ON NONRANDOMIZED FRACTIONAL WEIGHING DESIGNS

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1. Summary and introduction. It is known that, for the estimation of p individual weights, the optimum weighing design [3] for a chemical balance is given [4] by a Hadamard matrix X of dimensions $p \times p$, when it exists. If r rows of X are used for the weighing operations, the resultant design matrix X_1 of dimensions $r \times p$ will be a fraction of the full design matrix X, and will necessarily be singular. While it is not possible, with such a fractional weighing design, to furnish unique and unbiased estimates of the individual weights, it may be practicable, however, to afford a unique and unbiased estimate of a linear function of the weights.

In a recent paper, Zacks [6] has considered questions of admissibility of "randomization procedures" for such fractional weighing designs and has indicated a few basic results in this direction proceeding on the same lines as followed in [2].

We furnish, in this paper, some results of connected interest with respect to such fractional weighing designs without resorting to any randomization procedure in the selection of rows of the full design matrix X. Apart from this connection, the results are expected to have an importance of their own. We have spelled out here the structure of the estimate of the estimable linear function along with its variance, bringing out the connection of this variance with the variance as obtainable with the full design matrix. And, in the process, it has been indicated, in relation to the fraction used, to what extent we can afford to be arbitrary in the selection of the components of λ_p which enters into the estimable linear function $\lambda_p'\beta_p$, where λ_p and β_p are $p \times 1$ column vectors representing the coefficients and the weights respectively. We have shown that, depending on λ_p, we can obtain, with a fraction, the same precision for the estimate without having to perform all the weighing operations as required in a full design matrix. In such situations, repetitions of the fraction would lead to increased precision as compared to the adoption of the full design matrix.

2. The statistical model and the characterization of classes of estimable linear functions under fractional weighing designs. In general, results of N weighing operations fit into the linear model $Y_N = X\beta_p + e_N$, where Y_N is an $N \times 1$ random observed vector of the recorded results of weighings; $X = (x_{ij})$, $i = 1, 2, \cdots, N$, $j = 1, 2, \cdots, p$, is an $N \times p$ matrix of known quantities; $x_{ij} = +1, -1$ or 0, if, in the ith weighing operation, the jth object is placed respectively in the left pan, right pan, or in none; β_p is a $p \times 1$ vector ($p \leqq N$) representing the weights of the objects; e_N is an $N \times 1$ unobserved random vector such that $E(e_N) = 0$ and $E(e_N e_N') = \sigma^2 I_N$. X represents the weighing design matrix.

It has been stated before that a Hadamard matrix X of dimensions $p \times p$

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$(p = N)$, when it exists, would represent the best possible weighing design [4] for the estimation of individual weights in a chemical balance. Such a matrix will be referred to in this paper as the weighing design matrix of full rank (being denoted by WDFR). We would consider in this paper the status of its fractions as weighing designs, consisting of r rows $(1 \leqq r \leqq p)$, and shall denote them as FWD (fractional weighing designs) of rank r.

Partitioning the WDFR as

$$X = \left[\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right] = \left[\begin{array}{c} X_1 \\ \hline X_2 \end{array}\right], \tag{2.1}$$

where the dimensions of X_{11}, X_{12}, X_{21}, X_{22}, X_1, and X_2 are $r \times r$, $r \times (p - r)$, $(p - r) \times r$, $(p - r) \times (p - r)$, $r \times p$ and $(p - r) \times p$ respectively, and remembering that $XX' = pI_p$, we have

$$X_{11}X_{21}' + X_{12}X_{22}' = 0. \tag{2.2}$$

When the FWD is chosen as X_1 of dimensions $r \times p$ as indicated in (2.1), the normal equations for the least squares estimates would be given by

$$S\hat{\beta}_p = X_1'Y_r, \tag{2.3}$$

that is, by

$$\left[\begin{array}{c|c} S_{11} & S_{12} \\ \hline S_{21} & S_{22} \end{array}\right] \hat{\beta}_p = \left[\begin{array}{c|c} X_{11}'X_{11} & X_{11}'X_{12} \\ \hline X_{12}'X_{11} & X_{12}'X_{12} \end{array}\right] \hat{\beta}_p = \left[\begin{array}{c} X_{11}' \\ \hline X_{12}' \end{array}\right] Y_r,$$

where $X_{11}'X_{11} = S_{11}$, $X_{11}'X_{12} = S_{12}$, etc., and Y_r is the $r \times 1$ column vector of the recorded results of weighings obtained from r weighing operations. We shall assume that X_{11}, which is of dimensions $r \times r$, is of rank r, and that it is possible to have such an X_{11}. Rank of X_{11} being r would imply that the rank of X_{22} is $(p - r)$. In other words, if X_{11}^{-1} exists, X_{22}^{-1} would also exist, and vice versa.

A general solution of the normal equations (2.3) would be given by

$$\hat{\beta}_p = BX_1'Y_r + (I_p - H)Z_p, \tag{2.4}$$

where B is a g − inverse of S, and Z_p is any arbitrary vector. To get B and H, we would apply the method of "sweep-out" [5] to the matrix S and apply the same operations to a unit appended matrix, until S reduces to H and the unit matrix to B. Defining $S^- = P'\Delta_s^-P$, where $PSP' = \Delta_s$, we shall have, as proved in [1], $H = BS = S^-S$, and $H_{12} = -P_{21}'$, where H and P have the following forms:

$$H = \left[\begin{array}{c|c} I_r & H_{12} \\ \hline 0 & 0 \end{array}\right], \qquad P = \left[\begin{array}{c|c} I_r & 0 \\ \hline P_{21} & I_{p-r} \end{array}\right]. \tag{2.5}$$

From (2.2) and the above, we shall have H_{12} as given by

$$H_{12} = -X_{21}'(X_{22}')^{-1}. \tag{2.6}$$

It is well known that when, in general, a design matrix X is not of full rank, an estimable linear function will have a unique solution, if and only if there exists a solution for b_p in the equations $Sb_p = \lambda_p$. It is also known [5] that $\lambda_p'\hat{\beta}_p$ will be unique for all $\hat{\beta}_p$ satisfying $S\hat{\beta}_p = X_1'Y_r$, if $\lambda_p'H = \lambda_p'$. In fact, any of these two conditions would imply [1] the other. In this note, however, we shall use the latter condition for unique estimability, since from this condition, we can conveniently find λ_p as given by

$$\mu_p'H = \lambda_p', \tag{2.7}$$

where μ_p is any arbitrary vector. (This result was indicated by Dr. S. Searle of Cornell University. The proof is simple and straight forward, and follows readily from the fact that H is idempotent.)

3. Least squares estimators of estimable linear functions under FWD. If $\lambda_p'\beta_p$ is estimable, the unique estimate will be given by

$$\lambda_p'S^{-}X_1'Y_r = \lambda_r'X_{11}^{-1}Y_r, \tag{3.1}$$

where λ_r represents the first r components of λ_p.

The above result can be easily obtained from (2.5) by performing the required multiplication with S^{-}, and bringing in the condition that $\lambda_p'H = \lambda_p'$. The variance of the estimate will be given by

$$V(\lambda_p'\hat{\beta}_p) = \sigma^2\lambda_p'S^{-}\lambda_p = \sigma^2\lambda_r'S_{11}^{-1}\lambda_r. \tag{3.2}$$

Results (3.1) and (3.2) indicate that the last $(p - r)$ components of λ_p will not enter into the expression of the estimate or its variance, whatever may be the estimable linear function $\lambda_p'\beta_p$.

As all possible linear functions will not be estimable in a FWD, a question arises as to the extent of freedom of choice of the components of λ_p. This is answered by the following theorem:

THEOREM 1. *If the rank of the* FWD *matrix as given by* X_1 *is* r, *the first* r *components of* λ_p, λ_r, *can be chosen arbitrarily, the latter* $(p - r)$ *components being fixed by the relationship,* $-X_{22}^{-1}X_{21}\lambda_r = \lambda_{p-r}$.

PROOF. From (2.7), we have

$$\left[\begin{array}{c|c} I_r & 0 \\ \hline H_{12}' & 0 \end{array}\right]\mu_p = \lambda_p',$$

where μ_p can be chosen arbitrarily. Thus, we have $\mu_r = \lambda_r$, and $H_{12}'\mu_r = \lambda_{p-r}$, where μ_r denotes the first r components of μ_p. Hence, we get

$$-X_{22}^{-1}X_{21}\lambda_r = \lambda_{p-r}. \tag{3.3}$$

The above result indicates that we have a choice as to the selection of the first r components of λ_p. The latter $(p - r)$ components will be fixed by (3.3).

We prove below a theorem on estimability of an estimable linear function in the situation when the FWD is augmented by the inclusion of an additional row of X.

THEOREM 2. *If an* FWD *is augmented by the inclusion of an additional row of* X*, the linear function, which is estimable under the* FWD *with rank* r*, will still be estimable under the augmented* FWD *with rank* $r + 1$.

PROOF. In the FWD with rank r, the unique estimability of the linear function is characterized by the equation,

$$X_{22}\lambda_{p-r} + X_{21}\lambda_r = 0, \tag{3.4}$$

where the first r components of λ_p are arbitrarily chosen, and the latter $(p - r)$ components are determined by (3.4).

Let an additional row be added to the FWD raising the rank from r to $(r + 1)$, and let the matrices X_{22} and X_{21} correspondingly change to X_{22}^* and X_{21}^* respectively. Also, let the p components of λ_p be partitioned into sets of $(r + 1)$ and $(p - r - 1)$ components.

If equations (3.4) are written in full, the first equation dropped, and the value of the $(r + 1)$th component of λ_p is taken as given (which step is also otherwise consistent, because the rank of the augmented matrix is now raised to $r + 1$) along with the first r components of λ_p, equations (3.4) will reduce to

$$X_{22}^*\lambda_{p-r-1} + X_{21}^*\lambda_{r+1} = 0. \tag{3.5}$$

From (3.5) we see that the linear function would be estimable under the augmented design and that the values of λ_{p-r-1} will be determined by λ_{r+1}.

In the next theorem, we indicate a result on the variance of the estimable linear function.

THEOREM 3. *If* $\lambda_p'\beta_p$ *is estimable under a* FWD *of rank* r $(1 \leqq r \leqq p)$*, then the variance of its least squares estimator is equal to the variance of the least squares estimator of* $\lambda_p'\beta_p$ *under a* WDFR.

PROOF. The variance of the estimate is given by (3.2) and is equal to

$$\begin{aligned} \sigma^2\lambda_r'S_{11}^{-1}\lambda_r &= \sigma^2\lambda_r'(X_{11}'X_{11})^{-1}\lambda_r \\ &= \sigma^2\lambda_r'[p^{-1}I_r + p^{-1}X_{21}'(X_{22}X_{22}')^{-1}X_{21}]\lambda_r \\ &= \sigma^2p^{-1}[\lambda_r'\lambda_r + \lambda_r'X_{21}'(X_{22}X_{22}')^{-1}X_{21}\lambda_r]. \end{aligned} \tag{3.6}$$

Again, variance of this estimate as obtained from WDFR is

$$\begin{aligned} \sigma^2p^{-1}\lambda_p'\lambda_p &= \sigma^2p^{-1}[\lambda_r' \vdots \lambda_{p-r}']\begin{bmatrix}\lambda_r \\ \hline \lambda_{p-r}\end{bmatrix} \\ &= \sigma^2p^{-1}[\lambda_r' \vdots -\lambda_r'X_{21}'(X_{22}')^{-1}]\begin{bmatrix}\lambda_r \\ \hline -X_{22}^{-1}X_{21}\lambda_r\end{bmatrix} \\ &= \sigma^2p^{-1}[\lambda_r'\lambda_r + \lambda_r'X_{21}'(X_{22}X_{22}')^{-1}X_{21}\lambda_r]. \end{aligned} \tag{3.7}$$

(3.6) is the same as (3.7)

If any row of X is λ_p', the variance will be σ^2, independent of r. This would give an invariance for the variance in respect of such estimable linear functions. Thus, this theorem shows that we cannot alter the variance of the estimate of such estimable linear functions by altering r in the FWD. That is to say that the variance remains unchanged even if r is increased to $r + 1$, the estimable function still remaining estimable as per Theorem 2.

4. Selection of rows. When all the components of λ_p are arbitrarily preassigned, the condition for estimability (as distinguished from *unique* estimability) is given by $X'b_p = \lambda_p$, and the corresponding estimate is given by $b_p'Y_p$. From this, we get b_p as given by

$$b_p = p^{-1}X\lambda_p . \tag{4.1}$$

However, λ_p may be such that some of the components of b_p are zeros. Let us suppose, that the first r components of b_p are non-zeros, and the latter $(p - r)$ components are zeros. In such a situation, the last $(p - r)$ components of Y_p will not enter into the estimate, as the corresponding b-coefficients are zeros. Thus, the design matrix will necessarily be fractional. The last $(p - r)$ zero-components of b_p, as obtained from (4.1) give rise to the condition

$$[X_{21} \vdots X_{22}] \left[\frac{\lambda_r}{\lambda_{p-r}}\right] = 0$$

or,

$$-X_{22}^{-1}X_{21}\lambda_r = \lambda_{p-r} . \tag{4.2}$$

We note that (4.2), which gives the relationship among the components of λ_p, is the same as (3.3) and that, in such a situation, the resultant weighing design is fractional with rank r. Under these circumstances, therefore, we may adopt, as detailed in the foregoing sections, a FWD of rank r to derive the advantage of the *same* precision for the estimate as obtainable from WDFR without actually adopting the full matrix for the design.

If the fraction of rank r is repeated, we shall have the same precision repeated each time. Thus, when the fraction r is such that $mr = p$, where m is a positive integer, it might be desirable, for increased precision, to have the same fraction repeated m times in preference to the adoption of the full matrix for the design.

In this context, the interested reader may refer to Zacks' paper [6] which deals with randomized procedures for estimating any given linear function of the parameters β_p. In some special cases, the optimal procedures studied by Zacks are non-randomized and are the same as those studied in the present paper, although the two papers treat two different problems.

5. Concluding remarks. Results presented in this paper would indicate that non-randomized fractional weighing designs may be fruitfully used for the estimation of certain types of linear functions with the same efficiency as is available under the full design. The fraction to be adopted has however to be adjusted in accordance with the vector of coefficients, λ_p, of the linear function.

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