ON PARTIALLY LINKED BLOCK DESIGNS[1]

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1. Introduction and summary. Youden [8] introduced a new class of designs called Linked Block (LB) designs, where a LB design is defined as that in which any two blocks of the design have the same number of treatments in common. Later, the same author together with Connor [9] presented a set of designs called 'chain block designs' useful in the field of Physical Sciences. These designs can be considered as 'not fully linked block designs'. In this paper we shall extend the idea of LB designs in yet another way by defining Partially Linked Block (PLB) designs. The basic motive in introducing PLB designs is to generalise the notion of LB designs and also to see how far the existing designs yield new PBIB designs. Consider an arrangement of v treatments in b blocks of k plots each ($k < v$), such that each treatment occurs at most once in any block and altogether in r blocks. This is called an incomplete block design and is denoted by $D(v, b, k, r)$. We have $bk = vr$. When $b = v$, the design becomes symmetric. The incidence matrix $N = (n_{ij})$ $(i = 1, 2, \cdots, v; j = 1, 2, \cdots, b)$, where $n_{ij} = 1$ or 0 according as the ith treatment occurs or does not occur in the jth block characterises such a design completely.

2. Definition and preliminaries. In general, we can define a PLB design as follows:

A $D(v, b, k, r)$ is said to be a PLB design if its dual is a Partially Balanced Incomplete Block (PBIB) design, where by a dual we mean a design obtained from the original one by considering blocks as treatments and treatments as blocks [6]. This definition implies a grouping of blocks. For almost all practical purposes it will be sufficient if we consider a scheme of blocks having at the most three groups.

The PLB *designs*. Let us consider a $D(v, b, k, r)$, where $b = b_1b_2$ (b_1 and b_2 being integers greater than 1). Let I_m denote an identity matrix of order m and E_{mm}, a matrix of 1's of order $m \times m$. A $D(v, b, k, r)$ is called a Rectangular PLB (R-PLB) design if the b blocks can be arranged in the form of a $b_1 \times b_2$ rectangle

$$\begin{array}{cccc} B_{11} & B_{12} & \cdots & B_{1b_2} \\ B_{21} & B_{22} & \cdots & B_{2b_2} \\ \vdots & \vdots & \cdots & \vdots \\ B_{b_11} & B_{b_12} & \cdots & B_{b_1b_2} \end{array} \tag{2.1}$$

so that any two blocks in the same row have μ_1 treatments in common, any two blocks in the same column have μ_2 treatments in common and blocks not in the same row or column have μ_3 treatments in common, where μ_1, μ_2, μ_3 are each

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$\leqq k$ and $\mu_1 \neq \mu_3$ and $\mu_2 \neq \mu_3$. When $b_1 = b_2$ and $\mu_1 = \mu_2 (\neq \mu_3)$ in (2.1) we have an L_2-association scheme for blocks [1] and we shall call a design having such an arrangement of blocks a L_2-PLB design. If in (2.1), $\mu_1 = \mu_3 (\neq \mu_2)$ or $\mu_2 = \mu_3 (\neq \mu_1)$ we shall call it a GD-PLB design.

3. Characteristic roots of $N'N$. It is obvious that when the design is of R-PLB type, $N'N$ has the form $I_{b_1} \times (A - B) + E_{b_1 b_1} \times B$, where

$$A = (k - \mu_1) I_{b_2} + \mu_1 E_{b_2 b_2} \tag{3.1}$$
$$B = (\mu_2 - \mu_3) I_{b_2} + \mu_3 E_{b_2 b_2},$$

'$\times$' denoting the Kronecker product. μ_1, μ_2 and μ_3 (each $\leqq k$) are non-negative integers (including zero) connected by the relation

$$(b_2 - 1)\mu_1 + (b_1 - 1)\mu_2 + (b_1 - 1)(b_2 - 1)\mu_3 = k(r - 1).$$

The matrix $N'N$, when it has the form given in (3.1) has the eigenvalues
(i) $\theta_0 = k + (b_2 - 1)\mu_1 + (b_1 - 1)\mu_2 + (b_1 - 1)(b_2 - 1)\mu_3$
(ii) $\theta_1 = k - \mu_1 + (b_1 - 1)(\mu_2 - \mu_3)$
(iii) $\theta_2 = k - \mu_2 + (b_2 - 1)(\mu_1 - \mu_3)$
(iv) $\theta_3 = k - \mu_1 - \mu_2 + \mu_3$,
with multiplicities 1, $(b_2 - 1)$, $(b_1 - 1)$ and $(b_1 - 1)(b_2 - 1)$ respectively [7].

Advantages of the PLB *designs.* There are two methods available for analysing experiments. One is the Q method and the other the P method [3]. The choice is to be made according to their ease. For the LB designs analysis by the P method is very easy [5]. But in a LB design $b \leqq v$ so that it requires as many varieties as there are blocks. PLB designs are the next alternativeness in such cases both from the point of view of ease of analysis and existence of the design.

Next we shall consider certain necessary conditions for the various known types of designs to be the PLB designs.

We know that the non-zero roots of NN' and $N'N$ are the same. A design, belonging to a particular class like the BIB, PBIB etc., has a set of known non-zero roots for $N'N$ with known multiplicities. If this design is also a PLB, then $N'N$ has another set of roots. Since $N'N$ cannot simultaneously possess two different sets of roots, there must exist some equality relationships between them and their multiplicities. The theorems herein are proved in this way.

4. BIB and the PLB designs. As symmetric BIB designs are known to be LB designs, we shall exclude them. If N denotes the incidence matrix of a BIB design having parameters v, b, r, k and λ, then $N'N$ has got the non-zero roots (i) $d_0 = rk$ and (ii) $d_1 = (r - \lambda)$ with multiplicities 1 and $(v - 1)$ respectively. Hence

THEOREM 4.1.1. *A necessary condition for a* BIB *design with the given parameters and with $b \neq v$ to be a* R-PLB *is that either* (i) $k \geqq \max [(1 - (1/b_1)), (1 - (1/b_2))]\, d_1$, (ii) *$d_1$ is divisible by both b_1 and b_2*, (iii) *$(k^2/v) + (d_1/b)$ is an integer and* (iv) $2b_1 = (b - v + 2) \pm ((b - v + 2)^2 - 4b)^{\frac{1}{2}}$, $2b_2 = (b - v + 2) \mp$

$((b - v + 2)^2 - 4b)^{\frac{1}{2}}$ *or* (i) $k \geqq \max((1/b_1), (1/b_2))\, d_1$, (ii) d_1 *is divisible by both* b_1 *and* b_2, (iii) $(k^2/v) - (2d_1/b)$ *is an integer and* (iv) $2b_1 = (v + 1) \pm ((v + 1)^2 - 4b)^{\frac{1}{2}}$, $2b_2 = (v + 1) \mp ((v + 1)^2 - 4b)^{\frac{1}{2}}$.

THEOREM 4.1.2. *A necessary condition for a* BIB *design with the given parameters and with* $b \neq v$ *to be a* GD-PLB *is that either* (i) $k \geqq d_1$, (ii) k^2/v *is an integer and* (iii) b *is divisible by* $(b - v + 1)$ *or* (i) $k \geqq v\, d_1/b$, (ii) $v\, d_1/b$ *is an integer and* (iii) b *is divisible by* v.

THEOREM 4.1.3. *A necessary condition for a* BIB *design with the given parameters and with* $b \neq v$ *to be a* L_2-PLB *is that either* (i) $k \geqq 4d_1/(v + 1)$ *and* (ii) $2\, d_1$ *is divisible by* $(v + 1)$ *or* (i) $k \geqq [(b - v)/(b - v + 2)]\, d_1$ *and* (ii) $2\, d_1$ *is divisible by* $(b - v + 2)$.

COROLLARY 4.1.1. *A necessary condition for a* BIB *design with the given parameters and with* $b \neq v$ *to be a* R-PLB, GD-PLB *or* L_2-PLB *is that* d_1 *is a composite number.*

Note that a GD-PLB is the general case of affine resolvability. All affine resolvable designs are GD-PLB's, the converse being not true.

5. Group divisible (GD) designs and the PLB's. If N denotes the incidence matrix of a GD design having parameters $v = mn, b, r, k, m, n, \lambda_1, \lambda_2$, then NN' has the roots (i) $g_0 = rk$, (ii) $g_1 = (r - \lambda_1)$ and (iii) $g_2 = (rk - v\lambda_2)$ with multiplicities $1, m(n - 1)$ and $(m - 1)$ respectively [2]. The GD designs are divided into three subclasses: (a) Singular GD if $g_1 = 0$, (b) SR-GD if $g_1 > 0$ and (c) Regular GD if $g_1 > 0, g_2 > 0$. A singular GD design has $b \geqq m$ and a SR-GD has $b \geqq (v - m + 1)$. But when $b = m$ and $b = (v - m + 1)$, a S-GD and a SR-GD are known to be LB designs [4]. In the remaining cases viz., $b > m$ and $b > (v - m + 1)$ for each type of design we get exactly the same type of results as in the case of BIB, the only difference being that d_1 is to be replaced by the respective non-zero root and v to be replaced by m or $(v - m + 1)$ as the case may be.

For a Regular GD design $b \geqq v$. Hence we consider the two cases viz., (i) $b = v$ and (ii) $b > v$.

THEOREM 5.1.1. *A necessary condition for a Regular* GD *design with the given parameters and with* $b = v$ *to be a* R-PLB *is that either* (i) $n(\lambda_2 - \lambda_1)$ *is divisible by* b_1 *and* b_2, (ii) $(k^2/v) - [(g_1 - n(\lambda_1 - \lambda_2))/v]$ *is an integer and* (iii) $2b_1 = (v - m + 2) \pm [m^2(n - 1)^2 - 4(m - 1)]^{\frac{1}{2}}$, $2b_2 = (v - m + 2) \mp [m^2(n - 1)^2 - 4(m - 1)]^{\frac{1}{2}}$ *or* (i) $n(\lambda_1 - \lambda_2)$ *is divisible by both* b_1 *and* b_2, (ii) $(k^2/v) - [g_1 + 2n(\lambda_1 - \lambda_2)]/v$ *is an integer,* (iii) $2b_1 = (m + 1) \pm [(m + 1)^2 - 4mn]^{\frac{1}{2}}$, $2b_2 = (m + 1) \mp [(m + 1)^2 - 4mn]^{\frac{1}{2}}$ *and* (iv) $m \leqq (2n - 1) - 2[n(n - 1)]^{\frac{1}{2}}$ *or* $m \geqq (2n - 1) + 2[n(n - 1)]^{\frac{1}{2}}$.

THEOREM 5.1.2. *A necessary condition for a* R-GD *with the given parameters and with* $b = v$ *to be a* GD-PLB *is that either* (i) $\mu_1 = \lambda_1, \mu_2 = \lambda_2 = \mu_3, b_1 = m, b_2 = n$ *or* (ii) $\mu_1 = \mu_3 = \lambda_2, \mu_2 = \lambda_1, b_1 = n, b_2 = m$.

THEOREM 5.1.3. *A necessary condition for a* R-GD *with the given parameters and with* $b = v$ *to be a* L_2-PLB *is that* $b = v = 4, m = n = 2$.

COROLLARY 5.1.1. *A necessary condition for a* R-GD *design with the given parameters and with* $b = v$ *to be a* R-PLB, GD-PLB *or* L_2-PLB *is that* $(g_1 - g_2)$ *is a composite number.*

THEOREM 5.1.4. *A necessary condition for a* R-GD *design with the given parameters and with* $b > v$ *to be a* R-PLB *is that either* (i) g_1 *and* g_2 *are divisible by* m *and* $(v - m + 1)$ *respectively,* (ii) $(k^2/v) - [(2g_1 - n(\lambda_2 - \lambda_1))/b]$ *is an integer and* (iii) $k \geqq \max(g_1/m, g_2/(v - m + 1))$ *or* (i) g_1 *and* $n(\lambda_1 - \lambda_2)$ *are divisible by* m *and* $(b - v + 1)$ *respectively and* (ii) $k \geqq g_1 + n(\lambda_1 - \lambda_2)/(b - v + 1)$.

6. The structure of PLB designs. Let $S_\alpha^{(i)}$ denote the number of times the treatment α occurs in the ith row of the scheme (2.1) and $m_\alpha^{(j)}$ denote the number of times it occurs in the jth column of (2.1). $S_\alpha^{(i)}C_2$ = the number of times the treatment α occurs as a common treatment between any pair of blocks in the ith row. So for $m_\alpha^{(j)}C_2$. Then

$$\textstyle\sum_{\alpha=1}^{v} S_\alpha^{(i)} = b_2 k, \qquad \sum_{\alpha=1}^{v} S_\alpha^{(i)}C_2 = \frac{1}{2}[\mu_1 b_2(b_2 - 1)].$$

$$\therefore \textstyle\sum_{\alpha=1}^{v} S_\alpha^{(i)}(S_\alpha^{(i)} - 1) = \mu_1 b_2(b_2 - 1)$$

$$\therefore \qquad \textstyle\sum_{\alpha=1}^{v} (S_\alpha^{(i)})^2 = \mu_1 b_2(b_2 - 1) + b_2 k.$$

Now,

$$\begin{aligned}\textstyle\sum_{\alpha=1}^{v} (S_\alpha^{(i)} - \bar{S}^{(i)})^2 &= \textstyle\sum_{\alpha=1}^{v} (S_\alpha^{(i)})^2 - v^{-1}(\sum_{\alpha=1}^{v} S_\alpha^{(i)})^2 \\ &= \mu_1 b_2(b_2 - 1) + b_2 k - b_2^2 k^2/v \\ &= b_2(k + \mu_1(b_2 - 1) - b_2 k^2 b_1/b_1 v) \\ &= (b_2/b_1)(b_1 k + \mu_1 b_1(b_2 - 1) - k - \mu_1(b_2 - 1) \\ &\quad - (b_1 - 1)\mu_2 - (b_1 - 1)(b_2 - 1)\mu_3) \\ &= [b_2(b_1 - 1)/b_1](k - \mu_2 + (b_2 - 1)(\mu_1 - \mu_3)) \\ &= (b_2/b_1)(b_1 - 1)\theta_2 .\end{aligned} \tag{6.1}$$

Similarly,

$$\textstyle\sum_{\alpha=1}^{v} (m_\alpha^{(j)} - \bar{m}^{(j)})^2 = (b_1/b_2)(b_2 - 1)\theta_1 ; \tag{6.2}$$

where $\bar{S}^{(i)} = v^{-1} \sum_{\alpha=1}^{v} S_\alpha^{(i)}$ and $\bar{m}^{(j)} = v^{-1} \sum_{\alpha=1}^{v} m_\alpha^{(j)}$. From the above we get

THEOREM 6.1. *Let* N *be a design belonging to the* R-PLB *type. Then* (a) *a necessary and sufficient condition that each treatment occurs the same number of times in the* ith *row of* (2.1) *is*

$$r/b_1 = [\mu_1(b_2 - 1)/k] + 1 = m \tag{6.3}$$

where m *is a positive integer, in which case each treatment occurs exactly* m *times in each row of the scheme* (2.1); (b) *similar condition for columns of* (2.1) *is that*

$$r/b_2 = [\mu_2(b_1 - 1)/k] + 1 = n, \tag{6.4}$$

where n is a positive integer, in which case each treatment occurs exactly n times in each column of the scheme (2.1).

PROOF. From (6.1) it follows that each treatment occurs the same number of times in each of the rows of the scheme (2.1) if and only if $\theta_2 = 0$. Let $\theta_2 = 0$, in which case $S_1^{(i)} = S_2^{(i)} = \cdots = S_v^{(i)} = b_2k/v = r/b_1$. Further

$$\begin{aligned} r &= 1 + k^{-1}[(b_2 - 1)\mu_1 + (b_1 - 1)\mu_2 + (b_1 - 1)(b_2 - 1)\mu_3] \\ &= 1 + k^{-1}[(b_2 - 1)(\mu_1 - \mu_3) + (b_1 - 1)\mu_2 + b_1(b_2 - 1)\mu_3] \\ &= 1 + k^{-1}[\mu_2 - k + (b_1 - 1)\mu_2 + b_1(b_2 - 1)\mu_3] \\ &= \mathrm{k}^{-1}[b_1\mu_2 + b_1(b_2 - 1)\mu_3], \qquad \text{as } \theta_2 = 0. \end{aligned}$$

Hence, $r/b_1 = k^{-1}(b_2 - 1)\mu_1 + 1 = m$.

Sufficiently. We have

$$\begin{aligned} \theta_2 &= k - \mu_2 + (b_2 - 1)(\mu_1 - \mu_3) \\ &= k - \mu_2 + (m - 1)k - (b_2 - 1)\mu_3 \\ &= [mk(b_1 - 1) + (b_2 - 1)\mu_1 - k(r - 1)]/(b_1 - 1) \\ &= [k/(b_1 - 1)](m(b_1 - 1) + (m - 1) - r + 1) \\ &= [k/(b_1 - 1)(b_1m - r) \\ &= 0, \end{aligned}$$

which implies that the distribution of the treatments into the sets of blocks is as required.

Now, in the case of L_2-PLB, the only modification required in the above theorem is to put $b_2 = b_1$. The above theorem holds also in the case of GD-PLB. The proofs in both cases are along the same lines as before.

EXAMPLE 1. Consider the SR-GD design with parameters $b = v = 8$, $r = k = 4$, $m = 4$, $n = 2$, $\lambda_1 = 0$, $\lambda_2 = 2$. This is seen to be a GD-PLB with $\mu_1 = 0$, $\mu_2 = 3$, $b_1 = 4$, $b_2 = 2$ so that

$$r/b_1 = k^{-1}(b_2 - 1)\mu_1 + 1 = 1$$

for this design.

It should be noted that Equation (6.3) combines together three different conditions, viz., (i) r/b_1 is a positive integer, (ii) $k^{-1}(b_2 - 1)\mu_1$ is a positive integer and (iii) the former integer is greater than the latter by 1. The following examples show that there exist PLB designs for which one of the above conditions is violated.

EXAMPLE 2. Consider the R-GD design with parameters $b = v = 6$, $r = k = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$. This is a GD-PLB with $\mu_1 = 2$, $\mu_2 = 1$, $b_1 = 3$, $b_2 = 2$. Hence $r/b_1 = \frac{3}{3} = 1$, $k^{-1}(b_2 - 1)\mu_1 = \frac{2}{3}$, not an integer. Thus Condition (i) above is satisfied.

EXAMPLE 3. Consider the R-GD design with parameters $v = b = 8$, $r = k = 3$,

$\lambda_1 = 0, \lambda_2 = 1$. This is seen to be a GD-PLB with $\mu_1 = 0, \mu_2 = 1, b_1 = 4, b_2 = 2$, Thus $r/b_1 = \frac{3}{4}$, not an integer, $k^{-1}(b_2 - 1)\mu_1 = 0$.

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