NOTE ON A THEOREM OF KINGMAN AND A THEOREM OF CHUNG

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Let $P = \{p_{ij}\}$ be the matrix of transition probabilities of an irreducible, aperiodic Markov chain. It is known (see [3]) that if the chain is transient, the interated probabilities $\{p_{ij}^{(n)}\}$ may tend geometrically to zero, in which case there is a common value $R > 1$ such that, for all i, j, $\{p_{ij}^{(n)}R^n\}$ tends to a finite limit as $n \to \infty$, but $\{p_{ij}^{(n)}r^n\}$ is divergent for $r > R$. Kingman [2] has called this the case of "geometric transience," and shown that, under the conditions below, if $\{u_i\}$ is an initial distribution, and C some set of states, the quantities

$$P_j(n) = \sum u_i p_{ij}^{(n)}$$

and

$${}_iQ_C(n) = \textstyle\sum_{j\varepsilon C} p_{ij}^{(n)}$$

satisfy $\lim_{n\to\infty} [P_j(n)]^{1/n} = \lim_{n\to\infty} [{}_iQ_C(n)]^{1/n} = 1/R$.

(Kingman discusses a continuous time process, but his results apply with the obvious changes in the present context.)

The conditions to be satisfied by $\{u_i\}$ and C are stated in terms of solutions to the inequalities

$$R\sum p_{ij}\beta_j \leqq \beta_i \qquad (\beta_i > 0), \tag{1}$$

$$R\sum \alpha_i p_{ij} \leqq \alpha_j \qquad (\alpha_j > 0). \tag{2}$$

(It can be shown that non-trivial solutions to these inequalities always exist. We shall call them right and left R-subinvariant vectors respectively.) Kingman's condition on the vector $\{u_i\}$ is that it should satisfy the condition $\sum u_i\beta_i < \infty$ for some right R-subinvariant vector $\{\beta_i\}$, and the condition on the set of states C is that it should satisfy the condition $\sum_{i\varepsilon C} \alpha_i < \infty$ for some left R-subinvariant vector $\{\alpha_i\}$.

The purpose of this note is to use the general theory developed in [4] to show that in fact these conditions imply a stronger result, namely the convergence of the quantities $P_j(n)R^n$ and ${}_iQ_C(n)R^n$ to finite limits. We shall also apply the results of [4] to two theorems of Chung's concerning the convergence of functionals of a Markov chain.

The discussion in [4] concerns the convergence of the more general sums

$$P_j(n; R) = \textstyle\sum_i u_i t_{ij}^{(n)} R^n;$$

$$Q_i(n; R) = \textstyle\sum_j t_{ij}^{(n)} v_j R^n;$$

$$S(n; R) = \sum\sum u_i t_{ij}^{(n)} v_j R^n;$$

Received 5 May 1966.

where T is any (not necessarily stochastic) irreducible, aperiodic, non-negative matrix with convergence parameter R, and $\{u_i\}$, $\{v_j\}$ are any (not necessarily non-negative) vectors.[1] As before, it can be shown that there always exist positive left and right R-subinvariant vectors, say $\{\alpha_i\}$, $\{\beta_i\}$, while suitable conditions on the vectors $\{u_i\}$ and $\{v_i\}$ take the form

(3) $\sum |u_i|\beta_i < \infty$ for some right R-subinvariant vector $\{\beta_i\}$,

(4) $\sum \alpha_i|v_i| < \infty$ for some left R-subinvariant vector $\{\alpha_i\}$.

Then it is proved that

(A) (3) is a sufficient condition for the convergence to a finite limit of the sequence $P_j(n; R)$ $(n \to \infty)$;

(B) (4) is a sufficient condition for the convergence to a finite limit of the sequence $Q_i(n; R)$;

(C) (3), (4), and the supplementary condition *either* $|u_i| \leqq K\alpha_i$ for some $K < \infty$, *or* $|v_i| \leqq K'\beta_i$ for some $K' < \infty$, are sufficient to ensure the convergence to a finite limit of the sequence $S(n; R)$.

When the appropriate conditions are satisfied, the limits can be computed by interchanging the limit and summation operations, i.e. they are zero whenever the matrix is R-transient or R-null, and equal respectively to

$$(\sum u_k\beta_k/\sum \alpha_k\beta_k)\alpha_j, \quad \sum \alpha_k v_k/\sum \alpha_k v_k)\beta_i, \quad (\sum \alpha_k v_k)(\sum \alpha_k\beta_k)/\sum \alpha_k\beta_k$$

when the matrix is R-positive, (when the vectors $\{\alpha_i\}$ and $\{\beta_i\}$ are uniquely defined (up to constant factors), strictly invariant, and satisfies the condition $\sum \alpha_k\beta_k < \infty$).[2]

Applying these results to Kingman's problem, we see that under his conditions, the sums $P_j(n)R^n$ and ${}_iQ_C(n)R^n$ tend to finite limits which are zero if P is R-transient or R-null, and equal to $(\sum_k \pi_k\beta_k/\sum_k \alpha_k\beta_k)\alpha_j$ and $(\sum_{k\epsilon C} \alpha_k/\sum \alpha_k\beta_k)\beta_i$ respectively if P is R-positive.

As a second application, suppose that the chain is positive recurrent, and let $\{z_n\}$ denote the sequence of random variables whose transition matrix is described by P. Then as $n \to \infty$, the distribution of z_n tends to a limit $\{\pi_i\}$ which is a left invariant vector for P. If $f(\cdot)$ is any function from the state space onto the real, we shall call the sequence $y_n = f(z_n)$ a *functional* of the Markov chain. Applying criterion (B) for the matrix P, with $R = 1$, $\alpha_k = \pi_k$ and $v_k = f(k)^r$ $(r > 0)$, and supposing that initially $z_0 = i$, we obtain the following theorem for the convergence of the moments of y_n.

THEOREM. *Let $\{z_n\}$, $f(\cdot)\{\pi_j\}$ be defined as above. Then the moments $E(y_n^r)$ exist and tend to a finite limit if the corresponding absolute moment of the limit distribution, $E|y_\infty|^r = \sum \pi_j|f(j)|^r$, is finite, in which case $E(y_n^r) \to E(y_\infty^r)$.*

It is not difficult to show that the conditions (A) and (B) are necessary as well

[1] The terminology is that used in [3] and [4], to which the reader is referred for further explanation and a proof of these results.

[2] See footnote number 1.

as sufficient for the convergence of the sums $P_j(n; R)$, $Q_i(n, R)$ if the matrix is R-positive and the vectors are non-negative; hence the condition of the theorem is necessary if the function $f(\cdot)$ is non-negative.

By making use of (C) it is possible to extend the results to the case when the initial distribution is not restricted to a single state. For example, the conclusions of the theorem will continue to hold if, in addition to the condition $E|y_\infty|^r < \infty$, the initial distribution is dominated by some multiple of the limit distribution.

These results are only a slight extension of those of Chung [1], Theorems I. 14.5 and I. 15.4; the main point of our discussion is that it shows that the analytical content of the theorems can be obtained very readily by direct arguments.

REFERENCES

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