PAUL A. MEYER, *Probability and Potentials*. Blaisdell Publishing Company, Waltham, Massachusetts, 1966. xiii + 266 pp. $11.50.

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This effective and penetrating book has a somewhat deceptive title since the main import concerns the theory of martingales. This, indeed, occupies the center stage both textually and mathematically. The "Introduction to Probability" which precedes it is largely a quite advanced and elegant exposition of the prerequisites, while the section "Analytic Tools of Potential Theory" which follows is somewhat unmotivated and will probably seem rather dry to most readers.

In the middle portion of the book is to be found an excellent development of the theory of martingales along lines attributable to J. L. Doob and P. Meyer. The central result is the decomposition of a supermartingale into the sum of a martingale and a "natural" process with decreasing path functions. This is closely allied to potential theory, but has no classical analogue. The subtle results which follow its proof, however, depend equally upon the analysis of stopping times into their "accessible" and "inaccessible" components (roughly, a stopping time is accessible if it is the limit of a strictly increasing sequence of stopping times).

The last two sections of the "Applications" bring to bear the whole theory in Theorems 21 and 30. The latter result, concerning square integrable martingales, is part of a study which considers the structure of the class of all martingales defined relative to a fixed family of σ-fields. This theory is still incomplete and will probably have important consequences. Theorem 21, by contrast, which asserts the existence of a stopping time having values in a preassigned set of times depending on the path, is in itself a profound theorem. It is interesting and perhaps surprising that the proof of this theorem involves a heavy use of non-constructive methods (axiom of choice) although the applications for which it is required are often quite concrete.

The connection of martingales with the potential theory which follows them will be clear only to those familiar with the latter. The basis of their analogy is, of course, that in both cases an averaging principle is involved. Indeed, as is indicated in the appendix to Chapter IX, Section 2, there is *formally* no real difference between them if sufficiently abstract spaces are permitted. The real difference, then, is one of emphasis. The martingale theory emphasis is on the path functions and the underlying measure theory, while the potential theoretic emphasis is on the analytic properties of the kernels and the function space superstructure.

It is beyond the scope of a review to enumerate the material contained throughout this book. A few points which especially appeal to the reviewer are (a) the dual treatment of abstract and topological measure theory, with statements of where and why topological considerations are introduced, (b) the concise and original proofs of classical theorems as examples, seen in a new approach to analytic sets and in the proof of Daniell's theorem by means of Choquet capacities, (c) the remarks serving to interconnect the separate topics of martingales, discrete parameter Markov transition functions, and continuous parameter semigroups and resolvents.

To conclude, this is an erudite and definitive book which is of value primarily to specialists in the field. It contains no practical applications and much of it requires considerable advance knowledge of Bourbaki and other source material. As an access to a large amount of material and numerous recent results concerning the interrelations of abstract probability theory and potential theory, and as a modern treatment of martingales, it is truly excellent. The style is economical and clear, and the organization is well thought out. This is truly a research book, and represents a significant advance in the annals of probability theory and analysis.