SOME PROPERTIES OF STATISTICAL RELIABILITY FUNCTIONS

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0. Summary. Networks with independent components are considered. Assuming all components to have a probability p of functioning we study the properties of the reliability function $R(p)$ that the network functions. In particular, we investigate networks of high order. It is shown that an arbitrary (randomized) network can be approximated by a pure one with an approximation error of the order n^{-1}. Bounds are obtained for the maximum difference quotient and derivative of the first and second order of $R(p)$. As a corollary we obtain an asymptotic result concerning the best possible approximation of a function of Lipschitz type by reliability functions.

1. Introduction. Consider a network $\mathfrak{N}$ consisting of n components. To the ith component we associate a variable x_i which takes the value 1 or 0 according to whether this component functions or not. The vector $x = (x_1, x_2, x_3, \cdots, x_n)$ can then take 2^n different values. To each such value belongs a probability $\phi(x)$; ϕ is called the *structure function* of the network. If ϕ takes only the values 1 and 0 we speak of a *pure* network, otherwise of a *randomized* network.

In this paper we shall only deal with networks consisting of *independent* components, all of which have the *same* probability p of functioning. No doubt this is a severe restriction but this special case, that has received a good deal of attention in the literature, still presents a number of open questions. Once the probability distribution of the stochastic vector x has been specified we can form the *reliability function* $R(p) = E\phi(x)$. When designing the wiring diagram of a network the resulting $R(p)$ will have to be taken into account. This aspect of the design problem should be put into correspondence with the technological and economic background.

Starting from n components with the above properties one may ask what reliability functions can be realized by choosing among the possible networks that can be formed from these components. Given a certain real valued function $f(p)$, $0 \leqq p \leqq 1$, how well can it be approximated by a reliability function associated with a network of size n? To make this question precise we must specify in what sense the approximation should be understood, e.g. in the sense of uniform convergence. Some simple results in this direction are given in Sections 2 and 3. Theorem 2 gives a fairly obvious characterization of what functions can be represented (exactly) as reliability functions. It would be useful to have access to more explicit criteria.

The study of reliability functions can be said to have started in the pioneering work by von Neumann (1956) and Moore-Shannon (1956) in which the funda-

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mental importance of the problem was made clear. These early works contain much valuable information and should be read by anyone interested in the mathematical theory of reliability. They also contain a description of the technological background that motivates the mathematical problem. An important step forward was taken in Birnbaum-Esary-Saunders (1961) as well as in Esary-Proschan (1963).

Suppose, in a given context, that one has been led to certain desirable properties for the reliability function. Is it possible to realize these properties for networks of fixed order n? It seems natural to consider this as an approximation problem and in this paper we shall take this as a starting point. It will depend upon the practical set up which properties should be considered as desirable and the result will vary from case to case. It is believed that very often one will look for reliability functions that are as steep as possible in some sense. There is a meaningful analogy with the OC-curve in statistical quality control, and, while we will not pursue this idea any further in this publication, the reader is referred to a paper by Ajne and Grenander dealing with this.

We must specify what steepness properties of the reliability function we will use. We shall work both with the difference quotient $[R(x+h) - R(x-h)]/2h$ and with the first two derivatives $R'(x)$ and $R''(x)$. Of course, one may be interested in other steepness criteria, but as long as these are linear functionals it seems possible to deal with them by methods similar to that employed by us.

The randomized networks are certainly mathematically convenient to handle. In the literature, von Neumann (1956), one can find statements indicating that random wiring diagrams also present real advantages. However that may be, it has seemed necessary to investigate how well a randomized network can be approximated by a pure one in terms of their reliability functions. This is done in Theorem 1.

2. Approximation by pure networks. First we need an expression for $R(p)$. By definition $R(p) = E\phi(x)$. Let $l(x) =$ the number of components functioning. We have

$$E\phi(x) = \sum_{k=0}^{n} \sum_{x;\, l(x)=k} \phi(x)p^k(1-p)^{n-k}.$$

Let

$$A_k = \sum_{x;\, l(x)=k} \phi(x).$$

Since $0 \leqq \phi(x) \leqq 1$, and since the sum consists of $\binom{n}{k}$ terms, we have $0 \leqq A_k \leqq \binom{n}{k}$. When $\phi(x)$ is a nonrandomized structure function, A_k is an integer. We call x a path of size i, if $\phi(x) = 1$ and $l(x) = i$. Then A_k is the number of paths of size k. Even if $\phi(x)$ is randomized we call A_k a pathnumber of size k. That is, $R(p) = \sum_{k=0}^{n} A_k p^k (1-p)^{n-k}$, $0 \leqq A_k \leqq \binom{n}{k}$. If we define relative pathnumbers B_k by $B_k = A_k/\binom{n}{k}$, then we have

$$R(p) = \sum_{k=0}^{n} B_k \binom{n}{k} p^k(1-p)^{n-k} = EB_K,$$

where B_K is a real number, $0 \leqq B_K \leqq 1$, and K is binomially distributed (n, p).

It is easy to derive an asymptotic expression for the smallest error, when a reliability function of a randomized structure is approximated by one of a nonrandomized.

THEOREM 1. *Let $R(p)$ be a reliability function of a randomized structure function corresponding to a network with n components. If $A_0 = 0$ or 1 and $A_n = 0$ or 1, then there is a reliability function $R_1(p)$, of a nonrandomized structure, so that $R_1(p)$ approximates $R(p)$ uniformly. In fact $|R(p) - R_1(p)|$ is dominated by* $\max_p \frac{1}{2} \sum_{k=1}^{n-1} p^k(1-p)^{n-k} \sim (2ne)^{-1}$.

PROOF. $R(p) = \sum_{k=0}^{n} A_k p^k (1-p)^{n-k}$, $0 \leqq A_k \leqq \binom{n}{k}$. Define $\langle A_k \rangle$ to be the integer closest to A_k. Then we can take $R_1(p) = \sum_{k=0}^{n} \langle A_k \rangle p^k (1-p)^{n-k}$. But

$$\begin{aligned} |R(p) - R_1(p)| &\leqq \sum_{k=0}^{n} |A_k - \langle A_k \rangle| p^k (1-p)^{n-k} \\ &= \sum_{k=1}^{n-1} |A_k - \langle A_k \rangle| p^k (1-p)^{n-k} \\ &\leqq \tfrac{1}{2} \sum_{k=1}^{n-1} p^k (1-p)^{n-k} \\ &= \tfrac{1}{2}[p(1-p)^n - p^n(1-p)]/(1-2p). \end{aligned}$$

Write $f_n(p) = [p(1-p)^n - p^n(1-p)]/(1-2p)$. $f_n(p)$ is symmetric about $p = \frac{1}{2}$. We will find an asymptotic expression for $M_n = \max_{0 \leqq p \leqq \frac{1}{2}} f_n(p)$.

Given $\epsilon > 0$, we can find $\delta > 0$ such that $(1-2\delta)^{-1} < 1 + \epsilon$. After this we keep ϵ and δ fixed. Divide the interval $[0, \frac{1}{2}]$ into two parts: $[0, \delta)$ and $[\delta, \frac{1}{2}]$. When $p \in [0, \delta)$, $nf_n(p) \leqq np(1-p)^n/(1-2\delta)$, and when $p \in [\delta, \frac{1}{2}]$, $n \cdot f_n(p) = n \sum_{k=1}^{n-1} p^k (1-p)^{n-k} \leqq n \cdot \sum_{k=1}^{n-1} (1-\delta)^k (1-\delta)^{n-k} = n(n-1)(1-\delta)^n$. Let

$$\begin{aligned} g_n(p) &= np(1-p)^n/(1-2\delta), \qquad \text{for } 0 \leqq p < \delta, \\ &= n(n-1)(1-\delta)^n, \qquad \text{for } \delta \leqq p \leqq \tfrac{1}{2}. \end{aligned}$$

$n \cdot M_n \leqq \max g_n(p)$. We want to find $\max_{0 \leqq p \leqq \frac{1}{2}} g_n(p)$, when n tends to infinity. Let n be so large that $(n+1)^{-1} < \delta$, then $\max_{0 \leqq p < \delta} g_n(p) = (1-2\delta)^{-1} \cdot (1 - (n+1)^{-1})^{n+1}$ and $(1-2\delta)^{-1}(1-(n+1)^{-1})^{n+1} > n(n-1)(1-\delta)^n$ for n sufficiently large, that is,

$$\lim_{n \to \infty} \max_{0 \leqq p \leqq \frac{1}{2}} g_n(p) = (1-2\delta)^{-1} \cdot e^{-1}.$$

We have $n \cdot M_n \geqq n \cdot f_n(n^{-1})$ and

$\lim_{n \to \infty} n \cdot f_n(n^{-1}) =$

$$\lim_{n \to \infty} \{[(1-n^{-1})^n - (n^{-1})^{n-1}(1-n^{-1})]/(1-2/n)\} = e^{-1}.$$

This gives us

$$e^{-1} \leqq \liminf_{n \to \infty} nM_n \leqq \limsup_{n \to \infty} n \cdot M_n \leqq (1-2\delta)^{-1} e^{-1} < e^{-1}(1+\epsilon).$$

Since ϵ is arbitrary we have $\lim_{n \to \infty} n \cdot M_n = e^{-1}$, and $\max_p \frac{1}{2} \sum_{k=1}^{n-1} p^k (1-p)^{n-k} \sim (2ne)^{-1}$.

3. Exact representation as a reliability function.

THEOREM 2. *If $R(p)$ is a reliability function of a structure with n components, and we know the Taylor expansion of $R(p)$, $R(p) = a_0 + a_1 p + \cdots + a_n p^n$, then the relative path numbers B_k are given by*

$$B_k = a_0 + a_1\binom{k}{1}/\binom{n}{1} + a_2\binom{k}{2}/\binom{n}{2} + a_3\binom{k}{3}/\binom{n}{3} + \cdots + a_n\binom{k}{n}/\binom{n}{n},$$

$\binom{k}{n} = 0$ *if* $k < n$.

PROOF. There are B_k, $k = 1, \cdots, n$, such that $R(p) = EB_K$, where K is a binomially distributed random variable (n, p). K has the generating function $(q + pz)^n$.

We find the factorial moments $EK(K-1)\cdots(K-\nu+1) = p^\nu n(n-1)\cdots(n-\nu+1)$. If B_k has the expression given above, that is

$$B_k = a_0 + a_1 k/n + a_2 k(k-1)/n(n-1) + a_3 k(k-1)(k-2)/n(n-1)(n-2) + \cdots + a_n k(k-1)\cdots(k-n+1)/n!,$$

then $EB_K = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n$.

As both the Taylor expansion of $R(p)$ and the expansion $R(p) = \sum_{k=0}^{n} B_k \cdot \binom{n}{k} p^k (1-p)^{n-k}$ are unique, the theorem is proved. (It is easily seen that the last expansion is unique.)

Let $\mathcal{R}_n$ be the class of reliability functions of randomized structures with n components. If $R_1(p)$ and $R_2(p) \in \mathcal{R}_n$, then we can write $R_\nu(p) = EB_K^{(\nu)}$, $\nu = 1, 2$. $B_k^{(\nu)}$ are the relative path numbers, which are real and $0 \leqq B_k^{(\nu)} \leqq 1$. K is a binomially distributed random variable (n, p).

α is a real number $0 \leqq \alpha \leqq 1$.

$$\alpha \cdot R_1(p) + (1-\alpha)\cdot R_2(p) = \alpha EB_K^{(1)} + (1-\alpha)EB_K^{(2)} = E(\alpha B_K^{(1)} + (1-\alpha)B_K^{(2)}) = EC_K \in \mathcal{R}_n,$$

since C_k is real and $0 \leqq C_k \leqq 1$ for every k, that is, C_k are relative pathnumbers for some randomized structure.

Accordingly we have:

REMARK. $\mathcal{R}_n$ is a closed convex subset of the set $\mathcal{P}_n$ of all polynomials of the nth degree.

Define linear operators L_k, $k = 0, 1, 2, \cdots$, on $\mathcal{P}_n$ by

$$L_k x^\nu = \binom{k}{\nu}/\binom{n}{\nu}.$$

Then we can state

THEOREM 2a. *If $f \in \mathcal{P}_n$ and $0 \leqq L_k f \leqq 1$ for all k, then $f \in \mathcal{R}_n$.*

PROOF. $f = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ for some coefficients a_i.

$$L_k f = a_0 + a_1\binom{k}{1}/\binom{n}{1} + a_2\binom{k}{2}/\binom{n}{2} + \cdots + a_k\binom{k}{k}/\binom{n}{k}.$$

By Theorem 2 we can take $B_k = L_k f$, $f = \sum_{k=0}^{n} B_k p^k (1-p)^{n-k}$, where $0 \leqq B_k \leqq 1$, so that $f \in \mathcal{R}_n$.

4. Maximum derivative.

THEOREM 3.

$$\max_{\mathfrak{N}} R'(p) = \beta_1/2p(1-p) \sim [n/2p(1-p)\pi]^{\frac{1}{2}},$$

where β_1 is the first absolute moment about the mean of a binomial distribution (n, p).

PROOF. $R(p) = \sum_{k=0}^{n} A_k p^k (1-p)^{n-k}$, $0 \leqq A_k \leqq \binom{n}{k}$. Then

$$R'(p) = \sum_{k=0}^{n} A_k [kp^{k-1}(1-p)^{n-k} - (n-k)p^k(1-p)^{n-k-1}]$$
$$= [p(1-p)]^{-1} \sum_{k=0}^{n} A_k p^k (1-p)^{n-k}[k - np].$$

To get the maximum we take A_k as large as possible when $k - np$ is positive and A_k as small as possible for $k - np$ negative. So

$$\max_{\mathfrak{N}} R'(p) = [p(1-p)]^{-1} \sum_{k=[np]+1}^{n} \binom{n}{k} p^k (1-p)^{n-k}(k - np).$$

Now

$$\sum_{k=[np]+1}^{n} \binom{n}{k} p^k (1-p)^{n-k}(k-np) = -\sum_{k=0}^{[np]} \binom{n}{k} p^k (1-p)^{n-k}(k-np),$$

since the first moment about the mean equals zero. Furthermore

$$\beta_1 = E|k - np| = -\sum_{k=0}^{[np]} \binom{n}{k} p^k (1-p)^{n-k}(k-np)$$
$$+ \sum_{k=[np]+1}^{n} \binom{n}{k} p^k (1-p)^{n-k}(k-np).$$

That is $\sum_{k=[np]+1}^{n} \binom{n}{k} p^k (1-p)^{n-k}(k-np) = \frac{1}{2}\beta_1$ and $\max_{\mathfrak{N}} R'(p) = \beta_1/2p(1-p)$. It is known (see e.g. Cramér, p. 257) that $\beta_1 \sim [2np(1-p)/\pi]^{\frac{1}{2}}$.

5. Maximum difference quotient. The following theorem requires a good deal of calculation.

THEOREM 4. *For given x and h*

$$|\max_{\mathfrak{N}} \{[R(x+h) - R(x-h)]/2h\} - (2h)^{-1}| \sim (2\pi n)^{-\frac{1}{2}} f_n(x, h)$$
$$\cdot \{[(x^2 - h^2)/c^2]^c [((1-x)^2 - h^2)/(1-c)^2]^{1-c}\}^{n/2}, \quad as\ n \to \infty,$$

where $\mathfrak{N}$ stands for the set of all structures of order n and R for the corresponding reliability functions, $0 < x < 1$, $h > 0$, $x + h \leqq 1$, $x - h \geqq 0$, $f_n(x, h)$ is uniformly bounded for all n, and

$$c = \log[(1 - x + h)\cdot(1 - x - h)^{-1}]/\log[(1 - x + h)$$
$$\cdot (1 - x - h)^{-1}\cdot(x + h)(x - h)^{-1}]$$

PROOF. $[R(x+h) - R(x-h)]/2h = (2h)^{-1}\cdot \sum_{k=0}^{n} B_k \binom{n}{k} c_k$;

$$c_k = (x+h)^k(1-x-h)^{n-k} - (x-h)^k(1-x+h)^{n-k}$$
$$= (x-h)^k(1-x+h)^{n-k}$$
$$\cdot \{[(x+h)/(x-h)]^k[(1-x-h)/(1-x+h)]^{n-k} - 1\}.$$

Now $f_k = [(x+h)/(x-h)]^k \cdot [(1-x-h)/(1-x+h)]^{n-k}$ increases monotonically as k increases, $f_0 < 1$ and $f_n > 1$. This implies that there is a $k = k_1$ such that $c_k > 0$ when $k \geqq k_1$ and $c_k \leqq 0$ when $k < k_1$.

To obtain $\max_{\mathfrak{N}} \{[R(x + h) - R(x - h)]/2h\}$ we have to take $B_k = 0$, when $c_k < 0$, and $B_k = 1$, when $c_k > 0$. We get the maximum when the reliability function is a binomial sum, truncated at some appropriate k.

Now we want to find k_1. We think of k as a continuous variable for a moment, $c_k = 0$ when

$$\{[(x + h)/(x - h)]\cdot[(1 - x + h)/(1 - x - h)]\}^k \cdot[(1 - x - h)/(1 - x + h)]^n = 1;$$

therefore

$$k = n\cdot\log [(1 - x + h)(1 - x - h)^{-1}]/\log [(x + h)(x - h)^{-1} \cdot(1 - x + h)(1 - x - h)^{-1}].$$

Accordingly k_1 is of the form $[cn] + 1$, where c is a real number, $0 < c < 1$. ($[x]$ means the largest integer smaller than x). It can actually be shown that $x - h < c < x + h$. If for given x we regard c as a function of h, then we have to show

$$\begin{aligned} &(x - h) \log \{[(1 - x + h)/(1 - x - h)]\cdot[(x + h)/(x - h)]\} \\ &\quad < \log [(1 - x + h)/(1 - x - h)] \\ &\quad < (x + h) \log \{[(1 - x + h)/(1 - x - h)]\cdot[(x + h)/(x - h)]\} \end{aligned} \tag{1}$$

for $h > 0$. Let

$$g_1(h) = (x - h) \log \{[(1 - x + h)/(1 - x - h)]\cdot[(x + h)/(x - h)]\} - \log [(1 - x + h)/(1 - x - h)];$$

$$g_2(h) = (x + h) \log \{[(1 - x + h)/(1 - x - h)]\cdot[(x + h)/(x - h)]\} - \log [(1 - x + h)/(1 - x - h)].$$

As is easily seen $g_1(0) = g_2(0) = 0$, and $g_1'(h) < 0$ for $h > 0$, $g_2'(h) > 0$ for $h > 0$. That is, the inequalities (1) are shown.

We shall now analyse sums of the form $\sum_{[cn]+1}^{n} \binom{n}{k} p^k(1 - p)^{n-k}$.

(a) $c > p$. We can use a theorem due to Bahadur, that states: *Let* $B_n(k; p) = \sum_{r=k}^{n} \binom{n}{r} p^r q^{n-r}$, $q = 1 - p$, $A_n(k; p) = \binom{n}{k} p^k q^{n-k+1}(k + 1)/[k + 1 - (n + 1)\cdot p]$, *and* $x = (k - np)/(npq)^{\frac{1}{2}}$. *If* $np \leqq k \leqq n$ *and* $x \to \infty$, *as* $n \to \infty$, *then* $B_n(k; p) \sim A_n(k; p)$.

In this case we have

$$B_n([cn] + 1, p) \sim \binom{n}{[cn]+1} p^{[cn]+1} q^{n-[cn]-1+1} \cdot([cn] + 1 + 1)/[([cn] + 1 + 1) - (n + 1)p].$$

Write $[cn] + 1 = cn + \theta_n$, where $0 < \theta_n \leqq 1$. Then we have

$$(cn + \theta_n + 1)/(cn + \theta_n + 1 - np - p) \sim c/(c - p),$$

and by using Stirling's formula

$$\binom{n}{cn+\theta_n} p^{cn+\theta_n} q^{n-cn-\theta_n+1} \sim (2\pi n)^{-\frac{1}{2}}[(1 - c)/c]^{\frac{1}{2}}(p/c)^{nc}[q/(1 - c)]^{n(1-c)+1} \cdot [p(1 - c)/cq]^{\theta_n}.$$

Consequently

$$B_n([cn] + 1, p) \sim (2\pi n)^{-\frac{1}{2}}[c(1 - c)]^{\frac{1}{2}}(c - p)^{-1}(p/c)^{nc} \cdot [q/(1 - c)]^{n(1-c)+1}[p(1 - c)/cq]^{\theta_n}.$$

(b) $c < p$.

$$\sum_{k=[cn]+1}^{n} \binom{n}{k} p^k(1 - p)^{n-k} = 1 - \sum_{k=0}^{[cn]} \binom{n}{k} p^k(1 - p)^{n-k}$$
$$= 1 - \sum_{k=n-[cn]}^{n} \binom{n}{k}(1 - p)^k p^{n-k} = 1 - B_n([c'n)] + 1, q),$$

where $c' = 1 - c - n^{-1}$ and $\theta_n' = 1 + nc - [nc] = 2 - \theta_n$. If $c' > q$, $B_n([c'n] + 1, q)$ has an asymtotic expression analogous to (a).

Now we are able to calculate

$$(2h)^{-1} \sum_{k=[cn]+1}^{n} \binom{n}{k}\{(x + h)^k(1 - x - h)^{n-k} - (x - h)^k(1 - x + h)^{n-k}\}$$

for large n. Write Σ for this expression.

Assume n to be large enough to fulfill $n^{-1} < h$ and $c < x + h - n^{-1}$. Then

$$\Sigma = (2h)^{-1}\{1 - B_n([c'n] + 1, 1 - x - h) - B_n([cn] + 1, x - h)\} \sim (2h)^{-1},$$

since $c' = 1 - c - n^{-1} > 1 - x - h + n^{-1} - n^{-1} = 1 - x - h$ and $c > x - h$.

We want to analyse the difference between $\max_{\mathfrak{N}} \{[R(x + h) - R(x - h)]/2h\}$ and $(2h)^{-1}$. We have

$$|\max_{\mathfrak{N}} \{[R(x + h) - R(x - h)]/2h\} - (2h)^{-1}| = B_n([c'n] + 1, 1 - x - h) + B_n([cn] + 1, x - h),$$

where $c = \{1 + \log [(x + h)(x - h)^{-1}]/\log [(1 - x + h)(1 - x - h)^{-1}]\}^{-1}$.
But

$$B_n([c'n] + 1, 1 - x - h) \sim (2\pi n)^{-\frac{1}{2}}[(c + n^{-1})(1 - c - n^{-1})]^{\frac{1}{2}}$$
$$\cdot (x + h - n^{-1} - c)^{-1}[(1 - x - h)/(1 - c - n^{-1})]^{n(1-c)-1}$$
$$\cdot [(x + h)/(c + n^{-1})]^{nc+2}$$
$$\cdot [(1 - x - h)(c + n^{-1})/(1 - c - n^{-1})(x + h)]^{2-\theta_n}$$
$$\sim (2\pi n)^{-\frac{1}{2}}[c(1 - c)]^{\frac{1}{2}}(x + h - c)^{-1}$$
$$\cdot [(1 - x - h)/(1 - c)]^{n(1-c)+1}$$
$$\cdot e[(x + h)/c]^{nc}e^{-1} \cdot [(1 - x - h)c/(1 - c)(x + h)]^{-\theta_n},$$

and

$$B_n([cn]+1, x-h) \sim (2\pi n)^{-\frac{1}{2}}[c(1-c)]^{\frac{1}{2}}(c-x+h)^{-1} \cdot[(x-h)/c]^{nc}[(1-x+h)/(1-c)]^{n(1-c)+1} \cdot[(x-h)(1-c)/c(1-x+h)]^{\theta_n}.$$

Asymptotically $B_n([c'n]+1, 1-x-h)$ and $B_n([cn]+1, x-h)$ are of the same magnitude, since

$$\begin{aligned}
&[(1-x-h)/(1-x+h)]^{1-c}[(x+h)/(x-h)]^{c} \\
&= \exp\{(1-c)\log[(1-x-h)/(1-x+h)] + c\log[(x+h)/(x-h)]\} \\
&= \exp\left\{\frac{\dfrac{\log[(x+h)(x-h)^{-1}]}{\log[(1-x+h)(1-x-h)^{-1}]}\log[(1-x-h)(1-x+h)^{-1}]}{1+\log[(x+h)(x-h)^{-1}]/\log[(1-x+h)(1-x-h)^{-1}]} + \frac{\log[(x+h)(x-h)^{-1}]}{1+\log[(x+h)(x-h)^{-1}]/\log[(1-x+h)(1-x-h)^{-1}]}\right\} \\
&= \exp(0) = 1.
\end{aligned}$$

Accordingly,

$$\begin{aligned}
&|\max_{\mathfrak{N}}\{[R(x+h)-R(x-h)]/2h\} - (2h)^{-1}| \sim (2\pi n)^{-\frac{1}{2}}[(1-c)/c]^{\theta_n-\frac{1}{2}} \\
&\quad\cdot\{[(x^2-h^2)/c^2]^{c}[((1-x)^2-h^2)/(1-c)^2]^{1-c}\}^{n/2} \\
&\quad\cdot\{(x+h)^{\theta_n}(1-x-h)^{1-\theta_n}/(x+h-c) + (x-h)^{\theta_n}(1-x+h)^{1-\theta_n}/(c-x+h)\}, \\
&f_n(x,h) = [(1-c)/c]^{\theta_n-\frac{1}{2}}\{(x+h)^{\theta_n}(1-x-h)^{1-\theta_n}/(x+h-c) \\
&\quad + (x-h)^{\theta_n}(1-x+h)^{1-\theta_n}/(c-x+h)\},
\end{aligned}$$

and this expression is evidently uniformly bounded in n.

6. Maximum second derivative. We now derive a similar result for the second derivative as for the derivative.

THEOREM 5. $\max_{\mathfrak{N}} R''(p) \sim cn/p(1-p)$, *where* $c = (2/\pi e)^{\frac{1}{2}} \approx 0.48394$.

PROOF. $R(p) = \sum_{k=0}^{n} A_k p^k(1-p)^{n-k}$. By using Leibniz's formula for derivation we have

$$\begin{aligned}
R''(p) &= \textstyle\sum_{k=0}^{n} A_k p^k(1-p)^{n-k}[k(k-1)/p^2 - 2(k/p)(n-k)/(1-p) + (n-k)(n-k-1)/(1-p)^2] \\
&= (p^2q^2)^{-1}\textstyle\sum_{k=0}^{n} A_k p^k q^{n-k}P(k), \qquad \text{where } q = 1-p, \text{ and} \\
P(k) &= k^2 - k(q^2+2npq+(2n-1)p^2) + n(n-1)p^2 \\
&= k^2 - k(q+(2n-1)p) + n(n-1)p^2 \\
&= (k-np)^2 + p(k-np) - kq.
\end{aligned}$$

In order to get the maximum of $R''(p)$ we must know where $P(k)$ changes its sign. We have

$$P(0) = n(n-1)p^2 \geqq 0,$$
$$P(n) = (nq)^2 + pnq - nq = (nq)^2 - nq^2 \geqq 0,$$
$$P(k) = 0 \quad \text{when} \quad k^2 - k(q + (2n-1)p) + n(n-1)p^2 = 0.$$

The two roots are

$$k_1 = np + (q-p)/2 - [(p-q)^2 + 4npq]^{\frac{1}{2}}/2;$$
$$k_2 = np + (q-p)/2 + [(p-q)^2 + 4npq]^{\frac{1}{2}}/2.$$

The calculation above implies that $P(k) > 0$ when $k < k_1$ and $k > k_2$. Write $b(k;\, n,\, p) = \binom{n}{k}p^k q^{n-k}$. As before we get the maximum by taking $A_k = \binom{n}{k}$, when $P(k) > 0$, and $A_k = 0$, when $P(k) < 0$.

$$\begin{aligned}\text{Max } R''(p) &= (p^2q^2)^{-1}\{\textstyle\sum_{k=0}^{[k_1]} P(k)b(k;\, n,\, p) + \sum_{k=[k_2]+1}^{n} P(k)b(k;\, n,\, p)\} \\ &= (p^2q^2)^{-1}\{\textstyle\sum_{k=0}^{n} P(k)b(k;\, n,\, p) - \sum_{k=[k_1]+1}^{[k_2]} P(k)b(k;\, n,\, p)\} \\ &= (p^2q^2)^{-1}[npq - qnp - \textstyle\sum_{k=[k_1]+1}^{[k_2]} P(k)b(k;\, n,\, p)] \\ &= -(p^2q^2)^{-1} \textstyle\sum_{k=[k_1]+1}^{[k_2]} P(k)b(k;\, n,\, p).\end{aligned}$$

Now we want an asymptotic expression for $\sum_{k=[k_1]+1}^{[k_2]} P(k)\cdot b(k;\, n,\, p)$. Write $x_k = (k - np)/[npq]^{\frac{1}{2}}$. For $[k_1] + 1 \leqq k \leqq [k_2]$ we have

$$\begin{aligned}\xi_1 = (q - p + 2\theta_n')/2[npq]^{\frac{1}{2}} - [1 + (p-q)^2/4npq]^{\frac{1}{2}} &\leqq x_k \\ &\leqq (q - p - 2\theta_n'')/2[npq]^{\frac{1}{2}} + [1 + (p-q)^2/4npq]^{\frac{1}{2}} = \xi_2,\end{aligned}$$

where $0 < \theta_n' \leqq 1$, $0 \leqq \theta_n'' < 1$.

Let $F_n(x)$ be the distribution function and $q_n(x_k)$ the frequency of the stochastic variable $X = (B - np)/[npq]^{\frac{1}{2}}$, where B is binomially distributed (n, p). We have

$$\begin{aligned}n^{-1}\cdot P(k) &= n^{-1}(k - k_1)(k - k_2) \\ &= n^{-\frac{1}{2}}(x_k\cdot[npq]^{\frac{1}{2}} - (q-p)/2 + [(p-q)^2 + 4npq]^{\frac{1}{2}}/2) \\ &\quad \cdot n^{-\frac{1}{2}}(x_k\cdot[npq]^{\frac{1}{2}} - (q-p)/2 \\ &\quad - [(p-q)^2 + 4npq]^{\frac{1}{2}}/2) \\ &= pq(x_k^2 - 1) + O(n^{-\frac{1}{2}}).\end{aligned}$$

So

$$n^{-1} \textstyle\sum_{k=[k_1]+1}^{[k_2]} P(k)\cdot b(k;\, n,\, p) = pq\cdot \sum_{k=[k_1]+1}^{[k_2]} (x_k^2 - 1)q_n(x_k) + O(n^{-\frac{1}{2}}).$$

Write

$$\begin{aligned}f(x) &= x^2 - 1, \quad \text{for } -1 \leqq x \leqq 1, \\ &= 0, \qquad\quad \text{for } x < -1 \text{ and } x > 1,\end{aligned}$$

then

$$\sum_{k=[k_1]+1}^{[k_2]} (x_k^2 - 1)q_n(x_k) = \int_{-\infty}^{\infty} f(x)\, dF_n(x) + \int_{\xi_1}^{-1} (x^2 - 1)\, dF_n(x) + \int_{1}^{\xi_2} (x^2 - 1)\, dF_n(x).$$

$F_n(x)$ tends to $\phi(x)$, the normal distribution function, when n tends to infinity, by the De Moivre-Laplace' theorem, and $f(x)$ is continuous. This implies that $\int_{-\infty}^{\infty} f(x)\, dF_n(x) \to \int_{-\infty}^{\infty} f(x)\, d\phi(x)$, as $n \to \infty$. Furthermore $\int_{\xi_1}^{-1} (x^2 - 1)\, dF_n(x) \to 0$ as $n \to \infty$, since $|\xi_1 + 1| < 2/[npq]^{\frac{1}{2}}$ when $n >$ some n_0, and $x^2 - 1$ tends to 0 when x is between ξ_1 and -1. In the same way $\int_{1}^{\xi_2} (x^2 - 1)\, dF_n(x) \to 0$, as $n \to \infty$. That is

$$\lim_{n\to\infty} n^{-1} \cdot \sum_{k=[k_1]+1}^{[k_2]} P(k) \cdot b(k; n, p) = pq \cdot \int_{-1}^{1} (x^2 - 1)\, d\phi(x).$$

It can also be shown that this convergence is uniform in p, when $p \in (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)$ for an arbitrary ϵ satisfying $0 < \epsilon < \frac{1}{2}$. Observe, e.g., the following: $F_{n,p}(x)$ is as before the distribution function of a standardized binomially distributed (n, p) variable. Then

$$|F_{n,p}(x) - \phi(x)| < C \cdot [(p^3q + q^3p)/(pq)^{3/2}] \cdot \log n/n^{\frac{1}{2}},$$

where C is an absolute constant. (See, e.g., Cramér: Random variables and probability distributions, p. 78). This together with integration by parts gives us

$$\left| \int_{-1}^{1} (x^2 - 1)\, dF_n(x) - \int_{-1}^{1} (x^2 - 1)\, d\phi(x) \right|$$
$$= \left| \int_{-1}^{1} 2x\phi(x)\, dx - \int_{-1}^{1} 2xF_n(x)\, dx \right| \leqq 4C[(p^3q + q^3p)/(pq)^{3/2}] \cdot \log n/n^{\frac{1}{2}},$$

so that the convergence of $\int_{-\infty}^{\infty} f(x)\, dF_n(x)$ is uniform in p. We have thus the following asymptotic expression for max $R''(p)$:

$$\max_{\mathfrak{N}} R''(p) \sim (n/pq) \cdot \int_{-1}^{1} (1 - x^2)\, d\phi(x),$$

which yields uniformly in p when $p \in (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)$,

$$\int_{-1}^{1} (1 - x^2)\, d\phi(x) = -\int_{-1}^{1} \phi'''(x)\, dx = [2/\pi e]^{\frac{1}{2}} \approx 0.48394.$$

Corollary. *Let f be an arbitrary function satisfying $|f(x) - f(y)| \leqq M|x - y|$. If for every n, f can be approximated uniformly by a reliability function, R_n, of order n, in such a way that $|R_n - f| \leqq A/n$, then $A \geqq M^2[2\pi e]^{\frac{1}{2}}/2^7$.*

Proof. Write

$$g(x) = 0, \quad \text{for } 0 \leqq x \leqq \tfrac{1}{2},$$
$$= Mx - M/2, \quad \text{for } \tfrac{1}{2} < x \leqq 1.$$

Choose

$$f(x) = g(x), \quad \text{if } g(x) \leqq 1,$$
$$= 1, \quad \text{if } g(x) > 1.$$

Apparently $f(x)$ satisfies $|f(x) - f(y)| \leqq M|x - y|$. The second order difference quotient is then at $x = \frac{1}{2}$, if $\Delta x = \epsilon$,

$$(\Delta^2 f/\Delta x^2)_{x=\frac{1}{2}} = [f(\tfrac{1}{2} + \epsilon) - 2f(\tfrac{1}{2}) + f(\tfrac{1}{2} - \epsilon)]/\epsilon^2 = M\epsilon/\epsilon^2 = M/\epsilon.$$

For every x and hence especially for $x = \frac{1}{2}$ we have $|\Delta^2 f/\Delta x^2 - \Delta^2 R_n/\Delta x^2| \leqq 4A/n\epsilon^2$, if R_n is a reliability function satisfying $|R_n - f| \leqq A/n$. As is easily seen $(\Delta^2 R_n/\Delta x^2)_{x=\frac{1}{2}} = R_n''(\frac{1}{2} + \theta)$, where $|\theta| < \epsilon$. Write $L = [2/\pi e]^{\frac{1}{2}}$. Then we have for large n, by Theorem 5,

$$L\cdot n/(\tfrac{1}{4} - \theta^2) \sim \max_{\mathfrak{N}} R''(\tfrac{1}{2} + \theta)$$
$$\geqq |(\Delta^2 R_n/\Delta x^2)_{x=\frac{1}{2}}| = |(\Delta^2 f/\Delta x^2)_{x=\frac{1}{2}} + \delta| \geqq M/\epsilon - \delta,$$

where $|\delta| \leqq 4A/n\epsilon^2$. Hence $Ln/(\frac{1}{4} - \theta^2) \geqq M/\epsilon - 4A/n\epsilon^2$. We are allowed to choose $\epsilon = \alpha/n$. For the present, α is regarded as an arbitrary positive real number.

For $\eta > 0$ arbitrarily small we can choose $n > n_0(\alpha)$ so that $4L < L/(\frac{1}{4} - \theta^2) < 4L + \eta$, since $|\theta| < \alpha/n$. Then the inequality becomes $4L + \eta \geqq M/\alpha - 4A/\alpha^2$, that is $A \geqq \frac{1}{4}(M\alpha - (4L + \eta)\alpha^2)$. Now $\max_{\alpha>0}(M\alpha - (4L + \eta)\alpha^2) = M^2/4(4L + \eta)$. Accordingly, A must be $\geqq M^2/16(4L + \eta)$, and since η is arbitrary, $A \geqq M^2/16\cdot 4L = M^2/2^7\cdot[2\pi e]^{\frac{1}{2}}$.

It should be pointed out that the bound obtained in this corollary is probably far from the best possible one. For example, if f is approximated by Bernstein-polynomials we have

$$|E_x(f(k/n)) - f(x)| \leqq E_x|f(k/n) - f(x)| \leqq ME_x|k/n - x|$$
$$\leqq M[E_x(k/n - x)^2]^{\frac{1}{2}} = M[x(1 - x)/n]^{\frac{1}{2}} \leqq (M/2)\, n^{-\frac{1}{2}}.$$

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