MOST STRINGENT SOMEWHERE MOST POWERFUL TESTS AGAINST ALTERNATIVES RESTRICTED BY A NUMBER OF LINEAR INEQUALITIES

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0. Summary. The paper studies hypothesis testing problems for the mean of a vector variate having a multivariate normal distribution, in cases where the alternative is restricted by a number of linear inequalities. A new criterion which can be regarded as a generalization of the "maximin-r^2" criterion of Abelson and Tukey (cf. [1]) is introduced: we try to obtain tests which are "most stringent" among the "somewhere most powerful" tests, (Section 2). For an important class of testing problems (Section 3) such tests can be characterized by a (half-) line l_0 minimizing a maximum angle (Sections 4, 6 and 7). This (half-) line l_0 can be obtained by means of a method described by Abelson and Tukey (Section 5). The theory of this paper can be applied to a large number of actual testing problems. Such applications, (one of which is treated in Section 8), generalizations and details of the theory of this paper will be considered in the thesis [5].

1. Introduction. The subject is introduced by considering two applications which elucidate the formulation of the problems (Section 3) and of the criterions (Section 2).

First, let X_i have independent normal $N(\mu_i, \sigma_i^2)$ distributions, σ_i^2 being known, and $(i = 1, \cdots, k)$. Homogeneity of means $(\mu_1 = \mu_2 = \cdots = \mu_k)$ has to be tested against an alternative defined by a number of homogeneous linear inequalities, e.g. against an upward trend ($\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_k$, with at least one inequality strong). Paper [2] is a key paper in this connection. *Bartholomew* applies the likelihood ratio principle and obtains tests entailing rather complicated calculations. Power calculations suggest that no worth-while improvement on his tests is possible, (cf. [2], p. 239). In contrast to Bartholomew, *Abelson* and *Tukey* restricted their attention to tests based on linear contrasts $\sum_{i=1}^k c_i X_i$ ($\sum_{i=1}^k c_i = 0$). They describe a method to obtain a "maximin-r^2" contrast in the special case $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2$, (cf. [1]), r being the correlation coefficient between the c_i and the μ_i. Power calculations of Bartholomew (cf. [2], p. 268) suggest that the test of Abelson and Tukey against an upward trend cannot be improved upon to a worth-while extent: both Bartholomew's likelihood ratio test and that of Abelson and Tukey based on the maximin contrast seem to be useful for testing homogeneity of means against an upward trend.

Next we consider the following problem: let $(X_1, \cdots, X_k)$ have a multivariate normal distribution with known covariance matrix; the hypothesis $H: \mu_i = 0 (i = 1, \cdots, k)$ has to be tested against the alternative

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$K_1 : \mu_i \geqq 0 (i = 1, \cdots, k)$, (with at least one inequality strong). Paper [3] is a key paper in this connection. Following Bartholomew, *Kudô* applies the likelihood ratio principle to this problem (H, K_1).

With regard to the above-mentioned problems, application of the likelihood ratio tests (Bartholomew, Kudô) involves rather complicated calculations, especially if certain tables (cf. [2], p. 248) are irrelevant. Furthermore these tests are "nowhere most powerful"; whereas tests φ based on certain linear combinations of $X_1, \cdots, X_k$ give rise to simple calculations and moreover such tests φ will be "somewhere most powerful" (S.M.P.) which means that φ is M.P. (among the tests of given size α) for at least one simple hypothesis out of the alternative. We propose formulating this requirement first, so we restrict our attention to the class C of all S.M.P. size-α tests. Finally, we must choose the "best" test φ^* out of C. We shall select φ^* from C so that the maximum shortcoming over the whole alternative is minimized. In other words the *most stringent S.M.P. size-α* test must be obtained.

In the problem of testing homogeneity of means $(\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2)$ this criterion leads to the same tests as those of Abelson and Tukey (v. Sections 5 and 8). In Kudô's case the linear combination obtained has, as is to be expected, positive coefficients.

We now consider two further problems, each of them giving rise to a certain modification of the criterion "*most stringent S.M.P. size-α*".

First, let X_{ij}, $(j = 1, \cdots, n_i)$ be a sample from the normal $N(\mu_i, \sigma^2)$ distribution $(i = 1, \cdots, k)$ with σ^2 *unknown*. Homogeneity of means $(\mu_1 = \mu_2 = \cdots = \mu_k)$ has to be tested against an upward trend $(\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_k$, with at least one inequality strong). Unfortunately we cannot construct the class of S.M.P. size-α tests, owing to the nuisance parameter σ^2 and therefore we shall try to obtain the most stringent test among the S.M.P. *similar* size-α tests.

Next, using the same notation, we consider the *two-sided* problem where homogeneity of means has to be tested against an upward or a downward trend. In this case we shall try to obtain the most stringent test among the S.M.P. *unbiased* size-α tests.

2. Notation and definitions. We use the following notation:

$$\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} \exp(-2^{-1}u^2)\, du; \quad \Phi^{\times}(x) = 1 - \Phi(x); \quad u_\alpha = \{\Phi^{\times}\}^{-1}(\alpha);$$

further $t_{f;\alpha}$ is the solution of $P(T_f \geqq t_{f;\alpha}) = \alpha$ where T_f has Student's t-distribution with f degrees of freedom. R^d will denote a linear subspace of dimension d (through the origin) in the sample space R^n. A superscript$^{(1)}$ is used sometimes in order to denote that the corresponding symbol has one, and only one, fixed value.

We recapitulate some definitions, most of which are generally used. For that purpose, let (H, K) be a hypothesis testing problem, with composite alternative K. Let $\beta_\varphi(\theta) = E_\theta\{\varphi(X)\}$ denote the power in θ of the test φ.

The test φ is *of size* α, if

$$\sup_{\theta \varepsilon H} \beta_\varphi(\theta) \leqq \alpha$$

and is *similar of size* α, if

$$\beta_\varphi(\theta) = \alpha \qquad \text{for all} \qquad \theta \varepsilon H$$

and is *unbiased of size* α, if

$$\sup_{\theta \varepsilon H} \beta_\varphi(\theta) \leqq \alpha, \qquad \inf_{\theta \varepsilon K} \beta_\varphi(\theta) \geqq \alpha.$$

The *envelope power function* $\beta_D^*(\theta)$ of a class D of tests φ, is defined by

$$\beta_D^*(\theta) = \sup_{\varphi \varepsilon D} \beta_\varphi(\theta).$$

The *shortcoming* $\gamma_{\varphi,D}(\theta)$ of a test φ with respect to the class D, is defined by

$$\gamma_{\varphi,D}(\theta) = \beta_D^*(\theta) - \beta_\varphi(\theta).$$

A test φ in a class C of tests, is said to be *most stringent in C with respect to D for testing against* K, if test φ minimizes in C the maximum shortcoming with respect to D on the alternative K:

$$\sup_{\theta \varepsilon K} \gamma_{\varphi,D}(\theta) = \inf_{\Psi \varepsilon C} \sup_{\theta \varepsilon K} \gamma_{\Psi,D}(\theta).$$

In the special case $C = D$, we obtain the *most stringent* (D) test. On specializing further, we obtain the *most stringent size-*α test in case D is the class of size-α tests.

The minimax principle leading to the preceding definitions is sometimes quite unreasonable, a minimized maximum shortcoming on K often going with a large shortcoming for "many" alternatives $\theta \varepsilon K$, (cf. [4], p. 13). This objection to the two foregoing principles seems to be realistic for many problems of the form (H, K_1) and (H, K_2) that will presently be considered. For these problems, it seems reasonable to restrict our attention to the subclass C of D, containing the tests φ which have the shortcoming $\gamma_{\varphi,D}(\theta)$ equal to zero for some θ in K. Tests satisfying this condition are said to be *somewhere most powerful with respect to the class* D, (abbreviated: S.M.P. (D)). By specializing the definition of "most stringent in C with respect to D for testing against K", in case C is the class of S.M.P. (D) tests, we obtain the *most stringent* S.M.P. (D) test. The following special cases will be applied:

(i) D is the class of size-α tests; we obtain the *most stringent* S.M.P. *size-*α test;

(ii) D is the class of similar size-α tests; we obtain the *most stringent* S.M.P. *similar size-*α test;

(iii) D is the class of unbiased size-α tests; we obtain the *most stringent* S.M.P *unbiased size-*α test.

Obviously the most stringent S.M.P. (D) test has in general a larger maximum shortcoming than the most stringent (D) test, whereas the latter test has a larger shortcoming generally in a region inside the alternative. For many problems of the form (H, K_1) and (H, K_2) which will be considered, no clear-cut preference will exist for either one of the two principles mentioned above.

3. The formulation of the problems. Let $X = (X_1, \cdots, X_n)$ have the multivariate normal distribution $N(\xi, \Sigma)$ with pdf

$$(1) \qquad [|A|^{\frac{1}{2}}/(2\pi\sigma^2)^{\frac{1}{2}n}] \exp\{-(2\sigma^2)^{-1} \textstyle\sum_{i=1}^n \sum_{j=1}^n a^{ij}(x_i - \xi_i)(x_j - \xi_j)\}$$

where the matrix A is nonsingular and known; $\Sigma = \sigma^2 A^{-1}$.

Problems will be considered where σ^2 is known, in which case we take σ^2 to be equal to 1, and also where σ^2 is unknown. The outcomes $x = (x_1, \cdots, x_n)$ of the random vector X and the vector $\xi = (\xi_1, \cdots, \xi_n)$ of means, can be regarded as points in the same n dimensional space R^n.

The vector ξ of means is known to lie in a subset of a given s-dimensional hyperplane V^s in R^n $(s \leqq n)$ defined by the $(n - s)$ equalities

$$(2) \qquad b^{0h} + \textstyle\sum_{i=1}^n b^{ih}\xi_i = 0 \qquad (h = 1, \cdots, n - s)$$

where $[b^{ih}](i = 1, \cdots, n; h = 1, \cdots, n - s)$ is a matrix of rank $n - s$.

The hypothesis H is to be tested that ξ lies in a given $(s - r)$-dimensional hyperplane V^{s-r} in V^s, defined by

$$\text{Hypothesis } H\colon \quad b^{0h} + \textstyle\sum_{i=1}^n b^{ih}\xi_i = 0 \quad (h = n - s + 1, \cdots, n - s + r)$$

where $1 \leqq r \leqq s$; $[b^{ih}](i = 1, \cdots, n; h = 1, \cdots, n - s + r)$ is a matrix of rank $n - s + r$, (cf. [4] Chap. 7).

We shall first derive tests for H against the following "*one-sided*" alternative

$$\text{Alternative } K_1: \quad b^{0h} + \textstyle\sum_{i=1}^n b^{ih}\xi_i \geqq 0 \quad (h = n - s + 1, \cdots, n - s + r),$$

with at least one inequality strong;

corresponding to a subset of V^s, which subset will be denoted also by K_1.

We shall describe some transformations simplifying the formulation of the problem defined above. We state here that the results of our investigations will be put in forms which do not depend on the particular transformations used, so the theory can be applied without an explicit construction of these transformations.

First, we choose the origin of R^n in V^{s-r} defined by the hypothesis H, thus obtaining a problem where

$$(3) \qquad b^{0h} = 0 \qquad (h = 1, \cdots, n - s + r)$$

holds true. So we can assume (3) in what follows. In this case all hyperplanes V^t become linear subspaces R^t, containing the origin.

Next, the problem can be written in a simpler form, by introducing a new basis in R^n. (Transformation into independent normal variates has also been used by Kudô, cf. [3], p. 404). Denoting the points $(x_1, \cdots, x_n)$ of R^n by x, we define an inner product in R^n by means of the bilinear form

$$(4) \qquad (x, y) = \textstyle\sum_{i=1}^n \sum_{j=1}^n a^{ij}x_iy_j.$$

Orthogonality $x \perp y$ is defined by $(x, y) = 0$, the norm $\|x\|$ is defined by $\|x\|^2 = (x, x)$ and the metric is defined by $d(x, y) = \|x - y\|$. We can construct an orthonormal basis $f_1, \cdots, f_n$ for R^n, e.g., by using the Gram-Schmidt orthogonalization process, such that $f_{n-s+r+1}, \cdots, f_n$ span the linear subspace R^{s-r} defined by the hypothesis H and $f_{n-s+1}, \cdots, f_n$ span R^s defined by the Equalities (2). The problem can be reformulated by means of the coordinates $Y_1, \cdots, Y_n$ with respect to the basis $f_1, \cdots, f_n$, of the sample point X: $Y_i = (X, f_i)$ and by means of the new coordinates $\eta_1, \cdots, \eta_n$ of the vector ξ of mean values. The original coordinates $X_1, \cdots, X_n$ of the sample point X having the normal $N(\xi, \Sigma)$ distribution (1), the new coordinates $Y_1, \cdots, Y_n$ of X will have independent normal $N(\eta_i, \sigma^2)$ distributions given by the pdf

$$(5) \qquad (2\pi\sigma^2)^{-\frac{1}{2}n} \exp\{-(2\sigma^2)^{-1} \textstyle\sum_{i=1}^n (y_i - \eta_i)^2\},$$

since

$$\textstyle\sum_{i=1}^n \sum_{j=1}^n a^{ij}(x_i - \xi_i)(x_j - \xi_j) = \|x - \xi\|^2 = \sum_{i=1}^n (y_i - \eta_i)^2.$$

The vector ξ of means $\eta_1, \cdots, \eta_n$ is known to lie in the s-dimensional linear subspace R^s, defined by the equalities

$$(6) \qquad \eta_i = 0 \qquad (i = 1, \cdots, n - s)$$

and the hypothesis H is to be tested, that ξ lies in the subspace R^{s-r} of R^s, defined by

$$\text{Hypothesis } H: \quad \eta_i = 0 \qquad (i = n - s + 1, \cdots, n - s + r),$$

whereas the alternative K_1 becomes a subset in R^s of the form

$$\text{Alternative } K_1: \quad \textstyle\sum_{i=n-s+1}^{n-s+r} d^{ih}\eta_i \geqq 0, \qquad (h = 1, \cdots, r),$$

with at least one inequality strong;

where $[d^{ih}](i = n - s + 1, \cdots, n - s + r; h = 1, \cdots, r)$ is a matrix of rank r.

4. Problem $\{(H, K_1), \sigma^2 = 1\}$. By applying the Neyman-Pearson fundamental lemma and Theorem 7 of [4], p. 91 we obtain the M.P. size-α test

$$(7) \qquad \textstyle\sum_{i=n-s+1}^{n-s+r} \eta_i^{(1)} Y_i \geqq u_\alpha \{\sum_{i=n-s+1}^{n-s+r} \eta_i^{(1)2}\}^{\frac{1}{2}}$$

for testing the hypothesis H against the simple alternative $\eta^{(1)} = (\eta_1^{(1)} \cdots \eta_n^{(1)})$ in K_1; so $\eta^{(1)}$ satisfies

$$\eta_i^{(1)} = 0 \quad (i = 1, \cdots, n - s); \qquad \textstyle\sum_{i=n-s+1}^{n-s+r} d^{ih}\eta_i^{(1)} \geqq 0 \quad (h = 1, \cdots, r).$$

Evidently the critical region is a half-space of n dimensions, bounded by a hyperplane of $(n - 1)$ dimensions parallel to R^{s-r} defined by H.

But Test (7) is uniformly M.P. size-α for testing the hypothesis H against the auxiliary alternative:

$$\text{Alternative } A_1: \quad \eta_i = 0 \qquad (i = 1, \cdots, n - s)$$

$$\eta_i = \theta\eta_i^{(1)} \qquad (i = n - s + 1, \cdots, n - s + r), \theta > 0$$

which corresponds with a half-R^{s-r+1} through $\eta^{(1)}$ and bounded by the R^{s-r} defined by the hypothesis H.

Test (7) can be rewritten in a form without reference to both the particular basis for R^n and the particular point $\eta^{(1)}$ in A_1. For that purpose, let R^r be the r dimensional subspace perpendicular to R^{s-r} in R^s. The intersection of an alternative K with R^r will be denoted by K'. So A_1' is the half-line l of points $(\eta_1, \cdots, \eta_n)$ satisfying

$$\eta_i = 0 \quad (i = 1, \cdots, n - s, n - s + r + 1, \cdots, n), \qquad \eta_i = \theta\eta_i^{(1)} \quad (i = n - s + 1, \cdots, n - s + r),$$

where $\theta > 0$. Let X^I denote the projection of the sample point X on to the R^1 spanned by l. X^I is determined by

$$\theta = \{\textstyle\sum_{i=n-s+1}^{n-s+r} \eta_i^{(1)2}\}^{-1} \textstyle\sum_{i=n-s+1}^{n-s+r} \eta_i^{(1)} Y_i$$

hence

$$\|X^I\| = \{\textstyle\sum_{i=n-s+1}^{n-s+r} \eta_i^{(1)2}\}^{-\frac{1}{2}} |\textstyle\sum_{i=n-s+1}^{n-s+r} \eta_i^{(1)} Y_i|.$$

Let OX^I be defined by $\|X^I\|$ in case $X^I \varepsilon\, l(\theta > 0)$ and by $-\|X^I\|$ in case $X^I \not\varepsilon\, l(\theta \leqq 0)$. The uniformly M.P. size-α test (7) for problem $\{(H, A_1), \sigma^2 = 1\}$ obtains the following form

$$OX^I \geqq u_\alpha. \tag{8}$$

In order to find the most stringent S.M.P. size-α test for Problem $\{(H,K_1), \sigma^2 = 1\}$, we must determine the half-line l_0 corresponding with this test.

The intersection K_1' of K_1 and R^r is a polyhedral angle with vertex O and edges $e_1, \cdots, e_r$ defined by:

$$e_g : \textstyle\sum_{i=n-s+1}^{n-s+r} d^{ih}\eta_i = 0 \quad (h = 1, \cdots, r; h \neq g); \qquad \textstyle\sum_{i=n-s+1}^{n-s+r} d^{ig}\eta_i > 0. \tag{9}$$

The critical region and hence the power function $\beta_\varphi(\theta)$ of each S.M.P. size-α test φ in class C and also the envelope power function $\beta_D^*(\theta) = \beta_C^*(\theta)$ are invariant under translations parallel to R^{s-r}. So we can confine our attention to the power functions and the envelope power function over K_1'.

The maximum shortcoming over K_1' of the S.M.P. size-α test φ, which is uniformly M.P. size-α against the half-line l in K_1', can be studied by considering the maximum shortcoming over each half-line m in K_1':

$$\sup_{\theta\varepsilon K_1} \gamma_{\varphi,D}(\theta) = \sup_{\theta\varepsilon K_1'} \gamma_{\varphi,C}(\theta) = \sup_{m\subset K_1'} \{\sup_{\theta\varepsilon m} \gamma_{\varphi,C}(\theta)\}. \tag{10}$$

The power of the test φ which is uniformly M.P. size-α against the half-line l in K_1', is equal to $\Phi^\times(d)$ in the point Q on the half-line m; d denotes the (signed) distance between Q and the boundary $OX^I = u_\alpha$ of the critical region (8). We have

$$d = u_\alpha - OQ \cos\{\Psi(l, m)\}$$

where $\Psi(l, m)$ is the angle between l and m. Hence

$$\beta_\varphi(Q) = \Phi^\times[u_\alpha - OQ \cos \{\Psi(l, m)\}].$$

The envelope power function $\beta_D^*(Q) = \beta_C^*(Q)$ is obtained when $\Psi(l, m) = 0$. Hence, the maximum shortcoming of test φ which is uniformly M.P. size-α against the half-line l, is determined by

$$\begin{aligned} \sup_{\theta\varepsilon m} \gamma_{\varphi,C}(\theta) &= \sup_{OQ>0} \{\beta_C^*(Q) - \beta_\varphi(Q)\} \\ &= \sup_{OQ>0} [\Phi^\times(u_\alpha - OQ) - \Phi^\times[u_\alpha - OQ \cos \{\Psi(l, m)\}]] \\ &=^{\text{def}} M_{1,\alpha}\{\Psi(l, m)\} \end{aligned} \tag{11}$$

on the half-line m. The function $M_{1,\alpha}(\Psi)$ introduced above is strictly increasing for $0 \leqq \Psi \leqq \frac{1}{2}\pi$; $M_{1,\alpha}(\frac{1}{2}\pi) = (1 - \alpha)$ and $M_{1,\alpha}(\Psi) = 1$ for $\Psi > \frac{1}{2}\pi$. The maximum shortcoming over K_1 of test φ is determined by

$$\sup_{\theta\varepsilon K_1'} \gamma_{\varphi,C}(\theta) = M_{1,\alpha}[\sup_{m \subset K_1'} \{\Psi(l, m)\}]$$

and we obtain the most stringent S.M.P. size-α test φ_0 provided that the corresponding half line l_0 minimizes $\sup_m \{\Psi(l, m)\}$.

THEOREM. *The most stringent* S.M.P. *size-α test φ_0 for Problem* $\{(H, K_1), \sigma^2 = 1\}$ *is given by* (8), *where X^I is the projection of the sample point X on to the R^1 spanned by the half-line l_0 satisfying*

$$\sup_{m \subset K_1'} \Psi(l_0, m) = \inf_{l \subset K_1'} \sup_{m \subset K_1'} \Psi(l, m). \tag{12}$$

The maximum shortcoming $\sup_{\theta \subset K_1} \gamma_{\varphi_0,D}(\theta)$ *of the test φ_0 is determined by* $M_{1,\alpha}(\Psi_0)$, *where Ψ_0 is the minimum mentioned above*:

$$\Psi_0 =^{\text{def}} \inf_{l \subset K_1'} \sup_{m \subset K_1'} \Psi(l, m). \tag{13}$$

A method of obtaining the half-line l_0 will be considered in the following section.

5. The determination of l_0 by means of a method of Abelson and Tukey. It is fairly obvious that the half-line l_0 satisfying (12) can be obtained by first determining the half-line l in R^r which satisfies

$$\Psi(l, e_1) = \Psi(l, e_2) = \cdots = \Psi(l, e_r) \tag{14}$$

(l is the axis of the circumscribed semi-cone of revolution of the polyhedral angle K_1'). In the case when $l \subset K_1'$ it is obvious that l is the half-line l_0 satisfying (12). We notice that $l \subset K_1'$ holds true for the greater number of the applications we have in mind. Further (14) admits simple explicit solutions in these cases and so we obtain tests involving simple computations, (v. Section 8 and [5]).

In case that $l \not\subset K_1'$, we have to search for a polyhedral angle $K'_{g_1,\cdots,g_f}$ with the f edges $e_{g_1}, \cdots, e_{g_f}(1 \leqq g_1 < g_2 < \cdots < g_f \leqq r)$ (so $K'_{g_1,\cdots,g_f}$ is a f-dimensional "face" of K_1') such that the half-line l', in the R^f spanned by these edges, which is determined by

$$\Psi(l', e_{g_1}) = \Psi(l', e_{g_2}) = \cdots = \Psi(l', e_{g_f}) =^{\text{def}} \Psi_{g_1,\cdots,g_f}$$

(l' is the axis of the circumscribed semi-cone of revolution of the polyhedral angle $K'_{g_1,\cdots,g_f}$), satisfies (i) $l' \subset K'_{g_1,\cdots,g_f}$ and (ii) $\Psi(l', e_g) \leqq \Psi_{g_1,\cdots,g_f} (g = 1, \cdots, r)$. In this case l' is the half-line l_0 satisfying (12).

Indeed, *Abelson and Tukey* (cf. [1], pp. 1353–54, 1366–68) have proved that the half-line l_0 can be obtained along the preceding lines.

6. Problem $\{(H, K_1), \sigma^2$ **unknown**$\}$. By arguments similar to those of Section 4, we can derive the most stringent S.M.P. similar size-α test for Problem $\{(H, K_1), \sigma^2 \text{ unknown}\}$.

By applying an obvious modification of Theorem 1 in [4], p. 161, we obtain the uniformly M.P. similar size-α test φ:

$$OX^I/\|X^I - X^{II}\| \geqq t_{n-s+r;\alpha}/(n - s + r - 1)^{\frac{1}{2}} \tag{15}$$

for Problem $\{(H, A_1), \sigma^2 \text{ unknown}\}$ where the alternative A_1 is defined in Section 4; X^I is the projection of the sample point X on to the R^1 spanned by the half-line $l = A_1'$, X^{II} is the projection of the sample point X on to the R^{n-s+r} perpendicular to the R^{s-r} defined by the hypothesis H, in the sample space R^n. So the class C of S.M.P. similar size-α tests for Problem $\{(H, K_1), \sigma^2 \text{ unknown}\}$ is determined by (15) for l varying over K_1'.

We observe that the critical region belonging to Test (15) consists of the points whose orthogonal projections on to R^{n-s+r} are inner or boundary points of asemi-cone of revolution with axis l and semi-angle

$$\Lambda_1 = \cot^{-1}\{(n - s + r - 1)^{-\frac{1}{2}} t_{n-s+r-1;\alpha}\}. \tag{16}$$

It can be proved that the maximum shortcoming $\sup_{\theta\varepsilon m} \gamma_{\varphi,C}(\theta)$ of Test (15) over the half-line m in K_1', is a non-decreasing function of $\Psi(l, m)$ which strictly increases for $\Psi(l, m) \leqq \Lambda_1$ and is constantly equal to 1 for $\Psi(l, m) > \Lambda_1$.

Applying considerations similar to those of Section 4, we obtain the following result.

THEOREM. *In case* $\Psi_0 \leqq \Lambda_1$, *the most stringent* S.M.P. *similar size-*α *test* φ_0 *for Problem* $\{(H, K_1), \sigma^2$ *unknown*$\}$ *is determined by* (15), *taking for* l *the half-line* l_0 *satisfying* (12).

In case $\Psi_0 > \Lambda_1$, each S.M.P. similar size-α test has the maximum shortcoming on K_1 equal to 1, so that no uniquely determined most stringent S.M.P. similar size-α test exists for Problem $\{(H, K_1), \sigma^2 \text{ unknown}\}$ in this case.

7. The two-sided problems $\{(H, K_2), \sigma^2 = 1\}$ **and** $\{(H, K_2), \sigma^2$ *unknown*$\}$. The "two-sided" alternative K_2 is defined by

Alternative K_2: $b^{0h} + \sum_{i=1}^{n} b^{ih}\xi_i \geqq 0 \quad (h = n - s + 1, \cdots, n - s + r)$,

with at least one inequality strong,

or

$b^{0h} + \sum_{i=1}^{n} b^{ih}\xi_i \leqq 0 \quad (h = n - s + 1, \cdots, n - s + r)$,

with at least one inequality strong;

using the formulation of Section 3. So K_2 is the combination of a region of the form K_1 and the reflection of this region with respect to an arbitrary point of H. Problem $\{(H, K_2), \sigma^2 = 1\}$ does not present particular difficulties. We obtain the following result.

THEOREM. *The most stringent* S.M.P. *unbiased size-*α *test* φ_0 *for Problem* $\{(H, K_2), \sigma^2 = 1\}$ *is determined by*

$$\|X^I\| \geqq u_{\frac{1}{2}\alpha} \tag{17}$$

where X^I *is the projection of the sample point* X *on the line* l_0 *in* K_2', *satisfying*

$$\sup_{m \subset K_2'} \Psi(l_0, m) = \inf_{l \subset K_2'} \sup_{m \subset K_2'} \Psi(l, m) =^{\text{def}} \Psi_0 . \tag{18}$$

The maximum shortcoming $\sup_{\theta \epsilon K_2} \gamma_{\varphi_0, D}(\theta)$ *of test* φ_0, *with respect to the class* D *of unbiased size-*α *tests, is determined by* $M_{2,\alpha}(\Psi_0)$ *where*

$$M_{2,\alpha}(\Psi) = \sup_{OQ>O} [\Phi^\times(u_{\frac{1}{2}\alpha} - OQ) + \Phi^\times(u_{\frac{1}{2}\alpha} + OQ) - \Phi^\times(u_{\frac{1}{2}\alpha} - OQ \cos \Psi) - \Phi^\times(u_{\frac{1}{2}\alpha} + OQ \cos \Psi)]. \tag{19}$$

Problem $\{(H, K_2), \sigma^2 \text{ unknown}\}$ presents a particular difficulty which restricts the class of problems to which our criterion can find an exact application. We consider the theory for Problem $\{(H, K_2), \sigma^2 \text{ unknown}\}$ in order to indicate the origin of this difficulty.

First, applying Theorem 1 in [4], p. 161, we derive the uniformly M.P. unbiased size-α test φ

$$\|X^I\|/\|X^I - X^{II}\| \geqq t_{n-s+r-1;\frac{1}{2}\alpha}/(n - s + r - 1)^{\frac{1}{2}} \tag{20}$$

for Problem $\{(H, A_2), \sigma^2 \text{ unknown}\}$ where A_2 is the two-sided analogue of the alternative A_1 of Section 4, (see Section 6). Test (20) will be uniformly M.P., among the tests of the class D of unbiased size-α tests for Problem $\{(H, K_2), \sigma^2 \text{ unknown}\}$, for testing against A_2, if and only if Test (20) belongs to this class D. This holds true because D is a subclass of the class of unbiased size-α tests for Problem $\{(H, A_2), \sigma^2 \text{ unknown}\}$. Test (20) will be unbiased size-α for Problem $\{(H, K_2), \sigma^2 \text{ unknown}\}$ if, and only if,

$$\sup_{m \subset K_2'} \Psi(l, m) \leqq \Lambda_2 \tag{21}$$

where Λ_2 is a suitable angle. So the class C of S.M.P. unbiased size-α tests for Problem $\{(H, K_2), \sigma^2 \text{ unknown}\}$ is determined by (20) for l varying over K_2' and provided that

$$\sup_{l \subset K_2'} \sup_{m \subset K_2'} \Psi(l, m) \leqq \Lambda_2 \tag{22}$$

is satisfied. We obtain the following result.

THEOREM. *If* (22) *holds true, the most stringent* S.M.P. *unbiased size-*α *test* φ_0 *for Problem* $\{(H, K_2), \sigma^2$ *unknown*$\}$ *is determined by* (20), *taking for* l *the line* l_0 *satisfying* (18).

The determination of the angle Λ_2, for which (21) is necessary and sufficient in order that Test (20) belongs to the class D of unbiased size-α tests for Problem $\{(H, K_2), \sigma^2 \text{ unknown}\}$, is an intricate problem. We presume that the following formula

$$\Lambda_2 = \cot^{-1} \{(n - s + r - 1)^{-\frac{1}{2}} \max (1, t_{n-s+r-1;\frac{1}{2}\alpha})\} \tag{23}$$

holds true, (cf. [5]).

8. An application. The theory of Section 6 will be applied to the problem of testing *homogeneity of means* ($\mu_1 = \mu_2 = \cdots = \mu_k$) against an *upward trend* ($\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_k$, with at least one inequality strong), as mentioned at the end of Section 1. This problem is of the form $\{(H, K_1, \sigma^2 \text{ unknown}\}$ where X_v corresponds with X_{ij} when $v = \sum_{h=1}^{i} n_h + j$ $(v = 1, \cdots, n)$. The indices $v = 1, \cdots, n$ are subdivided into k blocks of $n_1, n_2, \cdots, n_k$ indices respectively. We have

$$s = k; \qquad r = k - 1; \qquad (x, y) = \sum_{v=1}^{n} x_v y_v = \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} y_{ij}$$

and

$$R^s = R^k \equiv \{(\mu_1 \cdots \mu_1 ; \mu_2 \cdots \mu_i \cdots \mu_k)\},$$

which notation indicates that the points of R^s have coordinates which are equal within each block. Similarly we have, always mentioning the jth coordinate of the ith block:

$$R^{s-r} = R^1 \equiv \{(\mu \cdots \mu; \mu \cdots \mu \cdots \mu)\};$$

$$R^r = R^{k-1} \equiv \{(u_1 \cdots \mu_1 ; \mu_2 \cdots \mu_i \cdots \mu_k)\}$$

where $\sum_{i=1}^{k} n_i \mu_i = 0$ has to be satisfied;

$$R^{n-s+r} = R^{n-1} \equiv \{(x_{11} \cdots x_{1n_1} ; x_{21} \cdots x_{ij} \cdots x_{kn_k})\}$$

where $\sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} = 0$ has to be satisfied.

The $r = (k - 1)$ edges e_g $(g = 1, \cdots, k - 1)$ of K_1' in $R^r = R^{k-1}$ are determined by

$$e_g \equiv \{(\mu_1 \cdots \mu_1 ; \mu_2 \cdots \mu_i \cdots \mu_k)\}$$

where $\mu_i = -\theta s_g^{-1}$ $(i \leqq g)$; $\mu_i = \theta(n - s_g)^{-1}$ $(i > g)$; $\theta > 0$, with the notation:

$$s_g = \sum_{i=1}^{g} n_i \qquad (g = 1, \cdots, k); \; s_0 = 0. \tag{24}$$

The "arbitrary" half-line

$$l \equiv \{(\theta w_1 \cdots \theta w_1 ; \theta w_2 \cdots \theta w_i \cdots \theta w_k)\}, \qquad \theta > 0 \tag{25}$$

is in K_1' provided that

$$\sum_{i=1}^{k} n_i w_i = 0 \qquad w_1 \leqq w_2 \leqq \cdots \leqq w_k . \tag{26}$$

The angles $\Psi(l, e_g)(g = 1, \cdots, k - 1)$ are determined by

$$\Psi(l, e_g) = \cos^{-1} \{(l, e_g)/\|l\| \, \|e_g\|\}$$
$$= \cos^{-1} \{-n^{\frac{1}{2}} \textstyle\sum_{i=1}^{g} n_i w_i / (\sum_{i=1}^{k} n_i w_i^2)^{\frac{1}{2}} s_g^{\frac{1}{2}} (n - s_g)^{\frac{1}{2}}\}.$$

Consequently l is the half-line l_0, making equal angles

$$\Psi_0 = \cos^{-1}\{(\textstyle\sum_{i=1}^{k} n_i w_i^2)^{-\frac{1}{2}}\} \tag{27}$$

with the edges e_g $(g = 1, \cdots, k-1)$, if the "weights" w_i $(i = 1, \cdots, k)$ satisfy the equations

$$-n^{\frac{1}{2}} \textstyle\sum_{i=1}^{g} n_i w_i = s_g^{\frac{1}{2}}(n - s_g)^{\frac{1}{2}} \qquad (g = 1, \cdots, k).$$

The solution

$$w_i = n^{-\frac{1}{2}} n_i^{-1}\{s_{i-1}^{\frac{1}{2}}(n - s_{i-1})^{\frac{1}{2}} - s_i^{\frac{1}{2}}(n - s_i)^{\frac{1}{2}}\} \qquad (i = 1, \cdots, k) \tag{28}$$

of these equations satisfies the Inequalities (26). So $l_0 \subset K_1'$ and l_0 satisfies (12).

The projection X^I of the sample point X on l_0 is determined by (25) where θ has to minimize $\sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \theta w_i)^2$. So

$$\theta = (\textstyle\sum_{i=1}^{k} n_i w_i^2)^{-1} \sum_{i=1}^{k} n_i w_i X_{i\cdot}\, ; \qquad OX^I = \theta(\sum_{i=1}^{k} n_i w_i^2)^{\frac{1}{2}} \tag{29}$$

where $X_{i\cdot}$ denotes the sample mean $n_i^{-1} \sum_{j=1}^{n_i} X_{ij}$.

The projection X^{II} of X on $R^{n-s+r} = R^{n-1}$ is determined by

$$X^{II} = (X_{11} - X_{\cdot\cdot}, \cdots, X_{ij} - X_{\cdot\cdot}, \cdots, X_{kn_k} - X_{\cdot\cdot})$$

where $X_{\cdot\cdot} = n^{-1} \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij}$; so we have

$$\|X^I - X^{II}\|^2 = \|X^{II}\|^2 - \|X^I\|^2 = \textstyle\sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - X_{\cdot\cdot})^2 - (OX^I)^2. \tag{30}$$

The Condition $\Psi_0 \leqq \Lambda_1$ (see (16) and (27)) can be written in the following form

$$t_{n-2;\alpha} \leqq (n-2)^{\frac{1}{2}}(\textstyle\sum_{i=1}^{k} n_i w_i^2 - 1)^{-\frac{1}{2}}. \tag{31}$$

Applying the theorem of Section 6, we obtain the following result.

COROLLARY. *The test*

$$\frac{(\sum_{i=1}^{k} n_i w_i^2)^{-\frac{1}{2}} \sum_{i=1}^{k} n_i w_i X_{i\cdot}}{\{\sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - X_{\cdot\cdot})^2 - (\sum_{i=1}^{k} n_i w_i^2)^{-1}(\sum_{i=1}^{k} n_i w_i X_{i\cdot})^2\}^{\frac{1}{2}}} \geqq \frac{t_{n-2;\alpha}}{(n-2)^{\frac{1}{2}}}$$

where the weights w_i $(i = 1, \cdots, k)$ are determined by (28) *and* (24), *is the most stringent* S.M.P. *similar size-α k-sample test against an upward trend if* (31) *holds true.*

Next we apply the second theorem of Section 7 to the two-sided k-sample problem of testing *homogeneity of means* against an *upward or a downward trend*, as mentioned at the end of Section 1. Obviously we can confine our attention to the Condition (22) merely, assuming (23). Some calculations show:

$$\begin{aligned}\sup_{l\subset K_2'} \sup_{m\subset K_2'} \Psi(l, m) &= \sup_{f,g=1,\cdots,k-1; f<g} \Psi(e_f, e_g) \\ &= \sup_{f,g=1,\cdots,k-1; f<g} [\cos^{-1}\{s_f^{\frac{1}{2}}(n - s_g)^{\frac{1}{2}}/(n - s_f)^{\frac{1}{2}} s_g^{\frac{1}{2}}\}] \\ &= \cos^{-1}\{n_1^{\frac{1}{2}} n_k^{\frac{1}{2}}/(n - n_1)^{\frac{1}{2}}(n - n_k)^{\frac{1}{2}}\}\end{aligned}$$

and consequently Condition (22) can be written in the following form (assuming (23))

$$\max(1, t_{n-2;\frac{1}{2}\alpha}) \leqq n^{-\frac{1}{2}}(n - n_1 - n_k)^{-\frac{1}{2}}(n - 2)^{\frac{1}{2}} n_1^{\frac{1}{2}} n_k^{\frac{1}{2}}, \tag{32}$$

so we obtain the following result.

COROLLARY. *The test*

$$\frac{(\sum_{i=1}^k n_i w_i^2)^{-\frac{1}{2}} |\sum_{i=1}^k n_i w_i X_{i\cdot}|}{\{\sum_{i=1}^k \sum_{j=1}^{n_i} (X_{ij} - X_{\cdot\cdot})^2 - (\sum_{i=1}^k n_i w_i^2)^{-1} (\sum_{i=1}^k n_i w_i X_{i\cdot})^2\}^{\frac{1}{2}}} \geqq \frac{t_{n-2;\frac{1}{2}\alpha}}{(n-2)^{\frac{1}{2}}}$$

where the weights w_i $(i = 1, \cdots, k)$ are determined by (28) *and* (24), *is the most stringent* S.M.P. *unbiased size-α k-sample test against an upward or a downward trend, if* (32) *holds true.*

We remark that the weights (28) constitute a generalization of the maximin-r^2 weights obtained by Abelson and Tukey in the special case $n_1 = n_2 = \cdots = n_k$ for the trend-problem where $\sigma^2 = 1$.

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