ON THE BLOCK STRUCTURE OF CERTAIN PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

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1. Introduction. In an earlier paper [3], the author gave the upper bounds for the number of disjoint blocks in (i) semi-regular GD designs, (ii) certain PBIB designs with two associate classes having a triangular association scheme, (iii) certain PBIB designs with two associate classes having an L_2 association scheme and (iv) certain PBIB designs with three associate classes having a rectangular association scheme. Later on, the author [4] gave bounds for the number of common treatments between two blocks of the above-mentioned designs. In this paper, we generalise the author's [3] results and give conditions under which no two blocks of the above-mentioned designs are (i) disjoint or (ii) the same set.

2. Semi-regular GD designs. An incomplete block design with v treatments each replicated r times in b blocks of size k is said to be group divisible (GD) [2], if the treatments $v = mn$ can be divided into m groups, each with n treatments, so that treatments belonging to the same group occur together in λ_1 blocks and treatments belonging to different groups occur together in λ_2 blocks $(\lambda_1 \neq \lambda_2)$. The primary parameters of such a design are $v = mn$, b, r, k, $n_1 = (n - 1)$, $n_2 = n(m - 1)$, λ_1, λ_2. They obviously satisfy the relations $bk = vr$, $(n - 1)\lambda_1 + n(m - 1)\lambda_2 = r(k - 1)$, $r \geqq \lambda_1$, $r \geqq \lambda_2$. Semi-regular GD designs [1] are further characterised by $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$.

From Theorem 2.1 of [3], we deduce Theorem 2.1.

THEOREM 2.1. *If in a semi-regular* GD *design,* $b = v - m + r$ *and* $v = 2k$, *where* k *is an odd integer, then no two blocks of this design are disjoint.*

THEOREM 2.2. *If in a given block of a semi-regular* GD *design with* $b > v - m + 1$ *has* d *blocks having a given number* l $(\leqq k)$ *of treatments common with it, then*

$$d \leqq b - 1 - [k(r - 1) - l(b - 1)]^2/Q,$$

where $Q = P + l^2(b - 1) - 2lk(r - 1)$, *and* $P = k^2[(v - k)\cdot(b - r) - (v - rk)(v - m)]/v(v - m)$. *Further, if* $d = b - 1 - [k(r - 1) - l(b - 1)]^2/Q$, *then* $[P - lk(r - 1)]/[k(r - 1) - l(b - 1)]$ *is an integer and the given block has* $[P - lk(r - 1)]/[k(r - 1) - l(b - 1)]$ *treatments common with each of the remaining* $(b - d - 1)$ *blocks.*

PROOF. We number the blocks $B_1, B_2, \cdots, B_b$. Let x_i denote the number of treatments common between B_1 and B_i, $i = 2, 3, \cdots, b$. Let $x_i = l$ for $i = 2, 3, \cdots, (d + 1)$.

Received 28 September 1964; revised 28 December 1965.

Then, from the author's [3] results (2.1) and (2.3), we obtain

$$\sum_{i=d+2}^{b} x_i = k(r-1) - dl, \tag{2.1}$$

$$\sum_{i=d+2}^{b} x_i^2 = P - dl^2, \tag{2.2}$$

where $P = k^2[(v-k)(b-r) - (v-rk)(v-m)]/v(v-m)$. Let $\bar{x} = \sum_{i=d+2}^{b} x_i/(b-d-1) = [k(r-1) - dl]/(b-d-1)$. As $\sum_{i=d+2}^{b} (x_i - \bar{x})^2 \geqq 0$, we have from (2.1) and (2.2),

$$dQ \leqq (b-1)Q - [k(r-1) - l(b-1)]^2, \tag{2.3}$$

where $Q = P + l^2(b-1) - 2lk(r-1)$. Since

$$\begin{aligned} Q &= k^2[(v-k)(b-r) - (v-rk)(v-m)]/v(v-m) - k^2(r-1)^2/(b-1) \\ &\quad + [k(r-1) - l(b-1)]^2/(b-1) \\ &= k^2(v-k)(b-r)(b-v+m-1)/v(v-m)(b-1) \\ &\quad + [k(r-1) - l(b-1)]^2/(b-1), \end{aligned}$$

it follows from the author's [3] result (2.7), that, when $b = v - m + 1$, $Q = 0$ and when $b > v - m + 1$, $Q > 0$. As for this design, $b > v - m + 1$, we have $Q > 0$. Hence, we have from (2.3)

$$d \leqq b - 1 - [k(r-1) - l(b-1)]^2/Q. \tag{2.4}$$

If the equality sign holds in (2.4), then all x_i's are equal, $i = d+2, \cdots, b$ and $x_i = [P - lk(r-1)]/[k(r-1) - l(b-1)]$ is an integer and the given block B_1 has $[P - lk(r-1)]/[k(r-1) - l(b-1)]$ treatments common with each of the remaining $(b-d-1)$ blocks.

The author's [3] earlier Theorem 2.1 follows as a corollary from the above theorem when $l = 0$.

From Theorem 2.2, we deduce the following Theorem:

THEOREM 2.3. *If in a semi-regular* GD *design*, $b = v - m + r$ *and* $v = 2k$, *then no two blocks of this design are the same set.*

PROOF. Let a block of the given design have d blocks having all the k treatments common with it. Then, using Theorem 2.2 for $b = v - m + r$ and $v = 2k$, we obtain

$$d \leqq (r-1)/(r+1) < 1. \tag{2.5}$$

Hence, $d = 0$, which proves the theorem.

Combining Theorems 2.1 and 2.3, we obtain the following theorem:

THEOREM 2.4. *If in a semi-regular* GD *design*, $b = v - m + r$ *and* $v = 2k$, *where* k *is an odd integer, then no two blocks of this design are* (i) *disjoint or* (ii) *the same set.*

3. PBIB designs with two associate classes having a triangular association

scheme. A PBIB design with two associate classes is said to have a triangular association scheme [2], if the number of treatments is $v = n(n - 1)/2$ and the association scheme is an array of n rows and n columns with the following properties:

(a) The positions in the principal diagonal are blanks;

(b) the $n(n - 1)/2$ positions above the principal diagonal are filled by the numbers $1, 2, \cdots, n(n - 1)/2$, corresponding to the treatments;

(c) the array is symmetric about the principal diagonal;

(d) for any treatment θ, the first associates are exactly those treatments which lie in the same row and the same column as θ.

The primary parameters of this design are $v = n(n - 1)/2$, b, r, k, λ_1, λ_2, $n_1 = 2n - 4$, $n_2 = (n - 2)(n - 3)/2$.

From Theorem 3.1 of [3], we deduce Theorem 3.1.

THEOREM 3.1. *If in a* PBIB *design with two associate classes having a triangular association scheme with* $rk - v\lambda_1 = n(r - \lambda_1)/2$ *and* $b = v - n + r$, *and* $v = 2k$, *where* k *is an odd integer, then no two blocks of this design are disjoint.*

THEOREM 3.2. *If in a* PBIB *design with two associate classes having a triangular association scheme with* $rk - v\lambda_1 = n(r - \lambda_1)/2$ *and* $b > v - n + 1$, *a given block has* d *blocks having a given number* $l(\leqq k)$ *of treatments common with it, then*

$$d \leqq b - 1 - [k(r - 1) - l(b - 1)]^2/Q,$$

where $Q = P + l^2(b - 1) - 2lk(r - 1)$ *and* $P = k^2[(v - k)(b - r) - (v - rk)\cdot(v - n)]/v(v - n)$. *Further, if* $d = b - 1 - [k(r - 1) - l(b - 1)]^2/Q$, *then* $[P - lk(r - 1)]/[k(r - 1) - l(b - 1)]$ *is an integer and the given block has* $[P - lk(r - 1)]/[k(r - 1) - l(b - 1)]$ *treatments common with each of the remaining* $(b - d - 1)$ *blocks.*

The proof is similar to that of Theorem 2.2.

The author's [3] earlier Theorem 3.1 follows as a corollary from the above theorem when $l = 0$.

From the above theorem, we deduce the following theorem:

THEOREM 3.3. *If in a* PBIB *design with two associate classes having a triangular association scheme with* $rk - v\lambda_1 = n(r - \lambda_1)/2$, $b = v - n + r$ *and* $v = 2k$, *then no two blocks of this design are the same set.*

The proof is similar to that of Theorem 2.3.

Combining Theorems 3.1 and 3.3, we obtain the following theorem:

THEOREM 3.4. *If in a* PBIB *design with two associate classes having a triangular association scheme with* $rk - v\lambda_1 = n(r - \lambda_1)/2$, $b = v - n + r$ *and* $v = 2k$, *where* k *is an odd integer, then no two blocks of this design are* (i) *disjoint or* (ii) *the same set.*

4. PBIB designs with two associate classes having L_2 association scheme. A PBIB design with two associate classes is said to have an L_2 association scheme [2], if the number of treatments is $v = s^2$, where s is a positive integer and the treatments can be arranged in an $s \times s$ square such that treatments in the same row or the same column are first associates; while others are second associates.

The primary parameters of this design are $v = s^2, b, r, k, \lambda_1, \lambda_2, n_1 = 2(s-1)$, $n_2 = (s-1)^2$.

From Theorem 4.1 of [3], we deduce the following theorem:

THEOREM 4.1. *If in a* PBIB *design with two associate classes having an L_2 association scheme and $rk - v\lambda_1 = s(r - \lambda_1)$, $b = v - 2s + r + 1$ and $v = 2k$, where k is an odd integer, then no two blocks of this design are disjoint.*

THEOREM 4.2. *If in a* PBIB *design with two associate classes having an L_2 association scheme and $rk - v\lambda_1 = s(r - \lambda_1)$ and $b > v - 2s + 2$, a given block has d blocks having a given number $l(\leqq k)$ of treatments common with it, then*

$$d \leqq b - 1 - [k(r-1) - l(b-1)]^2/Q,$$

where $Q = P + l^2(b-1) - 2lk(r-1)$ and $P = k^2[(v-k)(b-r) - (v-rk)\cdot(s-1)^2]/v(s-1)^2$. Further, if $d = b - 1 - [k(r-1) - l(b-1)]^2/Q$, then $[P - lk(r-1)]/[k(r-1) - l(b-1)]$ is an integer and the given block has $[P - lk(r-1)]/[k(r-1) - l(b-1)]$ treatments common with each of the remaining $(b - d - 1)$ blocks.

The proof is just similar to that of Theorem 2.2.

From the above theorem, we deduce the following theorem:

THEOREM 4.3. *If in* PBIB *design with two associate classes having an L_2 association scheme and $rk - v\lambda_1 = s(r - \lambda_1)$, $b = v - 2s + r + 1$ and $v = 2k$, then no two blocks of this design are the same set.*

The proof is similar to that of Theorem 2.3.

Combining Theorems 4.1 and 4.3, we obtain the following theorem:

THEOREM 4.4. *If in a* PBIB *design with two associate classes having an L_2 association scheme with $rk - v\lambda_1 = s(r - \lambda_1)$, $b = v - 2s + r + 1$ and $v = 2k$, where k is an odd integer, then no two blocks of this design are* (i) *disjoint or* (ii) *the same set.*

5. PBIB designs with three associate classes having a rectangular association scheme. A PBIB design with three associate classes is said to have a rectangular association scheme [5], if the number of treatments $v = v_1 v_2$ can be arranged in the form of a rectangle of v_1 rows and v_2 columns, so that the first associates of any treatment are the $(v_2 - 1)$ treatments of the same row, the second associates are the other $(v_1 - 1)$ treatments of the same column; while the remaining $p = (v_1 - 1)(v_2 - 1)$ treatments are the third associates. The primary parameters of this design are $v = v_1 v_2$, b, r, k, $n_1 = v_2 - 1$, $n_2 = v_1 - 1$, $n_3 = n_1 n_2$, λ_1, λ_2, λ_3. Vartak [5] has proved that the characteristic roots of NN' of this design are $\theta_0 = rk$, $\theta_1 = r - \lambda_1 + (v_1 - 1)(\lambda_2 - \lambda_3)$, $\theta_2 = r - \lambda_2 + (v_2 - 1)_3 (\lambda_1 - \lambda_3)$, $\theta_3 = r - \lambda_1 - \lambda_2 + \lambda_3$. Here, we consider this design with $\theta_1 = 0 = \theta_2$.

From Theorem 5.1 of [3], we deduce the following theorem:

THEOREM 5.1. *If in a* PBIB *design with three associate classes having a rectangular association scheme and $\theta_1 = 0 = \theta_2$, and $b = p + r$ and $v = 2k$ where k is an odd integer, then no two blocks of this design are disjoint.*

THEOREM 5.2. *If in a* PBIB *design with three associate classes having a rectangular association scheme and* $\theta_1 = 0 = \theta_2$ *and* $b > p + 1$, *a given block has d blocks having a given number* $l(\leqq k)$ *of treatments common with it, then*

$$d \leqq b - 1 - [k(r - 1) - l(b - 1)]^2/Q,$$

where $Q = P + l^2(b - 1) - 2lk(r - 1)$ *and* $P = k^2[(v - k)(b - r) - p(v - rk)]/vp$, *p being equal to* $(v_1 - 1)(v_2 - 1)$. *Further if* $d = b - 1 - [k(r - 1) - l(b - 1)]^2/Q$, *then* $[P - lk(r - 1)]/[k(r - 1) - l(b - 1)]$ *is an integer and the given block has* $[P - lk(r - 1)]/[k(r - 1) - l(b - 1)]$ *treatments common with each of the remaining* $(b - d - 1)$ *blocks.*

The proof is similar to that of Theorem 2.2.

The author's [3] earlier Theorem 5.1 follows as a corollary from the above theorem when $l = 0$.

From the above theorem, we deduce the following theorem:

THEOREM 5.3. *If in a* PBIB *design with three associate classes having a rectangular association scheme and* $\theta_1 = 0 = \theta_2$, $b = p + r$ *and* $v = 2k$, *then no two blocks of this design are the same set.*

Combining Theorems 5.1 and 5.3, we obtain the following theorem:

THEOREM 5.4. *If in a* PBIB *design with three associate classes having a rectangular association scheme with* $\theta_1 = 0 = \theta_2$, $b = p + r$, *and* $v = 2k$, *where k is an odd integer, then no two blocks of this design are* (i) *disjoint or* (ii) *the same set.*

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