J. Kubilius, *Probabilistic Methods in the Theory of Numbers* (translated from the Russian by Gretchen Burgie and Susan Schuur). Vol. 11 of Translations of Mathematical Monographs, American Mathematical Society, Providence, 1964. xviii + 182 pp. $8.60 ($6.45 to members).

Review by W. J. LeVeque

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The scope of this monograph is not quite so wide as the title indicates. The author has made no attempt to cover all the applications of probability to number theory, but has restricted himself to one topic: the probabilistic analysis of the behavior of additive number-theoretic functions. There is no mention, for example, of the application of probabilistic methods in the metric theory of Diophantine approximations, or in the theory of distribution modulo 1.

By an additive number theoretic function, one means a function defined on the positive integers and such that $f(mn) = f(m) + f(n)$ whenever m and m are relatively prime. If in addition $f(p^\alpha) = f(p)$ for every prime p and positive integer α, then f is said to be strongly additive. It follows immediately from the unique factorization theorem that if f is additive, then $f(n) = \sum_{p^\alpha \| n} f(p^\alpha)$, and that if f is strongly additive then $f(n) = \sum_{p|n} f(p)$. (Here $p \mid n$ means that p divides n, and $p^\alpha \| n$ that p^α is the exact power of p occurring in the factorization of n.) If $\delta_p(m)$ is 1 or 0 according as $p \mid m$ or not, the latter equation can be written in the form

$$f(m) = \sum_p \delta_p(m) f(p). \tag{1}$$

Now let $D\{a_n\}$ be the asymptotic density of the sequence $\{a_n\}$ of positive integers. If A and B are arithmetic progressions of moduli d and d', respectively, then $DA = d^{-1}$, $DB = d'^{-1}$, and if $(d, d') = 1$, then the Chinese remainder theorem implies that $D(A \cap B) = DA \cdot DB$. In particular,

$$D\{m: \delta_{p_1}(m) = \epsilon_1, \cdots, \delta_{p_r}(m) = \epsilon_r\} = \prod_{j=1}^r D\{m: \delta_{p_j}(m) = \epsilon_j\}, \tag{2}$$

where each of $\epsilon_1, \cdots, \epsilon_r$ is 0 or 1, and $p_1, \cdots, p_r$ are distinct primes. Roughly speaking, the basis of the present book is that, although density is not a countably additive measure, it should be possible to utilize the independence assertion (2) to analyze the behavior of f as a sum (1) of independent random variables. The difficulty with additivity can be temporarily circumvented by truncating the sum in (1), and considering the function $f(m)_r = \sum_{p<r} \delta_p(m) f(p)$, to which the standard limit theorems apply, but there is then a very delicate interchange of limit processes required to obtain information about $f(m)$ itself. This interchange depends, as usual, on obtaining uniform estimates for certain error terms; the latter are provided in this case by a (Brun or Selberg) "sieve" argument.

Chapter I of the present book is devoted to the basic arithmetic lemmas that will be needed later. These include estimates for certain sums in which the index runs over the primes $p \leqq n$, for large n, and sieve-type results estimating (for example) the number of elements of certain kinds of sequences of integers which are not divisible by any of a specified set of primes. In Chapter II, the author lays the probabilistic foundations for the later work by considering various ways in which a genuine probability measure can be associated with the positive integers, or a segment thereof, in such fashion that the function δ_p, or related functions, become independent random variables.

Chapter III is concerned with the law of large numbers. By an argument analogous to that for the Čebyšev inequality, one shows that if f is additive, then

$$\sum_{m=1}^{n} |f(m) - A(n)|^2 = O(nD^2(n)),$$

where $A(n) = \sum_{p \leqq n} f(p)/p$, $D(n) = (\sum_{p^\alpha \leqq n} |f^2(p^\alpha)|/p^\alpha)^{\frac{1}{2}}$. It follows that if $\lim_{n\to\infty} \Psi(n) = \infty$, then $\nu_n\{|f(m) - A(n)| \leqq D(n)\Psi(n)\} \to 1$ as $n \to \infty$, where $\nu_n(\cdots)$ is the frequency of positive integers $m \leqq n$ satisfying $\cdots$. Generalizations are indicated for laws of large numbers for $f(R(m))$ and $f(R(p))$, where R is a polynomial.

In Chapter IV, the author considers limit theorems for distribution functions $\nu_n(f(m) < x)$ for real additive f. In order to make profitable use of the truncated function $f(m)_r$, he restricts himself to what he calls functions of class H: Those additive f for which $D(n) \to \infty$ as $n \to \infty$, and for each of which there exists an unbounded increasing function $r(n)$ such that $\log r(n)/\log n \to 0$ and $D(r(n))/D(n) \to 1$ as $n \to \infty$. For such functions, limit laws for $f(m)$ and $f(m)_r$ can only exist simultaneously, and coincide when they exist. Kolmogoroff's formula for the possible limiting distributions applies, as well as Kubik's explicit characterization of the Kolmogoroff function $K(u)$ for sums of independent random variables which assume only two values. In particular, these limit laws include the following generalization of the well-known Erdös-Kac theorem: for a strongly additive function f such that $B(u) \to \infty$ and $\max_{p \leqq n} |f(p)| = o(B(n))$, $\nu_n\{[f(m) - A(n)]/B(n) < x\} \to \Phi(x)$, where Φ is the normal distribution function and $B(n) = (\sum_{p \leqq n} |f(p)|^2/p)^{\frac{1}{2}}$.

The case in which f is real and $B(n)$ converges as $n \to \infty$ is also treated, by a method of H. Delange, giving the theorem of Erdös and Wintner: a necessary and sufficient condition for the convergence of the distribution law $\nu_n\{f(m) \leqq x\}$ as $n \to \infty$ is the convergence of the series $\sum \|f(p)\|/p$ and $\sum \|f(p)\|^2/p$, where $\|f(p)\| = f(p)$ or 1, according as $|f(p)| < 1$ or $|f(p)| \geqq 1$. Local theorems, concerning $\nu_n\{f(m) = k\}$, are also given for rather severely restricted functions f.

In Chapter V, the author considers asymptotic laws for sums of additive functions with translated arguments, $\sum_1^s \{[f_j(m + a_j) - A_j(n)]/B_j(n)\}$, with fixed s. In Chapter VI, he specializes to certain kinds of functions for which the above sum has a normal limiting distribution, and obtains an estimate for the rate of approach to normality. In Chapter VIII, analogues of Kolmogoroff's inequality and the law of the iterated logarithm are obtained relative to the sequence of truncated functions $f(m)_1, f(m)_2, \cdots$ on the segment $\{1, 2, \cdots, n\}$.

Chapter VII is devoted to many-dimensional asymptotic laws, concerning the joint distribution of $(f_j(m) - A_j(n))/B_j(n)$, $n = 1, \cdots, s$.

Chapter IX is on a different footing from the other chapters. The question considered is the same as in Chapter VI, on the rate of convergence to normality, but now instead of obtaining an estimate for the error term, the goal is an asymptotic expansion of the distribution function $\Omega_n(x) = \nu_n\{(\omega(m) - \log\log n)/(\log\log n)^{\frac{1}{2}} < x\}$, where $\omega(m)$ is the number of distinct prime divisors of m, and the corresponding $A(n)$ and $B(n)$ are (essentially) $\log\log n$. Such an expansion cannot at present be obtained by the truncation method, and instead the author uses the method devised by Rényi and Turán to establish the conjecture, due to the reviewer, that for fixed x,

$$\Omega_n(x) - \Phi(x) = O(1/(\log\log n)^{\frac{1}{2}}).$$

This method, which has not yet been successfully applied to functions significantly different from $\omega(m)$, depends on the powerful analytic theory of the Riemann zeta-function. The asymptotic expansion obtained is of the kind one would expect from the classical Cramér expansion theorem, and it would be most interesting to obtain a probabilistic derivation of it.

The book closes with a brief chapter on additive number theoretic functions in the ring of Guassian integers, and a bibliography of 97 items.

The reviewer noticed misprints only on pp. 27, 37, and 72. Some of the displayed formulas, which were photographically reproduced from the Russian book, have a damaged appearance. The translation is both literate and accurate. The author's sure grasp of the two fields in which he is working has provided a most welcome monograph to those mathematicians interested in both probability and number theory.