0$ and $1 > p \geqq 0$, define

$$s = \text{first} \quad n \geqq 1 \quad \text{such that} \quad S_n \geqq cn^p.$$

Theorem 1. (i) *If for some* $0 < \delta < \mu/3$, $P[x_n \leqq m_n + n\delta] = 1$ *for all large* n, *then* $Es < \infty$.

(ii) *If* $E([(x_n - m_n)^+]^\alpha \mid \mathfrak{F}_{n-1}) \leqq K < \infty$ *for some* $\alpha > 1$ *and*

$$E(|x_n - m_n| \mid \mathfrak{F}_{n-1}) \leqq K < \infty,$$

then $ES < \infty$ *and*

$$\lim_{c\to\infty} \mu Es/(cEs^p) = 1 = \lim_{c\to\infty} ES_s/(cEs^p). \tag{3}$$

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PROOF. (i) Set $t = \min(s, k)$ for $k = 1, 2, \cdots$. Then by the Wald identity for martingales (see [5], p. 302; or [3]),

$$ET_t = ES_t = E(S_{t-1} + x_t) \leqq cEt^p + E(m_t + \delta t) + O(1).$$

Let $0 < \epsilon < \delta$. As $k \to \infty$, by (2)

$$ET_t \geqq (\mu - \epsilon)Et + O(1), \qquad Em_t = O(1) + o(Et).$$

Hence

$$(\mu - \epsilon)Et \leqq cEt^p + \delta Et + O(1) + o(Et),$$

$$\int_{[s \leqq k]} s\, dP + kP[s > k] = Et = O(1),$$

as $k \to \infty$. Therefore $P[s < \infty] = 1$ and $Es < \infty$.

(ii) For any $0 < \delta < \mu/6$, define $x_n' = \min(x_n, m_n + n\delta)$, $m_n' = E(x_n' \mid \mathfrak{F}_{n-1})$, and $T_n' = m_1' + \cdots + m_n'$. Let $I(A)$ be the indicator function of the set A. Then

$$\begin{aligned} 0 \leqq m_n - m_n' &= E((x_n - m_n - n\delta)I[x_n > m_n + n\delta] \mid \mathfrak{F}_{n-1}) \\ &\leqq E((x_n - m_n)I[x_n > m_n + n\delta] \mid \mathfrak{F}_{n-1}) \\ &\leqq E^{1/\alpha}([(x_n - m_n)^+]^\alpha \mid \mathfrak{F}_{n-1})P^{1/\alpha'}(x_n - m_n > n\delta \mid \mathfrak{F}_{n-1}) \quad (\alpha + \alpha' = \alpha\alpha') \\ &\leqq K(n\delta)^{-\alpha/\alpha'}. \end{aligned} \tag{4}$$

Therefore $\lim_{n\to\infty} T_n'/n = \mu$ uniformly on $\Omega - N$ and $P[x_n' < m_n' + 2n\delta] = 1$ for all large n. Define

$$t = \text{first} \quad n \geqq 1 \quad \text{such that} \quad x_1' + \cdots + x_n' \geqq cn^p.$$

Then $s \leqq t$. By (i), $Et < \infty$. Therefore $Es < \infty$ and it follows by the Wald identity again ([5], p. 302; or [3]) that

$$E(cs^p + x_s) \geqq ES_s = ET_s \geqq cEs^p. \tag{5}$$

Let $Z_n = \sum_1^n [(x_j - m_j)^+]^\alpha$. Then by Lemma 6 of [3],

$$E^\alpha(x_s - m_s)^+ \leqq EZ_s = E\sum_1^s E([(x_j - m_j)^+]^\alpha \mid \mathfrak{F}_{j-1}) \leqq KEs. \tag{6}$$

Since (2) implies that as $c \to \infty$, $Em_s = O(1) + o(Es)$ and $ET_s = O(1) + (\mu + o(1))Es$, we have

$$Ex_s = O(E^{1/\alpha}s) + o(Es) + O(1) \tag{7}$$

from (6); and

$$\lim_{c\to\infty} \mu Es/(cEs^p) = \lim ET_s/(cEs^p) = \lim ES_s/(cEs^p) = 1$$

from (5) and (7), since $\lim_{c\to\infty} Es = \infty$. The proof is completed.

When $p = 0$, part (ii) of Theorem 1 reduces to an elementary renewal theorem, which was proved in [2], in a slightly restricted form by requiring that $m_n = E(x_n)$ for each n.

3. The second moment. Assume that $Ex_n^2 < \infty$ for each n, let

$$V_n = \sum_1^n E((x_j - m_j)^2 \mid \mathfrak{F}_{j-1})$$

for $n = 1, 2, \cdots$, and define s as before. For a random variable y, put $\|y\| = (Ey^2)^{\frac{1}{2}}$.

THEOREM 2. *If (2) holds and* $E((x_n - m_n)^2 \mid \mathfrak{F}_{n-1}) \leqq K < \infty$, *then* $Es^2 < \infty$, $ES_s^2 < \infty$, *and as* $c \to \infty$,

$$ES_s^2 + ET_s^2 = EV_s + 2ES_sT_s, \tag{8}$$

$$\lim ES_s^2/ET_s^2 = 1, \tag{9}$$

$$\lim \mu^2 Es^2/(c^2 Es^{2p}) = 1, \tag{10}$$

$$\lim ES_s^2/(c^2 Es^{2p}) = 1, \tag{11}$$

$$\lim \mu Es^2/(cEs^{1+p}) = 1. \tag{12}$$

PROOF. (i) First, assume that for some $0 < \delta < \mu/8$ and $0 < M < \infty$, $P[x_n \leqq m_n + n\delta + M] = 1$ for all large n. Set $t = \min(s, k)$ for $k = 1, 2, \cdots$. Then by Theorem 1 and Lemma 6 of [3], $E(S_t - T_t)^2 = EV_t \leqq KEt$. Hence by Schwarz inequality

$$E_kS_t^2 + E_kT_t^2 \leqq KE_kt + 2\|T_t\|_k \cdot \|S_t\|_k, \tag{13}$$

where $E_ky = \int_{[s \leqq k]} y\, dP$ and $\|y\|_k = (E_ky^2)^{\frac{1}{2}}$ for a random variable y. Assume, on the contrary, that $Es^2 = \infty$. Then $\lim_{k\to\infty} E_kt^2 = \infty$ and (2) implies that

$$E_km_t^2 = O(1) + o(E_kt^2) = o(E_kt^2),$$

as $k \to \infty$. Hence

$$\|S_t\|_k \leqq \|ct^p + m_t + \delta t + M\|_k + O(1) \leqq c\|t^p\|_k + \delta\|t\|_k + o(\|t\|_k) = (\delta + o(1))\|t\|_k; \tag{14}$$

and from (2),

$$E_kT_t^2 = O(1) + (\mu^2 + o(1))E_kt^2 = (\mu^2 + o(1))E_kt^2. \tag{15}$$

By (13), (14) and (15), we have

$$1 + E_kS_t^2/E_kT_t^2 \leqq O(\|t\|_k^{-1}) + 2\|S_t\|_k/\|T_t\|_k \leqq O(\|t\|_k^{-1}) + (2\delta + o(1))/\mu = 2\delta/\mu + o(1).$$

Since $\delta < \mu/8$, we have a contradiction when k is large. Therefore $Es^2 < \infty$ and $E_kt^2 = O(1)$. Hence

$$\|S_t\|_k \leqq \|ct^p + m_t + \delta t + M\|_k + O(1) \leqq O(1) + \|m_t\|_k \leqq O(1) + o(E_kt^2) = O(1).$$

By Fatou's lemma, $ES_s^2 < \infty$ and from (13), $ET_s^2 < \infty$.

(ii) For the general case, let $x_n' = \min(x_n, m_n + n\delta + M)$ for arbitrary constants $\infty > M > 0$ and $0 < \delta < \mu/16$. Define m_n', T_n' and t as in the proof of part (ii) of Theorem 1. Then by (4) (for $\alpha = 2$), $0 \leqq m_n - m_n' \leqq K(n\delta)^{-1}$. Hence $P[x_n' \leqq m_n' + 2n\delta + M] = 1$ for all large n, and $\lim T_n'/n = \mu$ uniformly on $\Omega - N$. It is not too difficult to see that

$$E((x_n - m_n)^2 \mid \mathfrak{F}_{n-1}) - E((x_n' - m_n')^2 \mid \mathfrak{F}_{n-1}) \geqq 0.$$

Therefore $E((x_n' - m_n')^2 \mid \mathfrak{F}_{n-1}) \leqq K$. Since $t \geqq s$ and from part (i) $Et^2 < \infty$, we have $Es^2 < \infty$. By Theorem 1 and Lemma 6 of [3] again,

$$E(S_s - T_s)^2 = EV_s \leqq KEs. \tag{16}$$

For $\epsilon > 0$, (2) implies that there exists a constant $\infty > L > 0$ such that $ET_s^2 \leqq L + (\mu^2 + \epsilon)Es^2$. Hence $ET_s^2 < \infty$ and from (16), $ES_s^2 < \infty$. Thus (8) follows.

Now by (16),

$$|ES_s^2 - ET_s^2| \leqq E|S_s^2 - T_s^2| \leqq \|S_s - T_s\| \cdot \|S_s + T_s\| \leqq (KEs)^{\frac{1}{2}}\|S_s + T_s\|.$$

Since $ES_s^2 \geqq c^2Es^{2p}$, from (3)

$$|1 - ET_s^2/ES_s^2| \leqq (KEs/ES_s^2)^{\frac{1}{2}}(1 + \|T_s\|/\|S_s\|) = o(1) + o(\|T_s\|/\|S_s\|)$$

as $c \to \infty$. Hence (9) follows.

Since (2) implies that $ET_s^2 = O(1) + (\mu^2 + o(1))Es^2$ as $c \to \infty$, from (9)

$$\lim_{c\to\infty} \mu^2Es^2/ET_s^2 = 1 = \lim \mu^2Es^2/ES_s^2. \tag{17}$$

Let $Z_n = \sum_1^n (x_j - m_j)^2$. Applying Lemma 6 of [3], we have

$$E(x_s - m_s)^2 \leqq EZ_s = E\sum_1^s E((x_j - m_j)^2 \mid \mathfrak{F}_{j-1}) \leqq KEs.$$

From (2), $Em_s^2 = O(1) + o(Es^2) = o(Es^2)$ as $c \to \infty$. Hence

$$Ex_s^2 = E(x_s - m_s + m_s)^2 = o(Es^2), \qquad \|x_s\| = o(\|s\|). \tag{18}$$

Now from (18), as $c \to \infty$,

$$c\|s^p\| \leqq \|S_s\| \leqq \|cs^p + x_s\| \leqq c\|s^p\| + \|x_s\| = c\|s^p\| + o(\|s\|). \tag{19}$$

Therefore (10) follows from (17) and (19), and (11) follows from (17) and (10).

Now $ET_sS_s = O(1) + (\mu + o(1))EsS_s$ as $c \to \infty$. By the definition of s and (18), as $c \to \infty$,

$$cEs^{1+p} \leqq EsS_s \leqq cEs^{1+p} + Esx_s \leqq cEs^{1+p} + \|s\| \cdot \|x_s\| \leqq cEs^{1+p} + o(Es^2). \tag{20}$$

Since $EV_s \leqq KEs$, from (8), (9), (10), and (11), $\lim ES_sT_s/(\mu^2Es^2) = 1$. Hence $\lim EsS_s/(\mu Es^2) = 1$ and then (20) implies (12).

4. Corollaries and comments. In this section we assume that x_n is a sequence of random variables and $p = 0$. Define S_n, m_n, T_n, $\mathfrak{F}_n$ and s as in Section 2.

COROLLARY 1. *If* (2) *holds,* $E(x_n^2) < \infty$ *for each n, and*

$$E((x_n - m_n)^2 \mid \mathfrak{F}_{n-1}) \leqq K < \infty, \tag{21}$$

then $Es^2 < \infty$ *and*

$$\lim_{c\to\infty} Es^\alpha/c^\alpha = \mu^{-\alpha} \quad \text{for} \quad 0 \leqq \alpha \leqq 2. \tag{22}$$

PROOF. Since (21) implies $E(x_n - m_n)^2 \leqq K$, from (2) and (21) it follows [8] that $\lim S_n/n = \mu$ a.e. Hence

$$1 \leqq \liminf_{c\to\infty} S_s/c \leqq \liminf \mu s/c \leqq \limsup \mu s/c = \limsup \mu(s-1)/c$$
$$= \limsup S_{s-1}/c \leqq 1.$$

Therefore $\lim s/c = \mu^{-1}$ a.e. Since $p = 0$, from (10) we have that $E(s/c)^2 \leqq M < \infty$ for all large c. Hence (see [5], p. 629) for every $0 \leqq \alpha < 2$, $(s/c)^\alpha$ is uniformly integrable and

$$\lim_{c\to\infty} E|\mu^{-1} - s/c|^\alpha = 0, \qquad \lim Es^\alpha/c^\alpha = \mu^{-\alpha}. \tag{23}$$

Thus (22) follows from (23) and (10).

COROLLARY 2. *Let x_n be a sequence of independent, identically distributed random variables such that $Ex_1 > 0$ and $E(x_1 - Ex_1)^2 < \infty$. Then for every $c > 0$, as $n \to \infty$,*

$$P[S_1 < c, \cdots, S_n < c] = o(n^{-2}). \tag{24}$$

PROOF. Since $[s > n] = [S_1 < c, \cdots, Sn < c]$, $Es^2 < \infty$ implies (24) and thus Corollary 2 follows from Corollary 1.

(22) has been proved by Hatori [6] for every $\alpha > 0$, by requiring, in addition to the assumptions of Corollary 1, that x_n be independent, $P[x_n \geqq 0] = 1$ and $m_n \geqq L > 0$ for each n.

Under the conditions of Corollary 2, Morimura [9] proves that

$$P[S_1 < c, \cdots, S_n < c] = O(n^{-\delta})$$

for $0 \leqq \delta < (1 + 5^{\frac{1}{2}})/2$ and that there exists an example such that for some $D > 0$ and for each $\epsilon > 0$, $P[S_1 < c, \cdots, S_n < c] \geqq Dn^{-2-\epsilon}$ when n is large enough. Thus (24) is the best possible. Clearly, Corollary 2 completes Morimura's work.

The counter example in [9] satisfies the condition $Es^{2+\epsilon} = \infty$ for every $\epsilon > 0$, since $P[s > n] \neq o(n^{-2-\epsilon})$. Therefore (22) can not be extended to the cases where $\alpha > 2$, without some conditions such as $P[x_n \geqq 0] = 1$ imposed in [6].

COROLLARY 3. *Let x_n be a sequence of independent, identically distributed random variables such that $0 < Ex_1 = \mu \leqq \infty$ and $E(x_1^-)^2 < \infty$. Then (22) holds.*

PROOF. Let $0 \leqq \alpha \leqq 2$. For $0 < \mu' < \mu$, choose $0 < M < \infty$ so that $Ex_1' = D > \mu'$, where $x_n' = \min(x_n, M)$. Define $S_n' = x_1' + \cdots + x_n'$ and t = first $n \geqq 1$ such that $S_n' \geqq c$. By Corollary 1, $\limsup Es^\alpha/c^\alpha \leqq \lim Et^\alpha/c^\alpha = D^{-\alpha}$. Since μ' is arbitrary, $\limsup_{c\to\infty} Es^\alpha/c^\alpha \leqq \mu^{-\alpha}$. Hence (22) holds for $\mu = \infty$ and $E(s/c)^2 \leqq M < \infty$ for all large c. Now assume $0 < \mu < \infty$. By the strong law of large numbers, $\lim_n S_n/n = \mu$ a.e. Hence for $0 \leqq \alpha < 2$, as in the proof of

Corollary 1, (23) holds and therefore (22) holds. For the case $\alpha = 2$, by Theorem 2 of [2] $\lim_c Es/c = \mu^{-1}$. Hence $\limsup_c E(s/c)^2 \geqq \lim_c E^2(s/c) = \mu^2$. Therefore $\lim_c E(s/c)^2 = \mu^{-2}$ and the proof is completed.

Corollary 3 has been recently proved by Heyde [7] under the stronger condition that $E(x_1^-)^3 < \infty$.

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