ON SOME NONPARAMETRIC ESTIMATES FOR SHIFT IN THE BEHRENS-FISHER SITUATION[1]

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1. Introduction and summary. It is now widely recognized that from the point of view of robustness, nonparametric tests, such as the Wilcoxon or normal scores test, should be used in practice for the two-sample location problem instead of the classical t-test. But until recently there were no robust estimates for the difference between the two location parameters. In a recent paper [4] Hodges and Lehmann proposed a solution for this problem. After this paper it is clear that all the arguments that can be used in favor of the Wilcoxon test as against the classical t-test can be used in favor of the estimate med $(Y_j - X_i)$ for the difference between the locations as against the classical estimate $(\bar{Y} - \bar{X})$. In their paper quoted above, Hodges and Lehmann propose a whole class of nonparametric estimates corresponding to a class of nonparametric tests both for the two-sample and the one-sample location problems. To indicate this correspondence suppose $h(X, Y)$ is a test-statistic, nonparametric or otherwise for the equality of the locations of X and Y. After having observed (X, Y) we estimate the difference between the two location parameters by the amount of shift required to match the samples X and Y in such a way that $h(X, Y)$ is close to its expected value when the shift is zero. For a more precise definition of the estimates the reader is referred to (1.2) below. For a corresponding definition of the one-sample estimates the reader is referred to (3.3).

Since the difference between the two one-sample estimates for location is an estimate for the difference between the two location parameters in the two-sample problem the paper of Hodges and Lehmann throws open a whole class of estimates for location in the two-sample problem. The aim of this paper is two-fold. First, how do these estimates compare among themselves in the Behrens-Fisher situation where the scale parameters of the populations can possibly differ? Second, how do these estimates compare with the classical estimate when the scales differ? A basic requirement to be able to answer the above questions is the asymptotic normality of the estimates in the Behrens-Fisher situation and this is shown under fairly general conditions in Sections 2 and 3. In answer to the first question the following main result is proved: If the Ψ^*-score test is the best linear rank order test when the underlying distribution is F^* then the difference between the two one-sample estimates based on the Ψ^*-score test is more efficient than the

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simultaneous two-sample estimate based on the Ψ^*-score test. It follows as a particular case that the difference between the two one-sample estimates based on the normal scores test is more efficient than the simultaneous two-sample estimate based on the normal scores test when the prototype distribution is normal. In answer to the second question, the following result is shown. The differences between the one-sample Hodges-Lehmann estimates has all the advantages over the classical estimate in the case of inequality of variances as in the case of equality of variances. For a more precise statement the reader is referred to the end of Section 4.

1.1. *General assumptions and notation.* All the distributions that occur are assumed to be continuous. By the existence of a density we mean except possibly at a countable number of points of which any finite interval contains at most a finite number of points. To avoid repetition this is not stated explicitly. Φ as usual denotes the cdf of standard normal and φ its density. K stands for a generic constant.

1.2. *Preliminaries.* Suppose $X_1, X_2, \cdots, X_m$ is a random sample from a cdf F and $Y_1, Y_2, \cdots, Y_n$ is a random sample from a cdf $G(x - \Delta)$. Let $V^{(1)}, V^{(2)}, \cdots, V^{(N)}$ be an ordered sample from a cdf Ψ where $N = m + n$ and let

$$h(X, Y) = n^{-1} \sum_{j=1}^{n} E_{\Psi}(V^{(s_j)}) \tag{1.1}$$

where $s_1, s_2, \cdots, s_n$ are the ranks of $Y_1, Y_2, \cdots, Y_n$ in the combined sample. Let $E\{h(X, Y)\} = \mu$, independent of F and G when $F = G$ and $\Delta = 0$. Then a class of estimates proposed by Hodges and Lehmann in [4] for Δ is obtained by shifting $Y_1, Y_2, \cdots, Y_n$ by $\hat{\Delta}$ in such a way that $h(X, Y - \hat{\Delta})$ is close to μ. More precisely let

$$\hat{\delta}_1 = \sup\{\Delta : h(X, Y - \Delta) > \mu\} \quad \text{and} \quad \hat{\delta}_2 = \inf\{\Delta : h(X, Y - \Delta) < \mu\} \tag{1.2}$$

and

$$\hat{\Delta} = \tfrac{1}{2}(\hat{\delta}_1 + \hat{\delta}_2). \tag{1.3}$$

The estimate $\hat{\Delta}$ depends on the Ψ we start with and is shown in [4] to be med $(Y_j - X_i)$ if Ψ is uniform. For future reference, we state a few results about the distribution of $\hat{\Delta}$ when F and G are not necessarily the same but are symmetric about the same point. The proofs follow as in [4] and [5].

THEOREM 1.0. (Høyland) *Let $\hat{\Delta}$ be defined by* (1.3) *with h defined by* (1.1) *and suppose $F, G \in \mathfrak{F}_1$, the class of continuous distributions symmetric about the origin. Then $\hat{\Delta}$ is symmetrically distributed about Δ.*

LEMMA 1.1. *If $\hat{\Delta}$ is defined by* (1.1), (1.2) *and* (1.3), *then*

(i) $\hat{\Delta}(x, y + a) = \Delta(x, y) + a$ *for all x and y,*

(ii) $P_{\Delta}(\hat{\Delta} - \Delta \leqq u) = P_0(\hat{\Delta} \leqq u)$,

(iii) $P\{h(X, Y - a) < \mu\} \leqq P(\hat{\Delta} \leqq a) \leqq P\{h(X, Y - a) \leqq \mu\}$,

where P_{Δ} indicates that the probability is taken when the true distributions are $F(x)$ and $G(x - \Delta)$.

THEOREM 1.1. *Let* $a, c_1, c_2, \cdots$ *be real constants,* $\Delta_N = a/c_N$ *and suppose that*

$$\lim_{N\to\infty} P_N\{c_N(h_N - \mu_N) \leqq u\} = \Phi[(u + aB)/A] \tag{1.4}$$

where by P_N *we mean the probability when the true distributions are* F *and* $G(x + \Delta_N)$. *Then for fixed* Δ,

$$\lim_{N\to\infty} P_\Delta\{c_N(\hat{\Delta}_N - \Delta) \leqq a\} = \Phi(aB/A). \tag{1.5}$$

REMARK. If the conditions of Theorem 1.1 are satisfied, Theorem 1.1 shows that $\hat{\Delta}$ estimates consistently the shift in G, which is unrelated to F. But as we see later, the conditions of Theorem 1.1 are satisfied only when there is some relation between F and G such as symmetry about the same point and then Δ refers to both F and G.

2. Asymptotic normality of $h(X, Y)$.

DEFINITION 2.1. A pair of distributions F and G are said to overlap if the probability measures P_1 and P_2 induced by F and G on the real line are *not* mutually singular.

LEMMA 2.1. *Suppose* X_1, X_2 *are independent random variables with* cdf F *and* Y_1, Y_2 *are independent random variables with* cdf G. *If* F *and* G *overlap then* $P(X_1 < Y_1 < Y_2 < X_2) > 0$ *and* $P(Y_1 < X_1 < X_2 < Y_2) > 0$.

OUTLINE OF PROOF. First note that

$$P(X_1 < Y_1 < Y_2 < X_2) = \iint_{-\infty<x<y<\infty} F(x)\{1 - F(y)\}\, dG(x)\, dG(y). \tag{2.1}$$

Let $S_\epsilon = \{x: \epsilon < F(x) < 1 - \epsilon\}$, $0 \leqq \epsilon < \frac{1}{2}$. By using (2.1) it is easy to see that

$$P(X_1 < Y_1 < Y_2 < X_2) = 0 \Rightarrow P_2(S_\epsilon) = 0, \quad 0 < \epsilon < \tfrac{1}{2}, \Rightarrow P_2(S_0) = 0.$$

But $P_1(S_0^c) = 0$ and hence F and G do not overlap.

NOTATION. Let $X_1, X_2, \cdots, X_m$ be a random sample from the cdf F and $Y_1, Y_2, \cdots, Y_n$ from $G(x + \Delta_N)$ where $\Delta_N = a/N^{\frac{1}{2}}$. We denote $G(x + \Delta_N)$ by $G_N(x)$. Let $h(X, Y)$ be defined by (1.1). Let $J(u) = \Psi^{-1}(u)$, $0 < u < 1$, $\lambda_N = (m/N)$ and $0 < \lambda_0 = \lim_{N\to\infty} (\lambda/N) < 1$. Let

$$\begin{aligned} H_N(x) &= \lambda_N F(x) + (1 - \lambda_N)G_N(x), \\ H_0(x) &= \lambda_0 F(x) + (1 - \lambda_0)G(x), \\ a_N &= \textstyle\int_{-\infty}^{\infty} J\{H_N(x)\}\, dG_N(x), \\ a_0 &= \textstyle\int_{-\infty}^{\infty} J\{H_0(x)\}\, dG(x); \end{aligned} \tag{2.2}$$

$$\begin{aligned} \sigma_1^2 &= \iint_{-\infty<x<y<\infty} F(x)\{1 - F(y)\}J'\{H_0(x)\} \\ &\quad \cdot J'\{H_0(y)\}\, dG(x)\, dG(y), \\ \sigma_2^2 &= \iint_{-\infty<x<y<\infty} G(x)\{1 - G(y)\}J'\{H_0(x)\} \\ &\quad \cdot J'\{H_0(y)\}\, dF(x)\, dF(y), \end{aligned} \tag{2.3}$$

$$\sigma_{1N}^2 = \iint_{-\infty<x<y<\infty} F(x)\{1 - F(y)\}J'\{H_N(x)\} \cdot J'\{H_N(y)\}\, dG_N(x)\, dG_N(y),$$

$$\sigma_{2N}^2 = \iint_{-\infty<x<y<\infty} G_N(x)\{1 - G_N(y)\}J'\{H_N(x)\} \cdot J'\{H_N(y)\}\, dF(x)\, dF(y);$$

$$(2.4) \qquad A_N^2 = 2\lambda_0\{\sigma_{1N}^2 + [\lambda_0/(1 - \lambda_0)]\sigma_{2N}^2\}$$
$$A^2 = 2\lambda_0\{\sigma_1^2 + [\lambda_0/(1 - \lambda_0)]\sigma_2^2\}.$$

ASSUMPTIONS. (a) $J'(u) = J^{(1)}(u)$ and $J''(u) = J^{(2)}(u)$ exist for $0 < u < 1$ and there exist $0 < k, \delta > 0$ such that $|J(u)| \leqq K[u(1-u)]^{-\frac{1}{2}+\delta}$ and $|J^{(i)}(u)| \leqq K[u(1-u)]^{-i-\frac{1}{2}+\delta}$, $i = 1, 2$.

(b) Ψ has a bounded density ψ.

(c) F and G overlap.

(d) $aB = -\lim_{N\to\infty} N^{\frac{1}{2}}(a_N - a_0)$ exists and is finite.

THEOREM 2.1. *Under the above assumptions and with the above notation*

$$\lim_{N\to\infty} P_N[N^{\frac{1}{2}}\{h(X, Y) - a_0\} \leqq u] = \Phi[(u + aB)/A]$$

where P_N denotes the probability when the true distributions are F and $G(x + \Delta_N)$.

PROOF. Write

$$(2.5) \qquad N^{\frac{1}{2}}\{h(X, Y) - a_0\} = N^{\frac{1}{2}}\{h(X, Y) - a_N\} + N^{\frac{1}{2}}(a_N - a_0).$$

By Assumption (d) the second term on the right side tends to $-aB$. To evaluate the limit law of the first term we first show that $\sigma_{1N}^2 \to \sigma_1^2$ and $\sigma_{2N}^2 \to \sigma_2^2$. Let $0 < \nu_0 < \lambda_0 < \nu_1 < 1$. Put $G_N(x) = u$, $G_N(y) = v$ so that $x = G_N^{-1}(u)$, $y = G_N^{-1}(v)$. Write $H_N[G_N^{-1}(u)] = \xi_N(u)$. Then

$$(2.6) \qquad \begin{aligned} \sigma_{1N}^2 &= \iint_{0<u<v<1} F\{G_N^{-1}(u)\}[1 - F\{G_N^{-1}(v)\}]J'\{\xi_N(u)\}J'\{\xi_N(v)\}\, du\, dv \\ &= \iint_{0<u<v<1} R_N(u, v)\, du\, dv \quad \text{(say)}. \end{aligned}$$

By 7.A.5. of [1], $R_N(u, v) \leqq \nu_0^{-2}\xi_N(u)\{1 - \xi_N(v)\}J'\{\xi_N(u)\}J'\{\xi_N(v)\}$. Now

$$\begin{aligned} \xi_N(u) = H_N\{G_N^{-1}(u)\} &= \lambda_N F\{G_N^{-1}(u)\} + (1 - \lambda_N)u \\ &\geqq (1 - \lambda_N)u > (1 - \nu_1)u \qquad \text{for large } N; \end{aligned}$$

$$\begin{aligned} 1 - \xi_N(u) = 1 - H_N\{G_N^{-1}(u)\} &\geqq 1 - \lambda_N - (1 - \lambda_N)u \\ &= (1 - \lambda_N)(1 - u) > (1 - \nu_1)(1 - u) \qquad \text{for large } N. \end{aligned}$$

Assuming without loss of generality that $0 < \delta < \frac{1}{2}$ and using Assumption (a) and the above inequalities

$$\begin{aligned} R_N(u, v) &\leqq (K/\nu_0^2)[\xi_N(u)\{1 - \xi_N(v)\}][\xi_N(u)\{1 - \xi_N(u)\}]^{\delta-3/2} \\ &\quad \cdot [\xi_N(v)\{1 - \xi_N(v)\}]^{\delta-3/2} \\ &= (K/\nu_0^2)\{\xi_N(u)\}^{\delta-\frac{1}{2}}\{1 - \xi_N(u)\}^{\delta-3/2}\{\xi_N(v)\}^{\delta-3/2}\{1 - \xi_N(v)\}^{\delta-\frac{1}{2}} \end{aligned}$$

$$\leqq Ku^{\delta-\frac{1}{2}}(1-u)^{\delta-3/2}v^{\delta-3/2}(1-v)^{\delta-\frac{1}{2}},$$

independent of N. This can be seen to be integrable in the range $0 < u < v < 1$. Thus the integrand in σ_{1N}^2 is bounded in N and can easily be seen to be continuous in Δ_N. It follows by the dominated convergence theorem that $\sigma_{1N}^2 \to \sigma_1^2$. Similarly $\sigma_{2N}^2 \to \sigma_2^2$. We now show that $\sigma_1^2 > 0$. By simple algebra one gets $J'(u) = [\psi\{\Psi^{-1}(u)\}]^{-1} \geqq K > 0$ by Assumption (b). Thus by Assumption (c) and Lemma 2.1 we get $\sigma_1^2 \geqq KP[X_1 < Y_1 < Y_2 < X_2] > 0$. Similarly $\sigma_2^2 > 0$. It follows that for sufficiently large values of N, σ_{1N}^2 and σ_{2N}^2 are bounded away from 0 and $A_N^2 \to A^2$. Thus the conditions of Theorem 1 and Corollary 1 of [1] are satisfied and we get

$$\lim_{N\to\infty} \{P_N[N^{\frac{1}{2}}\{h(X, Y) - a_N\}/A_N] \leqq u\} = \Phi(u).$$

Since $A_N \to A$, an application of Slutsky's theorem gives

$$\lim_{N\to\infty} P[N^{\frac{1}{2}}\{h(X, Y) - a_N\} \leqq u] = \Phi(u/A). \tag{2.7}$$

Another application of Slutsky's theorem to (2.5) gives

$$\lim_{N\to\infty} P_N[N^{\frac{1}{2}}\{h(X, Y) - a_0\} \leqq u] = \Phi[(u + aB)/A]. \qquad \text{Q.E.D.}$$

Remark. A sufficient condition for the asymptotic normality of $h(X, Y)$ for Pitman alternatives is the uniform asymptotic normality for these alternatives. Assumptions (a), (b) and (c) are made so that Theorem 1 and Corollary 1 of [1] are applicable. One can state some conditions under which Assumption (d) holds but these seem to be restrictive and hence we examine each case separately.

Corollary 2.1. *Suppose* $\int J\{H_0\}\, dG = \int J(G)\, dG = \mu$ *independent of* F *and* G. *Then it follows fom Theorems* 1.1 *and* 2.1 *that*

$$\lim_{N\to\infty} P_N[N^{\frac{1}{2}}\{h(X, Y) - \mu\} \leqq u] = \Phi[(u + aB)/A]$$

and

$$\lim_{N\to\infty} P[N^{\frac{1}{2}}(\hat{\Delta} - \Delta) \leqq a] = \Phi(aB/A).$$

Remarks. We see below that Theorem 2.2 of [5] and the asymptotic normality of $\hat{\Delta}$ when $F = G$ (see [4], p. 609) which are apparently unrelated to each other, come out as special cases of our Theorem 2.1 and Corollary 2.1. Besides, Theorem 2.1 and Corollary 2.1 prove the asymptotic normality of $\hat{\Delta}$ whenever F and G are symmetric about the same point. It should be mentioned, however, that there is a slight difference between the conditions here and those of Theorem 2.2 of [5]. We had to impose the extra condition that F and G overlap to make sure that the asymptotic variance of $h(X, Y)$ is bounded away from zero.

Special Cases. (1) Ψ *is uniform*. In this case $J(u) = u$ and we have the following

Theorem 2.2. *Suppose* $\hat{\Delta} = \text{med}_{i,j}(Y_j - X_i)$ *and* $\Delta =$ *the median of the distribution of* $(Y - X)$ *and* (i) F *and* G *overlap* (ii) F *and* G *have bounded densities* f *and* g *respectively. Then* $\lim_{N\to\infty} P_\Delta\{N^{\frac{1}{2}}(\hat{\Delta} - \Delta) \leqq a\} = \Phi(aB/A)$, *where*

$$(2.8)\qquad A^2 = \lambda_0^2\{\lambda_0^{-1} \int G^2\, dF + (1 - \lambda_0)^{-1} \int F^2\, dG - [4\lambda_0(1 - \lambda_0)]^{-1}\},$$
$$B = \lambda_0 \int_{-\infty}^{\infty} g(x)f(x)\, dx.$$

Proof. We intend to apply Theorem 2.1 and Corollary 2.1. For this we first note that $\Delta = \text{med}\,(Y - X) \Leftrightarrow \int F\, dG = \frac{1}{2}$. Hence

$$\int \{\lambda_0 F + (1 - \lambda_0)G\}\, dG = \lambda_0 \int F\, dG + (1 - \lambda_0) \int G\, dG$$
$$= \tfrac{1}{2}\lambda_0 + \tfrac{1}{2}(1 - \lambda_0) = \tfrac{1}{2}$$

independent of λ_0. Thus the conditions of Corollary 2.1 are satisfied. Conditions (a), (b) and (c) of Theorem 2.1 are obviously satisfied. The only condition that needs to be checked is condition (d). Now,

$$\begin{aligned} N^{\frac{1}{2}}(a_N - a_0) &= N^{\frac{1}{2}}[\int_{-\infty}^{\infty} \{\lambda_N F(x) + (1 - \lambda_N)G(x + \Delta_N)\}\, dG(x + \Delta_N) \\ &\quad - \int_{-\infty}^{\infty} \{\lambda_0 F(x) + (1 - \lambda_0)G(x)\}\, dG(x)] \\ &= N^{\frac{1}{2}}[\int_{-\infty}^{\infty} \{\lambda_N F(x - \Delta_N) + (1 - \lambda_N)G(x)\}\, dG(x) \\ &\quad - \int_{-\infty}^{\infty} \{\lambda_N F(x) + (1 - \lambda_N)G(x)\}\, dG(x)] \\ &= N^{\frac{1}{2}}[\int_{-\infty}^{\infty} \lambda_N\{F(x - \Delta_N) - F(x)\}\, dG(x)]. \end{aligned}$$

Note that $\Delta_N = a/N^{\frac{1}{2}}$ and $N^{\frac{1}{2}}\{F(x - \Delta_N) - F(x)\}$ is bounded because of condition (ii) and the dominated convergence theorem allows us to proceed to the limit under the integral sign. Thus

$$\lim_{N\to\infty} N^{\frac{1}{2}}(a_N - a_0) = -a\lambda_0 \int_{-\infty}^{\infty} f(x)\, g(x)\, dx.$$

As for the constant A^2 we get from (5.1) of the appendix

$$\begin{aligned} A^2 &= 2\lambda_0[\tfrac{1}{2}\{\int G^2\, dF - \tfrac{1}{4}\} + [\lambda_0/2(1 - \lambda_0)]\{\int F^2\, dG - \tfrac{1}{4}\}] \\ &= \lambda_0^2\{\lambda_0^{-1} \int G^2\, dF + (1 - \lambda_0)^{-1} \int F^2\, dG - [4\lambda_0(1 - \lambda_0)]^{-1}\}. \qquad \text{Q.E.D.} \end{aligned}$$

(2) $F = G$.

Theorem 2.3. *Suppose* $\hat{\Delta}$ *is given by* (1.3) *with* $\mu = E\{h(X, Y)\}$ *when* $\Delta = 0$ *and* h *is given by* (1.1) *and*

(i) F *has a bounded density* f;

(ii) J *satisfies the regularity conditions* (a) *and* (b) *of Theorem* 2.1;

(iii) $f(x)J'\{F(x)\}$ *is bounded.*

Then

$$\lim_{N\to\infty} P_\Delta[N^{\frac{1}{2}}(\hat{\Delta} - \Delta) \leq a] = \Phi(aB/A)$$

where

$$(2.9)\qquad A^2 = [\lambda_0/(1 - \lambda_0)][\int_0^1 J^2(u)\, du - \{\int_0^1 J(u)\, du\}^2];$$
$$B = \lambda_0 \int J'\{F(x)\}f^2(x)\, dx.$$

Proof. It is easy to see that all the conditions of Theorem 2.1 and Corollary 2.1

are satisfied except Assumption (d) of Theorem 2.1 which we check now:

$$\begin{aligned} N^{\frac{1}{2}}(a_N - a_0) &= N^{\frac{1}{2}}[\textstyle\int_{-\infty}^{\infty} J\{\lambda_N F(x) + (1 - \lambda_N)F(x + \Delta_N)\}\, dF(x + \Delta_N) \\ &\quad - \textstyle\int J[F(x)]\, dF(x)] \\ &= N^{\frac{1}{2}}[\textstyle\int_{-\infty}^{\infty} J[\lambda F(x - \Delta_N) + (1 - \lambda)F(x)]\, dF(x) \\ &\quad - \textstyle\int_{-\infty}^{\infty} J[F(x)]\, dF(x)]. \end{aligned}$$

By making use of conditions (i) and (iii) it is easy to see as in Lemma 3(b) of [3] that

$$\lim_{N\to\infty} N^{\frac{1}{2}}(a_N - a_0) = -a\lambda_0 \textstyle\int J'\{F(x)\}f^2(x)\, dx.$$

As regards A^2 it follows from (4.19) of [1] that A^2 is given by (2.9).

(3) *Ψ symmetric, F and G symmetric about the same point.*

We first prove a lemma.

LEMMA 2.2. *If Ψ is symmetric and F and G are symmetric about the same point, then $\int J\{H_0(x)\}\, dG(x) = \mu$, the mean of Ψ, independent of λ_0.*

PROOF. Without loss of generality we assume that F, G and Ψ are symmetric about the same point μ. Now

$$\int_{-\infty}^{\infty} J\{H_0(x)\}\, dG(x) = \int_{-\infty}^{\mu} J\{H_0(x)\}\, dG(x) + \int_{\mu}^{\infty} J\{H_0(x)\}\, dG(x). \tag{2.10}$$

By putting $\mu - x = y$ in the first integral on the right side and using the symmetry of F and G we get

$$\begin{aligned} &\textstyle\int_{-\infty}^{\mu} J\{H_0(x)\}\, dG(x) \\ &\quad = \textstyle\int_0^{\infty} J[\lambda_0\{1 - F(\mu + y)\} + (1 - \lambda_0)\{1 - G(\mu + y)\}\, dG(\mu + y) \\ &\quad = \textstyle\int_0^{\infty} J[1 - \{\lambda_0 F(\mu + y) + (1 - \lambda_0)G(\mu + y)\}]\, dG(\mu + y) \end{aligned}$$

which, by (5.2.a) of the appendix,

$$\begin{aligned} &= 2\mu \textstyle\int_0^{\infty} dG(\mu + y) - \int_0^{\infty} J[\lambda_0 F(\mu + y) + (1 - \lambda_0)G(\mu + y)]\, dG(\mu + y) \\ &= \mu - \textstyle\int_0^{\infty} J[\lambda_0 F(\mu + y) + (1 - \lambda_0)G(\mu + y)]\, dG(\mu + y). \end{aligned} \tag{2.11}$$

Now, putting $z = x - \mu$ in the second integral on the right side of (2.10) we get

$$\int_0^{\infty} J[\lambda_0 F(\mu + z) + (1 - \lambda_0)G(\mu + z)]\, dG(\mu + z). \tag{2.12}$$

By adding (2.11) and (2.12) we get the desired result.

THEOREM 2.4. *Suppose $\hat{\Delta}$ is the estimate* (1.3) *and*

(i) *Ψ is symmetric and $J = \Psi^{-1}$ satisfies the regularity condition* (a) *of Theorem* 2.1,

(ii) *F and G overlap and are symmetric about the same point possessing densities f and g respectively,*

(iii) *$J'\{\lambda F(x) + (1 - \lambda)G(x)\}\{\lambda f(x) + (1 - \lambda)g(x)\}$ is bounded uniformly in λ in a neighborhood of λ_0.*

Then $\lim_{N\to\infty} P_\Delta\{N^{\frac{1}{2}}(\hat{\Delta} - \Delta) \leqq a\} = \Phi(aB/A)$ *with* A^2 *defined by* (2.4) *and* $B = \lambda_0 \int_{-\infty}^{\infty} J'\{H_0(x)\}f(x)g(x)\,dx$.

PROOF. We again intend to apply Theorem 2.1 and Corollary 2.1. The conditions of Corollary 2.1 are satisfied because of Lemma 2.2. It only remains to check condition (d) of Theorem 2.1 as the rest are obvious. Now, from Lemma 2.2,

$$(2.13)\quad N^{\frac{1}{2}}(a_N - a_0) = N^{\frac{1}{2}}[\int_{-\infty}^{\infty} J[\lambda_N F(x - \Delta_N) + (1 - \lambda_N)G(x)]\,dG(x) - \int_{-\infty}^{\infty} J[\lambda_N F(x) + (1 - \lambda_N)G(x)]\,dG(x)].$$

Now we show that

$$(2.14)\quad \theta^{-1}[J\{\lambda F(x - \theta) + (1 - \lambda)G(x)\} - J\{\lambda F(x) + (1 - \lambda)G(x)\}]$$

is bounded uniformly in λ in a neighborhood of λ_0 and in $\theta > 0$ so that the limit and integration can be interchanged in (2.13) and we can proceed to the limit as if λ_N and Δ_N were independent. Now, J, F and G being nondecreasing (2.14) is bounded above by 0, (2.14) is obviously bounded below by

$$A(x, \theta, \lambda) = \theta^{-1}[J\{\lambda F(x - \theta) + (1 - \lambda)G(x - \theta) - J\{\lambda F(x) + (1 - \lambda)G(x)\}]$$
$$= -J'\{\lambda F(x - \xi) + (1 - \lambda)G(x - \xi)\}[\lambda f(x - \xi) + (1 - \lambda)g(x - \xi)]$$

and by condition (iii) the above is bounded uniformly in λ and ξ. Thus integration and limit can be interchanged and by simple calculus one can see that $\lim_{N\to\infty} N^{\frac{1}{2}}(a_N - a_0) = -aB$ where B is given in the theorem. Q.E.D.

REMARKS. We shall use Theorem 2.4 only in the Behrens-Fisher situation where $G(x) = F(cx)$. Condition (iii) of the theorem is the crucial one and we examine this and the other conditions of the theorem now.

COROLLARY 2.2. *Suppose* $G(x) = F(cx)$ *and* $\Psi = \Phi$, $J_0 = \Phi^{-1}$. *Then all the conditions of Theorem* 2.4 *are satisfied for any distribution* F *for which* $f(x)J_0'\{F(x)\}$ *is bounded and then* $\hat{\Delta}$ *is asymptotically normal.*

PROOF. It is well known that J_0 satisfies the regularity condition (a) of Theorem 2.1. Condition (ii) of the above theorem is obviously satisfied. It follows from (5.3.b) of the appendix that condition (iii) of the theorem is satisfied. Q.E.D.

It may be mentioned that the condition of boundedness of $f(x)J_0'\{F(x)\}$ is not new and is assumed in [1]. The class of distributions F for which the above condition is satisfied is large and includes normal, logistic, Laplace and Cauchy distributions among others (see 5.3.c. of the appendix in this connection). The asymptotic variance of the estimate can be derived in each case from the general expression given in Theorem 2.4. We study these estimates in greater detail in a later section. Since we wish to compare $\hat{\Delta}$ with the estimates based on the one-sample tests we now turn to the one-sample case.

3. The one-sample case.

We now prove some theorems in the one-sample case similar to those of Section 2. If the proofs are entirely analogous to those of Section 2 they are omitted. The essential difference is that in the one-sample case

m and n are random with nonrandom sum whereas in the two-sample case m and n are not random.

Suppose $Z_1, Z_2, \cdots, Z_N$ is a random sample from a distribution $\Pi(x - \theta)$. Π being arbitrary here θ does not have any statistical significance. But later on we are going to fix the location of Π by some condition such as symmetry around the origin and then θ acquires statistical significance. Suppose $s_1, s_2, \cdots, s_n$ are the ranks of positive Z's among $|Z_1|, |Z_2|, \cdots, |Z_N|$. Let $V^{(1)}, V^{(2)}, \cdots, V^{(N)}$ be the ordered absolute values from the distribution Ψ. Let

$$h(Z) = N^{-1} \textstyle\sum_{j=1}^{n} E_\Psi(V^{(s_j)}). \tag{3.1}$$

Let $E\{h(Z)\} = \mu$ when Π is symmetric about $\theta = 0$. Let

$$\hat{\theta}_1 = \sup\{\theta: h(Z - \theta) > \mu\} \quad \text{and} \quad \hat{\theta}_2 = \inf\{\theta: h(Z - \theta) < \mu\} \tag{3.2}$$

and

$$\hat{\theta} = \tfrac{1}{2}(\hat{\theta}_1 + \hat{\theta}_2). \tag{3.3}$$

When Π is symmetric about 0, Hodges and Lehmann [4] proposed $\hat{\theta}$ as an estimate for θ and showed that this estimate has the same advantages over the sample mean as those of the corresponding tests based on h over the one-sample t-test. We will now show the asymptotic normality of θ whether Π is symmetric or not.

Let $\theta_N = a/N^{\frac{1}{2}}$ and $\Pi(x + \theta_N) = \Pi_N(x)$. Let $p_N = 1 - \Pi_N(0)$ and $p = 1 - \Pi(0)$ and

$$\begin{aligned}
F_N(x) &= [\Pi_N(x) - \Pi_N(0)]/[1 - \Pi_N(0)], && \text{for } x \geqq 0,\\
&= 0 && \text{otherwise};\\
G_N(x) &= [\Pi_N(0) - \Pi_N(-x)]/\Pi_N(0), && \text{for } x \geqq 0,\\
&= 0 && \text{otherwise};\\
F(x) &= [\Pi(x) - \Pi(0)]/[1 - \Pi(0)], && \text{for } x \geqq 0,\\
&= 0; && \text{otherwise},\\
G(x) &= [\Pi(0) - \Pi(-x)]/\Pi(0), && \text{for } x \geqq 0,\\
&= 0 && \text{otherwise}.
\end{aligned} \tag{3.4}$$

Let

$$\begin{aligned}
H_N^*(x) &= p_N F_N(x) + (1 - p_N) G_N(x);\\
H_0^*(x) &= pF(x) + (1 - p)G(x);\\
U_N &= 2 \textstyle\iint_{-\infty<x<y<\infty} G_N(x)\{1 - G_N(y)\} J'\{H_N^*(x)\}\\
&\quad \cdot J'\{H_N^*(y)\}\, dF_N(x)\, dF_N(y);\\
V_N &= 2 \textstyle\iint_{-\infty<x<y<\infty} F_N(x)\{1 - F_N(y)\} J'\{H_N^*(x)\}
\end{aligned}$$

$$\cdot J'\{H_N^*(y)\}\, dG_N(x)\, dG_N(y);$$

$$U = 2 \iint_{-\infty<x<y<\infty} G(x)\{1 - G(y)\}J'\{H^*(x)\}J'\{H^*(y)\}\, dF(x)\, dF(y);$$

$$V = 2 \iint_{-\infty<x<y<\infty} F(x)\{1 - F(y)\}J'\{H^*(x)\}J'\{H^*(y)\}\, dG(x)\, dG(y).$$

Let

$$\Psi_0(x) = \Psi(x) - \Psi(-x), \qquad \text{if } x \geqq 0,$$
$$= 0 \qquad \text{otherwise},$$

and

$$J_0(u) = \Psi_0^{-1}(u),\ J_0'(u) = J_0^{(1)}(u), \qquad \Delta_N(x) = F_N(x) - G_N(x),$$

$$\Delta(x) = F(x) - G(x),\ L_N^{(i)} = \int_{-\infty}^{\infty} \Delta_N^{\,i}(x)J_0^{(i)}\{H_N^*(x)\}\, dF_N(x), \qquad i = 0, 1;$$

$$L^{(i)} = \int_{-\infty}^{\infty} \Delta^{(i)}(x)J_0^{(i)}\{H^*(x)\}\, dF(x), \qquad i = 0, 1;$$

$$\mu_2^{(N)} = N^{-1}p_N(1 - p_N), \qquad \mu_2 = N^{-1}p(1 - p), \qquad \alpha_N = p_N L_N^{(0)},$$

$$\alpha_0 = pL^{(0)},$$

$$\beta_N^{\,2} = p_N(1 - p_N)[\{p_N U_N + (1 - p_N)V_N\} + \{L_N^{(0)} + p_N L_N^{(1)}\}^2],$$

$$\beta^2 = p(1 - p)[\{pU + (1 - p)V\} + \{L^{(0)} + pL^{(1)}\}^2].$$

Assumptions. (a) $J_0'(u) = J_0^{(1)}(u)$ and $J_0''(u) = J_0^{(2)}(u)$ exist for $0 < u < 1$ and there exist $K > 0$, $\delta > 0$ such that $|J_0(u)| \leqq K\{u(1 - u)\}^{-\frac{1}{2}+\delta}$ and $|J_0^{(i)}(u)| \leqq K\{u(1 - u)\}^{-i+\delta}$;

(b) Π admits a density π and $\Pi(0) \neq 0$ or 1;

(c) Ψ has a bounded density ψ;

(d) F and G overlap;

(e) $aB = -\lim_{N\to\infty} N^{\frac{1}{2}}(\alpha_N - \alpha_0)$ exists and is finite.

Theorem 3.1. *Under the above assumptions*

$$\lim_{N\to\infty} P_N[N^{\frac{1}{2}}\{h(Z) - \alpha_0\} \leqq u] = \Phi[(u + aB)/A]$$

where P_N denotes the probability when the true distribution is $\Pi(x + \theta_N)$.

Proof. The proof is quite analogous to Theorem 2.1. We here use the results of [2].

Corollary 3.1. *Suppose* $pL^{(0)} = p \int J_0\{H^*(x)\}\, dF(x) = \frac{1}{2} \int J_0(F)\, dF = \mu = \frac{1}{2}$ (mean of Ψ_0). *Then*

$$\lim_{N\to\infty} P_N[N^{\frac{1}{2}}\{h(Z) - \mu\} \leqq u] = \Phi[(u + aB)/\beta]$$

and for every fixed θ,

$$\lim_{N\to\infty} P_\theta[N^{\frac{1}{2}}(\hat{\theta} - \theta) \leqq a] = \Phi(aB/\beta).$$

Proof. The first part follows from Theorem 3.1 and the second part follows from the one-sample analogue of Theorem 1.1.

Special Cases. (1) Π *is symmetric about* 0.

In this case we have the following

THEOREM 3.2. *Suppose* $\hat{\theta}$ *is the estimate* (3.3) *with* $\mu = \frac{1}{2}$ (mean of Ψ_0) *and*

(i) J_0 *satisfies the regularity condition* (a) *of Theorem* 3.1*;*

(ii) Ψ *is symmetric and unimodal with density* ψ*;*

(iii) Π *admits a unimodal density* π *satisfying the following properties*

(a) $\pi(a)$ *is bounded in a neighborhood of the origin,*

(b) $J_0'\{\Pi(x)\}\pi(x)$ *is bounded.*

Then $\lim_{N\to\infty} P_\theta\{N^{\frac{1}{2}}(\hat{\theta} - \theta) \leqq a\} = \Phi(aB/\beta)$, *where* $B = \frac{1}{2}\int_0^\infty J_0'[Q(x)]q^2(x)\,dx$ *with*

$$Q(x) = \Pi(x) - \Pi(-x), \qquad x > 0,$$
$$= 0 \qquad \textit{otherwise}$$

and $q(x) = Q'(x)$ *and* $\beta^2 = \frac{1}{4}$ [*second moment of* Ψ_0].

PROOF. We intend to apply Theorem 3.1 and Corollary 3.1. Note first that in this case F and G defined by (3.4) coincide and $p = \frac{1}{2}$. We now check condition (e) of Theorem 3.1 as the rest are obvious. The proof for this is analogous to Lemma 6c.1 of [2]. We only mention the points of difference with the proof there. Lemma 6c.1 of [2] is proved only for $J_0(u) = \chi^{-1}(u)$, the inverse of the χ-distribution. But a careful examination of the proof shows that this applies for any J_0 such that J_0' is nondecreasing. In our case this is assured by condition (ii). Furthermore, our conditions iii(a) and iii(b) give us conditions (ii) and (iii) of Lemma 6c.1 of [2] for, then we can take $r(x) = \pi(x)$. It only remains to check the constants B and β. Note first that when $F = G$, $L^{(1)} = 0$ and hence

$$\begin{aligned}\beta^2 &= p(1-p)[pU + (1-p)V] + pq(L^{(0)})^2\\ &= \tfrac{1}{4}[\textstyle\int_0^1 J_0^2(u)\,du - \{\int_0^1 J_0(u)\,du\}^2] + \tfrac{1}{4}[\int_0^1 J_0(u)\,du]^2\\ &= \tfrac{1}{4}\textstyle\int_0^1 J_0^2(u)\,du = \tfrac{1}{4}\,[\text{second moment of } \Psi_0].\end{aligned}$$

Our constant B is the same as $I'(0)$ of Lemma 6c.1 of [2].

REMARKS. Conditions (i) and (ii) concern the Ψ used in the statistic and are satisfied for Wilcoxon and normal scores tests among others. Condition iii(b) is the crucial one and it can be shown that for the normal scores test this is satisfied whenever $J_0'\{F(x)\}f(x)$ is bounded, which is true for a large class of distributions.

DEFINITION 3.1 (Høyland). Let Z_1 and Z_2 be independent random variables with cdf Π. Then we call med $\frac{1}{2}(Z_1 + Z_2)$ the pseudomedian of Π.

(2) Ψ *is uniform in* [0, 1].

THEOREM 3.3. *Let* $\hat{\theta} = \text{med}_{i\leqq j}\,\frac{1}{2}(Z_i + Z_j)$ *and* θ *the pseudomedian of* Π *and suppose that*

(i) F *and* G *defined by* (3.4) *overlap,*

(ii) Π *admits a bounded density* π.

Then $\lim_{N\to\infty} P_\theta\{N^{\frac{1}{2}}(\hat{\theta} - \theta) \leqq a\} = \Phi(aB/\beta)$ *where*

$$B = \int \pi(-x)\pi(x)\,dx \quad \textit{and} \quad \beta^2 = [\int \pi^2(-x)\pi(x)\,dx - \tfrac{1}{4}].$$

PROOF. It is shown in [4] that when Ψ is uniform $\hat{\theta} = \text{med}_{i \leqq j} \frac{1}{2}(Z_i + Z_j)$. We now check condition (e) of Theorem 3.1. Let us first write α_N and α_0 in terms of Π_N and Π:

$$\alpha_N = p_N \int J_0\{H_N^*(x)\}\, dF_N(x) = \int J_0\{p_N F_N(x) + (1 - p_N G_N(x)\} \cdot d(p_N F_N(x)).$$

Recalling the definition of F_N and G_N in (3.4) we see that

$$\alpha_N = \int_0^\infty J_0\{\Pi_N(x) - \Pi_N(-x)\}\, d\Pi_N(x).$$

Similarly

$$\alpha_0 = \int_0^\infty J_0\{(x) - \Pi(-x)\}\, d\Pi(x).$$

Hence

$$\begin{aligned} N^{\frac{1}{2}}(\alpha_N - \alpha_0) &= N^{\frac{1}{2}}[\textstyle\int_0^\infty J_0\{\Pi(x+\theta_N) - \Pi(-x+\theta_N\}\, d\Pi(x+\theta_N) \\ &\qquad - \textstyle\int_0^\infty J_0\{\Pi(x) - \Pi(-x)\}\, d\Pi(x)]. \\ &= N^{\frac{1}{2}}[\textstyle\int_0^{\theta_N} \{\Pi(2\theta_N - x) - \Pi(x)\}\pi(x)\, dx \\ &\qquad - \textstyle\int_0^\infty \{\Pi(-x+2\theta_N) - \Pi(-x)\}\pi(x)\, dx]. \end{aligned} \tag{3.5}$$

Now,

$$\begin{aligned} 0 &\leqq \textstyle\int_0^{\theta_N} N^{\frac{1}{2}}\{\Pi(2\theta_N - x) - \Pi(x)\}\pi(x)\, dx \\ &\leqq a\{\Pi(2\theta_N) - \Pi(0)\}\theta_N^{-1}\, \Pi_0^{\theta_N}\, \pi(x)\, dx. \end{aligned}$$

π being bounded, $(\theta_N)^{-1} \int_0^{\theta_N} \pi(x)\, dx$ is bounded and $\Pi(2\theta_N) - \Pi\,(0) \to 0$ as $N \to \infty$. As for the second term, we have

$$\begin{aligned} N^{\frac{1}{2}}[\Pi(-x+2\theta_N) - \Pi(-x)] &= a\cdot(2\theta_N)^{-1}\{\Pi(-x+2\theta_N) - \Pi(-x)\} \\ &= 2a\{\pi(-x) + O(1)\} \end{aligned}$$

which is bounded by (ii) of the hypothesis. Thus the limit and integration can be interchanged and we see that

$$\lim_{N\to\infty} N^{\frac{1}{2}}(\alpha_N - \alpha_0) = 2a \int_0^\infty \pi(-x)\pi(x)\, dx = a \int_{-\infty}^\infty \pi(-x)\pi(x)\, dx.$$

Now suppose that the pseudomedian of Π is 0. Then θ is the pseudomedian of $\Pi(x - \theta)$ and the assertion follows if we check the rest of the conditions of Theorem 3.1 and Corollary 3.1. We only check the latter as the former are obvious. Suppose X_1 and X_2 are two independent observations from Π. Pseudomedian of Π is $0 \Leftrightarrow P(X_1 + X_2 > 0) = \frac{1}{2}$ and

$$\begin{aligned} P(X_1 + X_2 > 0) &= P(X_1 < 0, X_2 > 0)P(-X_1 < X_2 \mid X_1 < 0 < X_2) \\ &\quad + P(X_1 > 0, X_2 < 0)P(-X_2 < X_1 \mid X_2 < 0 < X_1) \\ &\quad + P(X_1 > 0, X_2 > 0) \end{aligned}$$

or

$$\tfrac{1}{2} = 2p(1-p) \int G\, dF + p^2$$

or

$$p(1-p) \int G\, dF = \tfrac{1}{4} - p^2/2 = \tfrac{1}{4} - p^2 \int F\, dF$$

or

$$[p \int \{pF + (1-p)G\}\, dF] = \tfrac{1}{2}\{\textit{mean of } R(0, 1)\}$$

Thus the conditions of Corollary 3.1 are satisfied. That the constant β reduces to the one given in the theorem can be shown after some computations. Q.E.D.

4. Comparison of the estimates. The theorems of Sections 2 and 3 enable us to compare the several estimates for shift when the prototype distributions are symmetric and satisfy certain regularity conditions. We shall discuss only the case of inequality of variances. We measure the performance of an estimate by the inverse of its asymptotic variance. We denote the estimate (1.3) based simultaneously on both the samples by $\hat{\Delta}_{\Psi 2}$. We denote the one-sample estimate for the location of Y's defined in (3.3) by $\hat{\theta}_{\Psi 2}$ and $\hat{\theta}_{\Psi 1}$ is the corresponding estimate for the locations of X's. We write $\hat{\Delta}_{\Psi 1}$ for $(\hat{\theta}_{\Psi 2} - \hat{\theta}_{\Psi 1})$. $\hat{\Delta}_{\Psi 2}$ and $\hat{\Delta}_{\Psi 1}$ are competitors as estimates for the shift Δ. $\hat{\Delta}_{U2} = \text{med}\,(Y_j - X_i)$ and $\hat{\Delta}_{U1} = \text{med}_{i \leqq j}\, \frac{1}{2}(Y_i + Y_j) - \text{med}_{i \leqq j}\, \frac{1}{2}(X_i + X_j)$ where U stands for the cdf of the uniform distribution. We denote the asymptotic relative efficiency of two estimates $\delta_1(X, Y)$ and $\delta_2(X, Y)$ by $e(\delta_1, \delta_2; \lambda_0, F, G)$ where $F(x)$ and $G(y - \Delta)$ are the underlying distributions with F and G symmetric about 0 and $\lambda_0 = \lim_{N\to\infty} (m/N)$. We denote by $F^{(c)}$ the distribution defined by $F^{(c)}(x) = F(cx)$, $0 < c < \infty$. Thus, when F and G differ in variances $G(X) = F^{(c)}(x)$ with $c \neq 1$. Ψ_0 is defined by

$$\begin{aligned} \Psi_0(x) &= \Psi(x) - \Psi(-x), \qquad && \text{if } x > 0, \\ &= 0 && \text{otherwise.} \end{aligned}$$

We denote Ψ_0^{-1} by J_0. In this section, J_1 stands for Φ^{-1} and J_{10} for χ^{-1}. Δ^* stands for the classical estimate $(\bar{Y} - \bar{X})$. The following two theorems are not entirely new and are implicit in Theorem 6 of [4]. We give them here with the precise conditions needed for future reference.

THEOREM 4.1. *Suppose* (i) Ψ *and* $J = \Psi^{-1}$ *satisfy the regularity conditions of Theorem* 3.2 *with* Ψ *symmetric about* 0; (ii) F *has a bounded symmetric density* f; (iii) $f(x)J'\{F(x)\}$ *is bounded. Then* $e(\hat{\Delta}_{\Psi 2}, \hat{\Delta}_{\Psi 1}; \lambda_0, F, F) = 1$ *irrespective of* λ_0 *and* F.

PROOF. It is easy to see that all the conditions of Theorems 2.3 and 3.2 are satisfied and it follows that the asymptotic variance of $N^{\frac{1}{2}}\hat{\Delta}_{\Psi 2}$ is given by

$$\{\lambda_0(1-\lambda_0)\}^{-1} \cdot \text{variance of } \Psi \cdot [\textstyle\int J'\{F(x)\} f^2(x)\, dx]^{-2} \tag{4.1}$$

and the asymptotic variance of $N^{\frac{1}{2}}\hat{\Delta}_{\Psi 1}$ is given by

(4.2) $\{\lambda_0(1 - \lambda_0)\}^{-1}\cdot$second moment of $\Psi_0[\int_0^\infty J_0'\{2F(x) - 1\}\{2f(x)\}^2\, dx]^{-2}$.

We shall show that the second factors in (4.1) and (4.2) are the same and this proves the result.

First note that the mean of Ψ being 0, variance of $\Psi = \int x^2\, d\Psi = \int |x|^2\, d\Psi = \int x^2\, d\Psi_0 =$ second moment of Ψ_0. Thus the numerators agree. As for the denominators making use of the equalities $J'(w) = J'(1 - w)$, $0 < w < 1$, and $J'(w) = 2J_0'(2w - 1)$, $\frac{1}{2} < w < 1$, we get

$$\begin{aligned}\int_{-\infty}^0 J'\{F(x)\}f^2(x)\, dx &= \int_{-\infty}^0 J'\{1 - F(x)\}f^2(x)\, dx \\ &= \int_{-\infty}^0 2J_0'\{1 - 2F(x)\}f^2(x)\, dx \\ &= \tfrac{1}{2}\int_0^\infty J_0'\{2F(y) - 1\}\{2f(y)\}^2\, dy.\end{aligned}$$

Similarly

$$\int_0^\infty J'\{F(x)\}f^2(x)\, dx = \tfrac{1}{2}\int_0^\infty J_0'\{2F(x) - 1\}\{2f(x)\}^2\, dx.$$

Adding the above two equalities completes the proof of the theorem.

THEOREM 4.2. *If F has a bounded symmetric density f $e(\hat{\Delta}_{U2}, \hat{\Delta}_{U1}; \lambda_0, F, F) = 1$ irrespective of λ_0 and F.*

PROOF. Theorems 2.2 and 3.3 are obviously applicable and the asymptotic variance of $N^{\frac{1}{2}}\hat{\Delta}_{U2}$ is given by

$$(4.3) \qquad \{12\lambda_0(1 - \lambda_0)\}^{-1}\{\int f^2(x)\, dx\}^{-2}.$$

The asymptotic variance of $N^{\frac{1}{2}}\hat{\Delta}_{U1}$ is given by

$$\{\lambda_0(1 - \lambda_0)\}^{-1}[\int F^2(-x)\, dF(x) - \tfrac{1}{4}][\int f(-x)f(x)\, dx]^{-2},$$

which can easily be seen to be equal to (4.3) because of symmetry of f.

THEOREM 4.3. *Suppose*

(i) *F has a bounded symmetric density f;*

(ii) *Ψ is symmetric and unimodal with density ψ and $J_0 = \Psi_0^{-1}$ satisfies the regularity conditions of Theorem* 3.1;

(iii) *$f(x)J_0'\{F(x)\}$ is bounded. Then*

$$e(\hat{\Delta}_{\Psi 1}, \Delta^*; \lambda_0, F, F^{(c)}) = \sigma^2[\int J'\{F(x)\}f^2(x)\, dx]^2$$

irrespective of c and λ_0 where σ^2 is the variance of F.

PROOF. The asymptotic variance of $N^{\frac{1}{2}}\Delta^*$ is given by

$$(4.4) \qquad \sigma^2\{(\lambda_0)^{-1} + [c^2(1 - \lambda_0)]^{-1}\}.$$

It is easy to see that the conditions of Theorem 3.2 are satisfied and from (4.2) which was shown to equal (4.1) the asymptotic variance of $N^{\frac{1}{2}}\hat{\Delta}_{\Psi 1}$ is given by

$$(4.5) \qquad \{(\lambda_0)^{-1} + [c^2(1 - \lambda_0)]^{-1}\}[\int J'\{F(x)\}f^2(x)\, dx]^2.$$

Dividing (4.4) by (4.5) gives the required result.

THEOREM 4.4. *If Ψ satisfies the regularity condition* (i) *of Theorem* 2.4 *and*

(i) *F admits a density and*

(ii) $J'\{\lambda F(x) + (1 - \lambda)F(cx)\}f(cx)$ *is bounded for* λ *in a neighborhood of* λ_0 *and* $-\infty < x < \infty$, *then*

$$e(\hat{\Delta}_{\Psi 2}, \Delta^*; \lambda_0, F, F^{(c)}) = \sigma^2(\lambda_0/c^2 + (1 - \lambda_0))(B^2/2A^2) \tag{4.6}$$

where $B = \int J'\{\lambda_0 F(x) + (1 - \lambda_0)F(cx)\}cf(x)f(cx)\,dx$ *and* $A^2 = \alpha_1^2 + \alpha_2^2$ *with*

$$\alpha_1^2 = \lambda_0 \iint_{-\infty<x<y<\infty} F(cx)\{1 - F(cy)\}J'\{\lambda_0 F(x) + (1 - \lambda_0)F(cx)\} \cdot J'\{\lambda_0 F(y) + (1 - \lambda_0)F(cy)\}\,dF(x)\,dF(y),$$

$$\alpha_2^2 = (1 - \lambda_0) \iint_{-\infty<x<y<\infty} F(x)\{1 - F(y)\} \cdot J'\{\lambda_0 F(x) + (1 - \lambda_0)F(cx)\} \cdot J'\{\lambda_0 F(y) + (1 - \lambda_0)F(cy)\}\,dF(cx)\,dF(cy).$$

PROOF. Theorem 2.4 is applicable and the asymptotic variance of $N^{\frac{1}{2}}\hat{\Delta}_{\Psi 2}$ is given by $\{\lambda_0(1 - \lambda_0)\}^{-1}(2A^2/B^2)$ where

$$A^2 = [\lambda_0 \iint_{-\infty<x<y<\infty} F(cx)\{1 - F(cy)\}J'\{\lambda_0 F(x) + (1 - \lambda_0)F(cx)\} \cdot J'\{\lambda_0 F(y) + (1 - \lambda_0)F(cy)\}\,dF(x)\,dF(y) + (1 - \lambda_0) \iint_{-\infty<x<y<\infty} F(x)\{1 - F(y)\}J'\{\lambda_0 F(x) + (1 - \lambda_0)F(cx)\} \cdot J'\{\lambda_0 F(y) + (1 - \lambda_0)F(cy)\}\,dF(cx)\,dF(cy)],$$

$$B = \int_{-\infty}^{\infty} J'\{\lambda_0 F(x) + (1 - \lambda_0)F(cx)\}f(x)cf(cx)\,dx.$$

On the other hand, from (4.4) the asymptotic variance of $N^{\frac{1}{2}}\Delta^*$ is given by $\sigma^2\{(\lambda_0)^{-1} + [c^2(1 - \lambda_0)]^{-1}\}$ and the result follows.

It follows from Corollary 2.2 that when $\Psi = \Phi$ all the conditions of Theorem 4.4 are satisfied for any distribution for which $f(x)J_1'\{F(x)\}$ is bounded.

We now turn to the comparison of $\hat{\Delta}_{\Psi 2}$ and $\hat{\Delta}_{\Psi 1}$ for a fixed Ψ.

LEMMA 4.1. *Suppose* F, G *and* Ψ *are symmetric and let*

$$I_1 = 2 \iint_{-\infty<x<y<\infty} F(x)\{1 - F(y)\}J'\{H_0(x)\}J'\{H_0(y)\}\,dG(x)\,dG(y)$$

be finite with H_0 *defined in* (2.2). *Then* $I_1 = \int B^2(t)\,dF(t)$ *with*

$$B(t) = \int_0^t J'\{H_0(y)\}\,dG(y). \tag{4.7}$$

PROOF. On account of (4.14) and (4.17) of [1] we need only show that $E\{B(x)\} = 0$ where F is the cdf of X. Because of symmetry of X it suffices to show that $B(t)$ is an odd function of t. Now using the properties $H(-y) = 1 - H(y)$ and $J'(u) = J'(1 - u)$ proved in the appendix we get

$$B(-t) = \int_0^{-t} J'\{H(-y)\}\,dG(y) = -\int_{-t}^{0} J'\{H(-y)\}\,dG(y) = -\int_0^t J'\{H(y)\}\,dG(y) = -B(t). \quad \text{Q.E.D.}$$

In the following theorem we write $H_c(x) = \lambda_0 F(x) + (1 - \lambda_0)F(cx)$.

THEOREM 4.5. *Suppose*

(i) Ψ *and* F *satisfy the regularity conditions of Theorem* 4.4;

(ii) $\lim_{x\to\infty} f(x)B(x) = 0 = \lim_{x\to\infty} f(x)B^*(x)$ *where* $B(x)$ *is defined by* (4.7) *with* $G(x) = F(cx)$ *and* $B^*(x) = \int_0^t J'\{H_{c-1}(t)\}c^{-1}f(c^{-1}t)\,dt$;

(iii) $\int \{f'(x)/f(x)\}^2\,dF(x) < \infty$. *Then*

$$e(\hat{\Delta}_{\Psi 2}, \Delta^*; \lambda_0, F, F^{(c)}) \leqq \sigma^2 E[f'(x)/f(x)]^2$$

irrespective of Ψ *and* λ_0.

PROOF. Observe first that $B^*(x) = \int_0^{c^{-1}x} J'\{H_c(t)\}\,dF(t)$ and hence if we define $B_1(x) = \int_0^x J'\{H_c(t)\}\,dF(t)$, α_1^2 of Theorem 4.4 reduces to $\lambda_0 \int B_1^2(x)\,dF(cx) = \lambda_0 \int B^{*2}(x)\,dF(x)$. Now using Lemma 4.1 in (4.6) we get

$$2A^2 = \lambda_0 \int B^{*2}(x)\,dF(x) + (1-\lambda_0)\int B^2(x)\,dF(x),$$

$$\begin{aligned} \{\lambda_0/c^2 + (1-\lambda_0)\}B^2 &= \lambda_0[\textstyle\int J'\{H_c(x)\}f(x)f(cx)\,dx]^2 \\ &\quad + (1-\lambda_0)[\textstyle\int J'\{H_c(x)\}f(x)cf(cx)\,dx]^2 \\ &= \lambda_0[\textstyle\int J'\{H_{c-1}(x)\}c^{-1}f(x)f(c^{-1}x)\,dx]^2 \\ &\quad + (1-\lambda_0)[\textstyle\int J'\{H_c(x)\}f(x)cf(cx)\,dx]^2. \end{aligned} \tag{4.8}$$

Now notice that because of the continuity of the integrands $dB(x)/dx = J'\{H_c(x)\}cf(cx)$ and $dB^*(x)/dx = J'\{H_{c^{-1}}(x)\}c^{-1}f(c^{-1}x)$. Thus

$$\begin{aligned} \{\lambda_0/c^2 + (1-\lambda_0)\}B^2 &= \lambda_0\{\textstyle\int [dB^*(x)/dx]f(x)\,d_x\}^2 \\ &\quad + (1-\lambda_0)\{\textstyle\int dB(x)/dx]f(x)\,dx\}^2. \end{aligned} \tag{4.9}$$

After integrating by parts and using condition (ii) of the theorem we see that

$$\int [dB^*(x)/dx]f(x)\,dx = \int B^*(x)\{-f'(x)/f(x)\}\,dF(x),$$

$$\int [dB(x)/dx]f(x)\,dx = \int B(x)\{-f'(x)/f(x)\}\,dF(x).$$

Now condition (iii) allows us to apply Schwarz's inequality to the above and we get

$$\begin{aligned} \lambda_0\{\textstyle\int [dB^*(x)/dx]f(x)\,dx\}^2 &\leqq \lambda_0 \textstyle\int B^{*2}(x)\,dF(x) \\ &\quad \cdot \textstyle\int \{f'(x)/f(x)\}^2\,dF(x), \\ (1-\lambda_0)\{\textstyle\int [dB(x)/dx]f(x)\,dx]^2 &\leqq (1-\lambda_0)\textstyle\int B^2(x)\,dF(x) \\ &\quad \cdot \textstyle\int \{f'(x)/f(x)\}^2\,dF(x). \end{aligned} \tag{4.10}$$

From (4.6), (4.8), (4.9) and (4.10) we get the result.

It is easy to see that (4.6) and $\sigma^2 E\{f'(x)/f(x)\}^2$ are both invariant under scale changes in F and there is no loss of generality in assuming, as we now do, that $\sigma^2 = 1$. Now suppose that F^* is the underlying distribution with density f^*. It follows from [1] that a locally best linear rank-order test, which we shall call the Ψ^*-score test is given by

$$\Psi^{*-1}\{F^*(x)\} = -[f^{*1}(x)/f^*(x)][E\{f^{*1}/f^*\}^2]^{-\frac{1}{2}}. \tag{4.11}$$

THEOREM 4.6. *Suppose F^* is the underlying distribution and Ψ^* is defined by* (4.11) *and further F^* and $J^* = \Psi^{*-1}$ satisfy the regularity conditions of Theorem* 4.4. *Then $e(\hat{\Delta}_{\Psi^*2}, \hat{\Delta}_{\Psi^*1}; \lambda_0, F^*, F^{*(c)}) \leqq 1$ irrespective of λ_0.*

PROOF. From Theorem 4.3 it follows that

$$(4.12) \quad e(\hat{\Delta}_{\Psi^*1}, \Delta^*; \lambda_0, F^*, F^{*(c)}) = [\int J^{*1}\{F^*(x)\}f^*(x)\, dF^*(x)]^2.$$
$$= [E(f^{*1}/f^*)]^2 \quad \text{from section 5 of [1].}$$

The result follows from (4.12) and Theorem 4.5.

COROLLARY 4.1. $e(\hat{\Delta}_{\Phi2}, \hat{\Delta}_{\Phi1}; \lambda_0, \Phi, \Phi^{(c)}) \leqq 1$.

PROOF. We intend to apply Theorem 4.6. It follows from the results of the appendix that condition (i) of Theorem 4.6 is satisfied. Condition (iii) of the theorem is obviously satisfied. Now we shall check condition (ii).

If $c > 1$,

$$\begin{aligned} 0 \leqq \varphi(x)B(x) &= c\varphi(x) \int_0^x J_1'\{H_c(t)\}\varphi(ct)\, dt \\ &\leqq c\varphi(x) \int_0^x J_1'\{\Phi(ct)\}\varphi(ct)\, dt \\ &= cx\varphi(x) \to 0 \quad \text{as} \quad x \to \infty. \end{aligned}$$

If $c < 1$,

$$\begin{aligned} 0 \leqq \varphi(x)B(x) &\leqq c\varphi(x) \int_0^x J'\{\Phi(t)\}\varphi(ct)\, dt \\ &= c\varphi(x) \int_0^x [\varphi(ct)/\varphi(t)]\, dt \\ &= c\varphi(x) \int_0^x \exp[(1 - c^2)t^x/2]\, dt \\ &\leqq c\varphi(x) \exp[(1 - c^2)x^2/2] \to 0 \quad \text{as} \quad x \to \infty. \quad \text{Q.E.D.} \end{aligned}$$

REMARKS. Corollary 4.1 tells us that when the underlying distribution is normal $\hat{\Delta}_{\Phi1}$ should be preferred to $\hat{\Delta}_{\Phi2}$. In the case of logistic distribution it can be seen that the upper bound for the efficiency of $\hat{\Delta}_{\Phi2}$ given in Theorem 4.5 reduces to $(\Pi^2/9) = 1.10$ and the efficiency of $\hat{\Delta}_{\Phi1}$ is 1.05 (see [3]). On the other hand it can be shown that the efficiency of $\hat{\Delta}_{\Phi2}(\hat{\Delta}_{\Psi2}$ for symmetric Ψ) drops down to $4\sigma^2 f^2(0) = (\Pi^2/12) = .82$ as the ratio of the two variances becomes very large. Thus in the case of logistic distribution also we can say that $\hat{\Delta}_{\Phi1}$ should be preferred to $\hat{\Delta}_{\Phi2}$. It can also be shown by using the results of [3] that when the underlying distribution is uniform the efficiency of $\hat{\Delta}_{\Phi1}$ is infinite whereas the efficiency of $\hat{\Delta}_{\Phi2}$ is finite for any $c \neq 1$.

In answer to the second question raised at the beginning of this paper we are now in a position to conclude the following: From Theorem 4.3 and the results of [1] it follows that whenever the prototype distribution is symmetric $\hat{\Delta}_{\Phi1}$ has all the advantages over Δ^* in the case of inequality of variances as in the case of equality. In particular $e(\hat{\Delta}_{\Phi1}, \Delta^*; \lambda_0, \Phi, \Phi^{(c)}) = 1$ and $e(\hat{\Delta}_{\Phi1}, \Delta^*; \lambda_0, F, F^{(c)}) \geqq 1$ for any F for which $J_0'\{F(x)\}f(x)$ is bounded. $e(\hat{\Delta}_{U1}, \Delta^*; \lambda_0, \Phi, \Phi^{(c)}) = 3/\Pi$ and for any F for which the density is bounded, $e(\hat{\Delta}_{U1}, \Delta^*; \lambda_0, F, F^{(c)}) \geqq .864$. The above conclusions support the use of Hodges-Lehmann estimates $\hat{\Delta}_{\Psi1}$ even in the case of inequality of variances.

5. Appendix. All the elementary results that are used in the paper are stated here explicitly. The proofs are omitted and can be seen after some calculations. J as before stands for Ψ^{-1}.

5.1. If $\int F\, dG = \frac{1}{2}$,

$$I = \int\int_{-\infty<x<y<\infty} F(x)\{1 - F(y)\}\, dG(x)\, dG(y) = [\int G^2\, dF - \tfrac{1}{4}].$$

5.2.a. If Ψ is symmetric about μ, then $J(1-u) = 2\mu - J(u)$ and hence if $J'(u)$ exists at $u = a$, then $J'(u)$ exists for $u = 1 - a$ and $J'(a) = J'(1-a)$.

5.2.b. If (i) Ψ is symmetric about μ and admits a unimodal density, (ii) $J''(u)$ exists at $u = a$, then $J''(a) \geqq a$ for $a \geqq \frac{1}{2}$, $J''(a) \leqq 0$ for $a \leqq \frac{1}{2}$.

5.3.a. Let $J_0 = \Phi_0^{-1}$. Then $J_0(u) \sim [-2\log(1-u)]^{\frac{1}{2}}$ and $\varphi[J_0(u)] \sim (1-u)[-2\log(1-u)]^{\frac{1}{2}}$ as $u \uparrow 1$.

5.3.b. With the notation of 5.3.a, if f is symmetric and $J_0'\{F(x)\}f(x)$ is bounded, then $J_0'\{\lambda F(x) + (1-\lambda)F(cx)\}\{\lambda F(x) + (1-\lambda)\, cf(cx)\}$ is bounded uniformly in x, λ and c for $-\infty < x < \infty$, λ in a neighborhood of λ_0 and $0 < c < 1$.

5.3.c. With the notation of 5.3.a, if $F(x)$ has density $f(x)$ and

$$f(x)[-2\log\{1 - F(x)\}]^{\frac{1}{2}}\{1 - F(x)\}^{-1}$$

is bounded, then $f(x)J_0'\{F(x)\}$ is bounded.

5.4. Suppose Ψ is symmetric and admits a unimodal density. Let Ψ_0 be defined by

$$\begin{aligned} \Psi_0(x) &= 0 && \text{if } x \leqq 0 \\ &= \Psi(x) - \Psi(-x) = 2\Psi(x) - 1 && \text{if } x > 0. \end{aligned}$$

Let $J_0(u) = \Psi_0^{-1}(u)$ for $0 < u < 1$. Then $J_0'(u) \uparrow u$.

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