BOOK REVIEWS

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D. R. Cox and H. D. Miller, *The Theory of Stochastic Processes*, Wiley, New York, 1965. x+398 pp, $11.50.

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In the preface the authors state that they have written the book for "statisticians and applied mathematicians interested in methods for solving particular problems, rather than for pure mathematicians interested in general theorems." With this aim they present a variety of scientific and technological applications, and through them approach the general theory. In this sense, this is an applied text on stochastic processes. Readers are assumed to have a good knowledge of elementary probability theory, advanced calculus and some matrix algebra.

In an introductory chapter basic notions of stochastic processes are illustrated with examples of random walks, Markov chains, Poisson process and queues. There is also a brief description of a stationary process, but all other examples are particular cases of a Markov process.

Chapter 2 deals with the random walk. The first two sections treat the simple case where the displacement in any single step is -1, 0 or $+1$. The unrestricted walk is analyzed by using the central limit theorem and the law of large numbers (however, the null case cannot be treated in this manner). The case where absorbing or reflecting barriers are present is then treated with the aid of generating functions. The remaining part of the chapter is devoted to the general random walk in discrete time. For the case where absorbing barriers are present, the extension of Wald's identity (due independently to H. D. Miller and J. H. B. Kemperman) is established, while for the walk with an impenetrable barrier at the origin, Lindley's integral equation approach is discussed. Both these topics are illustrated by several interesting examples.

In Chapter 3 two different treatments of Markov chains are given. The first one is the more standard treatment, based on Feller's theory of recurrent events. Here the main results are proved by the use of generating functions. The authors use the term 'recurrent' instead of 'persistent'; this latter term is favored by Feller, and is now in general use. There already exists a confusing list of terms describing the states of a Markov chain, to which the authors have added a new one: a state j is here defined to be *ephemeral* if $P_{ij} = 0$ for every i (p. 91). For

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finite Markov chains, a second treatment is given, based on the Jordan canonical form of a non-negative square matrix.

The next two chapters are concerned with Markov processes in continuous time. Chapter 4 treats the denumerable case. The discussion here is mainly concerned with special cases: Poisson process and its generalizations, and birth-and-death processes. The general formulation is given briefly. Chapter 5, entitled 'Markov processes with continuous state space,' deals almost entirely with the Wiener process: its diffusion equations, first passage times, behavior in the presence of absorbing or reflecting barriers, and transformations. For the general diffusion process, the Fokker-Planck equation is derived, starting from a stochastic differential equation (5.56), rather than from the more familiar postulates for the infinitesimal velocity and variance. Two examples of a mixed process (where changes of state occur continuously as well as by jumps) are then discussed. The compound Poisson process, which is an important one featuring in many applications, appears under the guise of a 'random walk in continuous time' (p. 238); its connection with the discrete case discussed earlier on page 155 is not indicated. (Actually, the term 'compound Poisson' does not appear anywhere in the book.)

Chapter 6 describes techniques to deal with the difficulties associated with non-Markovian processes. In particular the method of stages, use of supplementary variables, and imbedded Markov chain analysis are explained. It is interesting to note that all these techniques were first developed in queueing theory, where except in simple cases, the underlying processes are non-Markovian. It is regrettable, however, that renewal-theoretic methods do not find a place here (nor in Chapter 9): these represent a powerful tool which has been used extensively in the literature on queues and inventories.

Chapters 7 and 8 treat second order stationary processes. Covariance analysis is discussed in Chapter 7, and spectral analysis in Chapter 8. Moving averages, autoregressive schemes, and periodic processes in discrete time are investigated. There is an informal account of the Wold decomposition of a stationary process. Problems of prediction, filtering and regulation arising in stationary processes are formulated and the first two are treated in some detail.

The last chapter deals with point processes, namely, those whose realizations consist of a series of point events. This term seems to be rarely used in this country; what is actually discussed here is renewal theory and some of its ramifications. Laplace transform techniques are used to derive the asymptotic results for the renewal function. In particular the proof of the elementary renewal theorem given here has less intuitive appeal than the one given by Smith (1958) using the strong law of large numbers. Lifetime distributions are assumed to have densities, so the limit theorem $h(t) \to \mu^{-1}$ (where $h(t)$ is the renewal density and μ the mean lifetime) plays an important role in the derivation of other limiting distributions. Thus the authors have no use for Blackwell's renewal theorem, which is not mentioned even in passing; actually this theorem and its generalized versions are more frequently used in applications. The probabilistic proof of the integral equation (I.E.) of renewal theory would be more accurate only if $h(t)dt$ is interpreted as the expected number of renewals in the interval $(t, t + dt]$, rather than as the probability of a renewal at time t (p. 347). Moreover, in applications what is very often used is the fact that this I.E. has a unique bounded solution, a result which is not mentioned here.

A stationary renewal process is defined on page 340 as the one for which the initial lifetime distribution has the density $[1 - F(x)]\mu^{-1}$. The motivation for this definition appears 8 pages later; but here it would have been helpful if it was actually *shown* that the distribution of the forward recurrence time $V(t)$ is the same for all t in a stationary process, especially because this fact is used later on page 356. More careful arguments would be necessary in the derivation of the density function of $V(t)$ in the stationary case (*p.* 356); actually, the expression $xf(x)\mu^{-1}$ is the limiting density of an arbitrarily selected interval, and there is a curious paradox concerning this interval [see Feller (1966), pp. 10–13]. It would have been useful to indicate how the results established here specialize to the discrete case, and recall the results assumed or proved earlier in the chapter on Markov chains. Thus on page 98 it was proved that $n^{-1}R_n \rightarrow \mu_k$, where R_n is the number of times a persistent state k is occupied by a Markov chain in the time interval $[0, n]$, and μ_k is its mean recurrence time. This is actually the discrete time version of Doob's (1948) result which states that $t^{-1}N(t) \rightarrow \mu^{-1}$ with probability one, where $N(t)$ is the number of renewals in $(0, t]$—incidentally, this holds even when $\mu = \infty$ (the transient or persistent null case).

Consistent with the aim of the book, general theory has been de-emphasized. Thus, densities have been assumed to exist for the various distributions, presumably to avoid the use of the Stieltjes integral (which, however, makes a brief appearance on pages 243–246). This is a drawback in the chapters on random walk and renewal theory, where in many applications densities do not exist. Now, it is a part of the statisticians' business to deal with distributions not having densities, and it is not obvious that they would find the Stieltjes integral more difficult than matrix algebra which has been used fairly extensively in this book. Again, wherever general theorems are established it would have been more effective to display them under suitable headings, instead of packing them into single sections as on pages 91–98, and on pages 340–350. Nevertheless the chapter on Markov chains is probably the only one where a reasonably complete theory is described. A similar account of Markov processes in continuous time is not possible at this level, and the authors are content with the modest aim of establishing the two systems of Kolmogorov equations and indicating the difficulties associated with the existence and uniqueness of their solution. Readers are usually encouraged to believe that the pathological case does not arise in practice. Chapter 8 contains a good (although informal) account of the spectral representation of a stationary process, a topic which is rarely treated in introductory books on stochastic processes. The disappointing part of the book, however, is the last chapter, which is written in a heuristic style, less in evidence elsewhere in the book. The main results of renewal theory are sketched all too briefly, and are restricted to the case of non-negative random variables. More recent developments like fluctuation theory, with important applications to queues, are not discussed.

Now, applied texts usually seek to motivate the theory through practical applications of stochastic processes. While this is historically correct and pedagogically wise, there remains the question of overall balance between the theory and its applications to be presented in such a text. In the present book this balance is not as much in favor of theory as I would like to have seen. However, on the positive side it should be remarked that the book contains a wealth of interesting examples which have been carefully treated in fairly great detail. These are supplemented by exercises at the end of each chapter. The exposition is lucid and the readers are sure to get a very good working knowledge of the subject.

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