ON TWO EQUIVALENCE RELATIONS BETWEEN MEASURES[1]

By David A. Freedman

University of California, Berkeley

1. Summary. Let I be the closed unit interval, with the usual topology; Π the set of probabilities on I, with the weak* topology: $\mu_n \rightarrow \mu$ in Π if and only if $\int_I f\, d\mu_n \rightarrow \int_I f\, d\mu$ for each continuous, real-valued f on I. For μ, ν in Π, recall that $\mu \equiv \nu$ means: $\mu(A) = 0$ if and only if $\nu(A) = 0$ for all Borel subsets A of I. Of course, $\equiv$ is an equivalence relation. The graph of $\equiv$, namely the set of $(\mu, \nu) \varepsilon \Pi \times \Pi$ with $\mu \equiv \nu$, is Borel (2.11 of [2]). One result of this paper is: *there is no Borel selector for* $\equiv$; that is, there is no Borel subset of Π meeting each $\equiv$ class in exactly one point. Let $\Sigma(\equiv)$ be the σ-field of all Borel subsets of Π saturated under $\equiv$, that is, containing with μ all $\nu \equiv \mu$. If there were a Borel selector for $\equiv$, there would be a Borel function f on Π with $f(\mu) = f(\nu)$ if and only if $\mu \equiv \nu$, and $\Sigma(\equiv)$ would be separable. However,

(1) Proposition. $\Sigma(\equiv)$ *is inseparable.*

The proof of (1) is based on the following idea of Blackwell. Let $\mathfrak{F}$ be a σ-field, and P a probability on $\mathfrak{F}$. Say P is continuous if each $\mathfrak{F}$-atom has outer P-measure 0, and say P is $0 - 1$ if $P(A) = 0$ or 1 for all $A \varepsilon \mathfrak{F}$.

(2) Lemma (Blackwell). *If P is continuous and* $0 - 1$, $\mathfrak{F}$ *is inseparable.*

Thus, (1) follows from

(3) Theorem. *There is a continuous* $0 - 1$ *probability on* $\Sigma(\equiv)$.

Two proofs of (3) will be given in Sections 2 and 4 respectively. Section 3 contains a result on random distribution functions [3], which may be of independent interest, and which is used in Section 4.

Section 5 deals with the coarser equivalence relation $\approx$, where $\mu \approx \nu$ means: $\mu(A) = 0$ if and only if $\nu(A) = 0$ for all open subsets A of I. Now $\approx$ is induced by a Borel function (3.5 of [2]). More is true:

(4) Theorem. *There is a Borel selector for* $\approx$.

2. First proof of (3). Let I^Z be the set of functions from the positive integers Z to I. For f and g in I^Z, let $f \sim g$ if and only if there is a permutation π (possibly infinite) of Z with $f = g \circ \pi$. Of course, $\sim$ is an equivalence relation. Let W be the set of all $f \varepsilon I^Z$ which are one-to-one. Of course, $W \varepsilon \Sigma(\sim)$. As usual, $W\Sigma(\sim)$ is the σ-field of all subsets of W of the form $W \cap S$, $S \varepsilon \Sigma(\sim)$.

(5) Lemma. *There is a continuous* $0 - 1$ *probability Q on* $W\Sigma(\sim)$.

Proof. Take Lebesgue measure on the Borel σ-field of each factor I of I^Z, and form the infinite product measure. Let Q be the restriction of this product measure to $W\Sigma(\sim)$. Plainly, Q is a continuous probability. Finally, Q is $0 - 1$ by the Hewitt-Savage $0 - 1$ law (Theorem 11.3 of [5]). □

Received 20 December 1965.

[1] This paper was prepared with the partial support of the U. S. Army Research Office (Durham) DA-31-124-ARO-D-548; and a grant from the Alfred P. Sloan Foundation.

Let D be the set of discrete $\mu \varepsilon \Pi$, that is, those μ with $\sum_{x \varepsilon I} \mu\{x\} = 1$. Define functions s and p from D to I^Z by requirements (6) through (9), for all $\mu \varepsilon D$ and $i \varepsilon Z$:

(6) $\mu\{s(\mu)(i)\} = p(\mu)(i) > 0;$

(7) $\sum_{i \varepsilon Z} p(\mu)(i) = 1;$

(8) $p(\mu)(i) \geqq p(\mu)(i+1);$

(9) $p(\mu)(i) = p(\mu)(i+1)$ implies $s(\mu)(i) > s(\mu)(i+1)$.

It is not hard to verify that

(10) s is Borel.

Plainly, for μ and ν in D,

(11) $\mu \equiv \nu$ if and only if $s(\mu) \sim s(\nu)$.

Since a subset of W which is both analytic and complentary analytic is Borel (Section 34 of [4]), (10) and (11) imply

(12) $A \to s(A)$ is a σ-isomorphism of $D\Sigma(\equiv)$ onto $W\Sigma(\sim)$.

(By [6], $s(A)$ is not Borel for general Borel A.) For $A \varepsilon D\Sigma(\equiv)$, let $P(A) = Q[s(A)]$. By (5) and (12),

(13) P is a continuous $0-1$ probability on $D\Sigma(\equiv)$,

which proves (3).

3. A random distribution function. Let 2^Z be the set of functions from the positive integers Z to the two-point set $\{0, 1\}$, with the usual product structure. Write B for the set of all finite sequences of 0's and 1's (including the empty sequence $\varnothing$). For $b \varepsilon B$, let $\bar{b}$ be the set of all $f \varepsilon 2^Z$ which agree with b on its domain. Thus $\overline{\varnothing} = 2^Z, \overline{00} = \{f : f \varepsilon 2^Z, f(1) = f(2) = 0\}$. Let J^B be the set of all functions t from B to the open unit interval J, with the usual product structure. For $t \varepsilon J^B$, let $M^*(t)$ be the probability on 2^Z which satisfies

(14) $M^*(t)(\overline{b0}) = t(b)M^*(t)(\bar{b}).$

Map 2^Z onto I by sending f to $\sum_{n=1}^{\infty} f(n)2^{-n}$. This sends $M^*(t)$ to $M(t) \varepsilon \Pi$. Take Lebesgue measure λ on the Borel subsets of each factor J of J^B, and form the infinite product measure λ^B. Let $P_\lambda = \lambda^B M^{-1}$.

Plainly, M is a continuous and 1-1 map of J^B onto the set of $\mu \varepsilon \Pi$ which assign positive mass to all nonempty open subsets of I. Moreover, P_λ-almost all μ are continuous (4.4 of [3]). Let $\nu \varepsilon \Pi$.

(15) THEOREM. *P_λ-almost all μ are singular with respect to any probability ν.*

PROOF. Suppose without loss of generality that ν assigns positive mass to all nonempty open subsets of I. For each $x \varepsilon I$ and $n = 0, 1, \cdots$, let $I(n, x)$ be the unique interval $[0, 2^{-n}]$, $(2^{-n}, 2 \cdot 2^{-n}]$, $\cdots$, $(1 - 2^{-n}, 1]$ which contains x. Fix

$x \varepsilon I$. For $n = 0, 1, \cdots$, the ratios of $M(\cdot)[I(n+1, x)]$ to $M(\cdot)[I(n, x)]$ are independent and uniform on I with respect to λ^B, by (14). Therefore, the ratio of $M(t)[I(n, x)]$ to $\nu[I(n, x)]$ converges to a finite, positive limit for λ^B-almost no t. By Fubini's theorem, for λ^B-almost all t, the ratio of $M(t)[I(n, x)]$ to $\nu[I(n, x)]$ converges to a finite, positive limit for ν-almost no x. From the usual martingale argument (VII.8 of [1]), for any such t, $M(t) \perp \nu$. □

4. Second proof of (3). Let C be the set of all continuous $\mu \varepsilon \Pi$ (that is, $\mu\{x\} = 0$ for all $x \varepsilon I$) which assign positive measure to all non-empty open subsets of I. Plainly, $C \varepsilon \Sigma(\equiv)$. It is clear from (15) that

$$(16) \qquad P_\lambda \text{ is a continuous probability on } C\Sigma(\equiv).$$

To prove (3), it is now enough to prove

(17) LEMMA. *P_λ is $0 - 1$ on $C\Sigma(\equiv)$.*

PROOF. For t and u in J^B, say $t \sim u$ provided $t(b) = u(b)$ for all but finitely many b. Plainly, $\sim$ is an equivalence relation, and $t \sim u$ implies $M(t) \equiv M(u)$. So, if $A \varepsilon C\Sigma(\equiv)$, then $M^{-1}A \varepsilon \Sigma(\sim)$. By the Kolmogorov $0 - 1$ law (p. 102 of [1]), λ^B is $0 - 1$ on $\Sigma(\sim)$. □

5. Proof of (4). It is not much harder to prove (4) when I is any compact metric set. To avoid trivial complications, suppose I is infinite. For $\mu \varepsilon \Pi$, let $C(\mu)$ be the smallest closed subset of I having μ-measure 1. Let 2^I be the set of non-empty closed subsets of I, in the usual compact metric topology (Section 28 of [4]). For $K \varepsilon 2^I$ and $x \varepsilon I$, let $m(K, x)$ be the set of $y \varepsilon K$ whose distance from x is minimal. Clearly,

$$(18) \qquad K \to m(K, x) \quad \text{is upper semicontinuous,}$$

and

(19) the diameter of $m(K, x)$ is at most twice the distance from x to K.

Let $R = \{r_1, r_2, \cdots\}$ be a dense subset of I. Define the functions $R^{(0)}, R^{(1)}, \cdots, R^{(\infty)}$ from 2^I to 2^I as follows: $R^{(0)}(K) = K$; $R^{(n+1)}(K) = m(R^{(n)}(K), r_{n+1})$ for $n = 0, 1, \cdots,$; $R^{(\infty)}(K) = \bigcap_{n=0}^{\infty} R^{(n)}(K)$. By (19), $R^{(\infty)}(K)$ consists of a single point, call it $R^*(K)$. By (18), R^* is a Borel function from 2^I to I. Now, let $R_n = \{r_n, r_{n+1}, \cdots\}$ for $n = 1, 2, \cdots$, and let $\rho(K) \varepsilon \Pi$ assign mass 2^{-n} to $R_n^*(K)$ for $n = 1, 2, \cdots$. Plainly, ρ is a Borel mapping from 2^I to Π. By (19), $\{R_1^*(K), R_2^*(K), \cdots\}$ is a dense subset of K, so $C(\rho(K)) = K$. In particular, ρ is 1-1, so $\rho(2^I)$ is Borel (Section 43 of [4]) in Π, and is plainly a selector for $\approx$. This completes the proof of (4).

The situation is similar when I is a $G_{\delta\sigma}$, but I do not know what happens for general Borel I.

REFERENCES

[1] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.

[2] DUBINS, L. E. and FREEDMAN, D. A. (1964) Measurable sets of measures. *Pacific J. Math.* **14** 1211–1222.

[3] DUBINS, L. E. and FREEDMAN, D. A. (1965). Random distribution functions. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* To appear.

[4] HAUSDORFF, FELIX (1957). *Set Theory*. Chelsea, New York.

[5] HEWITT, EDWIN and SAVAGE, L. J. (1955). Symmetric measures on Cartesian products. *Trans. Amer. Math. Soc.* **80** 470–501.

[6] PURVES, ROGER (1965). Bimeasurable functions. *Fund. Math.* To appear.