SEQUENTIAL ESTIMATION OF THE MEAN OF A LOG-NORMAL DISTRIBUTION HAVING A PRESCRIBED PROPORTIONAL CLOSENESS

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0. Introduction and summary. It is a common practice, in engineering and applied sciences, to ask for an estimator of a parameter of a statistical distribution which, with high probability, does not deviate from the value of the parameter by more than a certain percentage of its absolute value. In other words, if θ is a parameter under consideration, and $\hat{\theta}$ is an estimator of θ it is required that, for given $0 < \delta < 1$, and $0 < \gamma < 1$,

$$P_\theta[|\hat{\theta} - \theta| < \delta|\theta|] \geqq \gamma, \qquad \text{for all } \theta. \tag{0.1}$$

This probability is called the proportional closeness of $\hat{\theta}$ (see Ehrenfeld and Littauer [5] p. 339).

In the present paper we study the problem of estimating the mean of a log-normal distribution, by a procedure which guarantees a prescribed proportional closeness. When the variance, σ^2, of the corresponding normal distribution is known, there is an efficient *fixed sample* estimation procedure having the required closeness property. The sample size required in this case is,

$$n_0 = \text{smallest integer} \geqq \chi_\gamma^2[1]\sigma^2 \log^{-2}(1 + \delta), \tag{0.2}$$

where $\chi_\gamma^2[1]$ denotes the γth fractile of the χ^2-distribution, with 1 degree of freedom. As indicated in the sequel, there is no such fixed sample procedure, when σ^2 is unknown. The prescribed closeness property can be, however, guaranteed if the estimation is based on at least two stages of sampling. The properties of two sequential estimation procedures, which asymptotically (as $\delta \to 0$) guarantee the prescribed proportional closeness, are presented in the present paper. One procedure is based on the maximum likelihood estimator of the mean, and is called the sequential M.L. procedure. The other procedure is based on the sample mean, and is called the sequential S.M. procedure.

Let v_n denote the maximum likelihood estimator of σ^2. The stopping rule for the sequential M.L. procedure defines the sample size, K, to be the first integer $k \geqq 2$, for which the following inequality is satisfied:

$$v_k \leqq (1 + (2/c_k)k \log^2(1 + \delta))^{\frac{1}{2}} - 1, \qquad 0 < \delta < 1 \tag{0.3}$$

where $\{c_k\}$ is a sequence of bounded, positive constants, approaching $\chi_\gamma^2[1]$ as $k \to \infty$. The sample size, N, in the sequential S.M. procedure, is the first integer $n \geqq 2$ such,

$$v_n \leqq \log(1 + (\delta/c_n)n), \qquad 0 < \delta < 1 \tag{0.4}$$

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It is proven that both stopping rules, (0.3) and (0.4), yield well-defined stopping variables, which are decreasing functions of δ, and have finite expectations for every $0 < \delta < 1$. The asymptotic orders of magnitude (a.s.) of K and of N are given, as well as the asymptotic order of magnitudes of their expectations (as $\delta \to 0$). It is shown that the efficiency of the sequential S.M. procedure, relative to that of the sequential M.L. procedure, decreases to zero as $\sigma^2 \to \infty$. That is,

$$\lim_{\delta \to 0, \sigma^2 \to \infty} E\{K\}/E\{N\} = 0. \tag{0.5}$$

Moreover, $\lim_{\delta \to 0} E\{K\}/E\{N\} < 1$ for all $0 < \sigma^2 < \infty$. This result establishes the uniform asymptotic superiority (with respect to all $0 < \sigma^2 < \infty$) of the sequential M.L. procedure over the sequential S.M. procedure. The sequential M.L. procedure studied in the present paper is not, however, asymptotically efficient in the Chow-Robbins sense. Chow and Robbins defined in [3] a sequential procedure to be asymptotically efficient, if

$$\lim_{\delta \to 0} \frac{E\{\text{sample size in sequential procedure}\}}{\text{minimum sample size required for } \sigma^2 \text{ known}} = 1. \tag{0.6}$$

It is proven that, for the sequential M.L. procedure

$$\lim_{\delta \to 0} E\{K\}/n_0 = 1 + \tfrac{1}{2}\sigma^2, \qquad 0 < \sigma^2 < \infty. \tag{0.7}$$

This limit is always greater than 1, and approaches infinity as $\sigma^2 \to \infty$. A sequential procedure for the log-normal case, which satisfies asymptotically the prescribed closeness condition, and is asymptotically efficient in the Chow-Robbins sense is still unavailable. The reason for this shortcoming is that we have actually to determine a fixed-width confidence interval for $(\mu + \frac{1}{2}\sigma^2)$, where (μ, σ^2) are the mean and variance of the normally distributed $Y = \log X$. Chow and Robbins [3], Gleser, Robbins and Starr [6], and Starr [7] show that a sequential estimation of the mean, based on the sample mean and the sample variance, provides an asymptotically efficient *fixed width confidence* procedure. This result is in contrast to the main result of the present paper, which shows that a prescribed *proportional closeness* sequential estimator of the meanof a log-normal distribution based on the sample mean is inefficient, and that there exists a more efficient sequential procedure, which is the one based on the M.L. estimator.

1. Preliminaries. Let $X_1, X_2, \cdots$ be a sequence of independent random variables, identically distributed like $\exp\{N(\mu, \sigma^2)\}$; i.e., $\log X \sim N(\mu, \sigma^2)$; $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$. As is well known (see Aitchison and Brown [1]) the expected value of X is

$$\xi = \exp\{\mu + \tfrac{1}{2}\sigma^2\} \tag{1.1}$$

and its variance is

$$\operatorname{Var}\{X\} = \xi^2[\exp\{\sigma^2\} - 1]. \tag{1.2}$$

If $X_1, \cdots, X_n$ is a fixed size sample from the given sequence, then

$$\bar{Y}_n = \textstyle\sum_{i=1}^n Y_i/n \text{ and } Q_n = \sum_{i=1}^n (Y_i - \bar{Y}_n)^2,$$

where $Y_i = \log X_i$ $(i = 1, \cdots, n)$, are complete sufficient statistics (both μ and σ^2 are unknown). The maximum likelihood (M.L.) estimator of $\xi = E\{X\}$ is

$$\hat{\xi}_n = \exp\{\bar{Y}_n + (2n)^{-1}Q_n\}. \tag{1.3}$$

It is easy to show that the Fisher information matrix is

$$J(\xi, \sigma^2) = \frac{n}{\sigma^2}\begin{bmatrix} \xi^{-2} & -\frac{1}{2}\xi^{-1} \\ -\frac{1}{2}\xi^{-1} & \frac{1}{4}\left(1 + \frac{2}{\sigma^2}\right)\end{bmatrix}. \tag{1.4}$$

It follows that the M.L. estimator $\hat{\xi}_n$ is distributed, asymptotically as $n \to \infty$, like $N(\xi, (\sigma^2/n)\xi^2(1 + \frac{1}{2}\sigma^2))$.

The following lemma of Chow and Robbins [3] and a theorem of Anscombe [2], are required for the proofs in the sequel.

LEMMA 1.1 (Chow and Robbins). *Let W_n $(n = 1, 2, \cdots)$ be any sequence of random variables such that $W_n > 0$ a.s., $\lim_{n\to\infty} W_n = 1$ a.s. Let $f(n)$ $(n = 1, 2, \cdots)$ be any sequence of constants such that*

$$f(n) > 0, \qquad \lim_{n\to\infty} f(n) = \infty, \qquad \lim_{n\to\infty} f(n)/f(n-1) = 1, \tag{1.5}$$

and define, for each $0 < t < \infty$,

$$N(t) = \text{smallest } n \geq 1 \text{ such that } W_n \leq f(n)/t. \tag{1.6}$$

Then $N(t)$ is a well-defined and non-decreasing function of t,

$$\lim_{t\to\infty} N(t) = \infty \text{ a.s.}, \qquad \lim_{t\to\infty} EN(t) = \infty, \tag{1.7}$$

and

$$\lim_{t\to\infty} f(N(t))/t = 1, \quad \text{a.s.}. \tag{1.8}$$

In order to state the theorem of Anscombe, the following notions are introduced. Let $\{Z_n\}$ $(n = 1, 2, \cdots)$ be any sequence of random variables, satisfying:

$$\lim_{n\to\infty} P[Z_n - \theta \leq x\tau_n] = F(x) \tag{1.9}$$

at all continuity points x of a distribution function F; where τ_n $(n = 1, 2, \cdots)$ are positive constants, $-\infty < \theta < \infty$. Furthermore, let $\{Z_n\}$ $(n = 1, 2, \cdots)$ be uniformly continuous in probability (see Anscombe [2] for definition). Let $a(t)$ be a decreasing positive function of t. Let $\hat{\tau}_n$ be a consistent estimate of τ_n, and define

$$N(t) = \text{smallest integer } n, \text{ such that } \hat{\tau}_n \leq a(t) \tag{1.10}$$

and

$$n(t) = \text{smallest integer } n, \text{ such that } \tau_n \leq a(t). \tag{1.11}$$

Moreover, assume that $\{\tau_n : n = 1, 2, \cdots\}$ is a decreasing sequence $\downarrow 0$, and $\lim_{n\to\infty} \tau_n/\tau_{n+1} = 1$. Assume that $N(t)$ is well-defined, and

$$\lim_{t\to\infty} N(t)/n(t) = 1, \quad \text{in probability.} \tag{1.12}$$

As proven by Anscombe in [2],

THEOREM 1.2. *Under the Conditions* (1.9) *and* (1.12), *if* Z_n *are uniformly continuous in probability, then*

$$\lim_{t\to\infty} P[Z_{N(t)} - \theta \leqq xa(t)] = F(x) \tag{1.13}$$

at all continuity points x *of* F.

2. The fixed sample size solution for the case of known σ^2, and the problem when σ^2 is unknown. The M.L. estimator of the mean of a log-normal distribution, when σ^2 is known, is

$$\hat{\xi}_n(\sigma^2) = \exp\{\bar{Y}_n + \tfrac{1}{2}\sigma^2\}, \qquad 0 < \sigma^2 < \infty. \tag{2.1}$$

For every fixed sample size, n, the proportional closeness of $\hat{\xi}_n(\sigma^2)$ is, for all $0 < \xi < \infty$ and all $0 < \sigma^2 < \infty$,

$$P[|\hat{\xi}_n(\sigma^2) - \xi| < \delta\xi] \geqq P[|N(0, 1)| \leqq n^{\frac{1}{2}}\sigma^{-1} \log(1 + \delta)], \quad 0 < \delta < 1. \tag{2.2}$$

Thus, when σ^2 is known, if the sample size is greater than

$$n_0 = \text{smallest integer } n \geqq \chi_\gamma^2[1]\sigma^2 \log^{-2}(1 + \delta), \tag{2.3}$$

the estimator $\hat{\xi}_n(\sigma^2)$ has the prescribed closeness.

We give now a heuristic argument to indicate that there is no fixed sample solution to the prescribed proportional closeness problem, when σ^2 is unknown. Suppose h_n is a fixed sample estimator of ξ such that for every $n \geqq n'$,

$$P[|h_n - \xi| < \delta\xi] \geqq \gamma, \qquad \text{for all } 0 < \xi < \infty, 0 < \sigma^2 < \infty. \tag{2.4}$$

Hence,

$$P[\log(1 - \delta) < \log h_n - (\mu + \tfrac{1}{2}\sigma^2) < \log(1 + \delta)] \geqq \gamma, \quad \text{for all } -\infty < \mu < \infty \tag{2.5}$$

and all $0 < \sigma^2 < \infty$. Suppose now that after the sample was drawn, the value of σ^2 is told to us. Then, $\Psi_n = \log h_n + \frac{1}{2}\sigma^2$ furnishes a fixed-width confidence interval $(\Psi_n + \log(1 - \delta), \Psi_n + \log(1 + \delta))$ for μ. But this contradicts Dantzig's result [4], according to which there is no fixed sample procedure which yields a fixed width confidence interval for μ. Although the assumption that σ^2 is known after we draw the sample is unrealistic, assuming this fact only improves our chances of finding a fixed sample size procedure which estimates ξ with a prescribed proportional closeness.

3. The sequential M.L. estimation procedure. As mentioned in the introduction, the sequential M.L. procedure has the following stopping rule: Given $X_1, \cdots, X_k$ $(k \geqq 2)$, determine the M.L. estimate of σ^2, namely $v_k = Q_k/k$. Continue sampling until the number of observations reaches the value,

$$K = \text{smallest integer } k \geqq \max\{2, c_k v_k(1 + \tfrac{1}{2}v_k) \log^{-2}(1 + \delta)\}, \tag{3.1}$$

$\{c_k\}$ is a sequence of bounded positive constants, $c_k \to \chi_\gamma^2[1]$. After K observations have been taken, estimate ξ by $\hat{\xi}_K = \exp\{\bar{Y}_K + \frac{1}{2}v_K\}$.

LEMMA 3.1. *The sample size K is a well-defined, decreasing function of δ, having the properties:*

$$E\{K\} < \infty \quad \text{for all } 0 < \delta < 1, 0 < \xi < \infty, 0 < \sigma^2 < \infty. \tag{3.2}$$

$$\lim_{\delta\to 0} K = \infty \quad \text{a.s.}, \qquad \lim_{\delta\to 0} E\{K\} = \infty, \tag{3.3}$$

and

$$\lim_{\delta\to 0} K \log^2(1+\delta)/\chi_\gamma^2[1]\sigma^2(1+\sigma^2/2) = 1 \quad \text{a.s.} \tag{3.4}$$

PROOF. Let,

$$W_k = v_k(1+\tfrac{1}{2}v_k)/\sigma^2(1+\sigma^2/2), \qquad k = 2, 3, \cdots \tag{3.5}$$

$$f(k) = k/c_k, \qquad k = 2, 3, \cdots \tag{3.6}$$

and

$$t = \sigma^2(1+\sigma^2/2)/\log^2(1+\delta). \tag{3.7}$$

Then $t \to \infty$ as $\delta \to 0$; $W_k \to 1$ a.s. as $k \to \infty$; $f(k) \to \infty$ and $f(k)/f(k-1) \to 1$ as $k \to \infty$. Furthermore, according to (3.5)–(3.7) the sample size K can be defined by (1.6), with the restriction that $K \geqq 2$. Hence, from Lemma 1.1, K is well-defined, decreasing function of δ, satisfying (3.3) and (3.4). To prove (3.2), define

$$D_k = c_k v_k(1+\tfrac{1}{2}v_k) - k\log^2(1+\delta), \qquad k = 2, 3, \cdots. \tag{3.8}$$

Let $d = \log^2(1+\delta)$, and let

$$D_k^* = c_k e^{v_k} - dk, \qquad k = 2, 3, \cdots. \tag{3.9}$$

The expected value of K, for every $0 < \delta < 1$, is

$$E\{K\} = \textstyle\sum_{k=1}^{\infty} P[K > k] \leqq 1 + \sum_{k=2}^{\infty} P[K > k]. \tag{3.10}$$

For every $k \geqq 2$,

$$\begin{aligned} P[K > k] &= P[\min_{2\leqq j\leqq k} D_j > 0] \\ &\leqq P[\min_{2\leqq j\leqq k} D_j^* > 0] = P[\min_{2\leqq j\leqq k}\{Q_j - j\log(jd/c_j)\} > 0] \end{aligned} \tag{3.11}$$

where $Q_j \sim \sigma^2\sum_{\nu=1}^{j-1} U_\nu^2 (2 \leqq j \leqq k)$, and where $\{U_\nu : \nu = 1, 2, \cdots\}$ is a sequence of independent random variables, identically distributed like $N(0, 1)$. Moreover,

$$P[\min_{2\leqq j\leqq k}\{Q_j - j\log(j\,d/c_j)\} > 0] \leqq P[Q_k > k\log(kd/c_k)]. \tag{3.12}$$

Let k^* be such that, for every $k \geqq k^*$, $\log(d/c_k)k - \sigma^2 > 0$. Such k^* exists, since $\{c_k\}$ is bounded. Then, for every $k \geqq k^*$, we obtain from Tchebytchev's inequality,

$$\begin{aligned} P[Q_k &> k\log(kd/c_k)] \\ &\leqq P[|Q_k - (k-1)\sigma^2| > k(\log(kd/c_k) - \sigma^2) + \sigma^2] \\ &\leqq 2\sigma^4(k-1)/ \\ &\quad [k^2(\log(kd/c_k) - \sigma^2)^2 + 2k\sigma^2(\log(kd/c_k) - \sigma^2) + \sigma^4] \\ &= O(1/k\log^2 k), \quad \text{as} \quad k \to \infty. \end{aligned} \tag{3.13}$$

Finally, from (3.10), (3.11), (3.12) and (3.13), we obtain (3.2).

In the following theorem we prove that the sequential M.L. estimation procedure has asymptotically the prescribed proportional closeness.

THEOREM 3.2. *The* M.L. *estimator* $\hat{\xi}_K$, *where* K *is determined sequentially by* (3.1), *satisfies asymptotically the prescribed closeness requirement*, i.e.,

(3.14) $\lim_{\delta\to 0} P[|\hat{\xi}_K - \xi| < \delta\xi] \geqq \gamma,$ *for all* $0 < \xi < \infty, 0 < \sigma^2 < \infty.$

PROOF. The proof of the theorem is based on Anscombe's Theorem 1.3. Let $Z_k = \bar{Y}_k + \frac{1}{2}v_k$ and $\tau_k = k^{-\frac{1}{2}}\sigma(1 + \frac{1}{2}\sigma^2)^{\frac{1}{2}}$. Since Z_k is the M.L. estimator of $(\mu + \frac{1}{2}\sigma^2)$, (1.9) is satisfied in the special form:

(3.15) $\lim_{k\to\infty} P[\bar{Y}_k + \frac{1}{2}v_k - (\mu + \frac{1}{2}\sigma^2) \leqq xk^{-\frac{1}{2}}\sigma(1 + \frac{1}{2}\sigma^2)^{\frac{1}{2}}] = \Phi(x),$

where $\Phi(x)$ denotes the standard normal integral. Since, for a given k, Z_k is the M.L. estimator of $\mu + \frac{1}{2}\sigma^2$, the sequence $\{Z_k\}$ $(k = 2, 3, \cdots)$ consists of uniformly continuous in probability random variables (see Anscombe [2]). $\hat{\tau}_k$ is a consistent estimator of τ_k. Let $t = \log^{-2}(1 + \delta)$, and $a^2(t) = (t\chi_\gamma^2[1])^{-1}$. According to (3.1), K is the smallest integer $k \geqq 2$ for which $\hat{\tau}_k \leqq a(t)$. Moreover, $k(t)$ is the smallest integer $k \geqq \chi_\gamma^2[1]\sigma^2(1 + \sigma^2/2)/\log^2(1 + \delta)$. According to (3.4), $\lim_{t\to\infty} K/k(t) = 1$ a.s.. Thus, according to Theorem 1.2, since $\log(1 + \delta) = a(t)\chi_\gamma[1]$,

(3.16) $\lim_{\delta\to 0} P[\bar{Y}_K + \frac{1}{2}v_K - (\mu + \frac{1}{2}\sigma^2) \leqq \log(1 + \delta)] = \Phi(\chi_\gamma[1]) = \frac{1}{2}(1 + \gamma).$

Similarly,

(3.17) $\lim_{\delta\to 0} P[\bar{Y}_K + \frac{1}{2}v_K - (\mu + \frac{1}{2}\sigma^2) \geqq -\log(1 + \delta)] = \frac{1}{2}(1 - \gamma).$

Hence,

$$\begin{aligned} &\lim_{\delta\to 0} P[|\hat{\xi}_K - \xi| < \delta\xi] \\ (3.18) \qquad &\geqq \lim_{\delta\to 0} P[|Y_K + \tfrac{1}{2}v_K - (\mu + \tfrac{1}{2}\sigma^2)| \leqq \log(1 + \delta)] = \gamma \\ &\text{for all } 0 < \xi < \infty, 0 < \sigma^2 < \infty \end{aligned}$$

We prove now that,

THEOREM 3.3. *In the sequential* M.L. *procedure, the sample size,* K, *satisfies:*

(3.19) $\lim_{\delta\to 0} E\{K\} \log^2(1 + \delta)/\chi_\gamma^2[1]\sigma^2(1 + \frac{1}{2}\sigma^2) = 1.$

PROOF. Let

(3.20) $k^0 =$ smallest integer $\geqq \chi_\gamma^2[1]\sigma^2(1 + \frac{1}{2}\sigma^2)/\log^2(1 + \delta).$

Consider the function $\log^2(1 + \delta)E\{K\}$. For all $0 < \delta < 1$,

$$(3.21) \quad \log^2(1 + \delta)E\{K\} = \log^2(1 + \delta)\sum_{k=1}^{k^0-1} P[K > k] + \log^2(1 + \delta)\sum_{k=k^0}^{\infty} P[K > k].$$

As proven in Lemma 3.1, $\sum_{k=1}^{\infty} P[K > k] < \infty$. Hence,

(3.22) $\lim_{\delta\to 0} \log^2(1 + \delta)\sum_{k=k^0}^{\infty} P[K > k] = 0.$

According to Lemma 3.1, $\lim_{\delta\to 0} K/k^0 = 1$ a.s.. Hence, since K is a decreasing function of δ, $\lim_{\delta\to 0} P[K = k^0] = 1$. Therefore, for any $\epsilon > 0$, there exists $\delta^0(\epsilon)$ such that, for all $0 < \delta < \delta^0(\epsilon)$

$$P[K > k) \geqq 1 - \epsilon, \qquad \text{for all } 1 \leqq k < k^0. \tag{3.23}$$

Thus, for all $0 < \delta < \delta^0(\epsilon)$,

$$\begin{aligned}(1 - \epsilon)[\chi_\gamma^2[1]\sigma^2(1 + \tfrac{1}{2}\sigma^2) - \log^2(1 + \delta)] \\ \leqq \log^2(1 + \delta) \textstyle\sum_{k=1}^{k0-1} P[K > k] \\ \leqq \chi_\gamma^2[1]\sigma^2(1 + \tfrac{1}{2}\sigma^2) - \log^2(1 + \delta)\end{aligned} \tag{3.24}$$

It follows that,

$$\begin{aligned}\chi_\gamma^2[1]\sigma^2(1 + \tfrac{1}{2}\sigma^2)(1 - \epsilon) &\leqq \lim_{\delta\to 0}\inf \log^2(1 + \delta) \textstyle\sum_{k=1}^{k0-1} P[K > k] \\ &\leqq \lim_{\delta\to 0}\sup \log^2(1 + \delta) \textstyle\sum_{k=1}^{k0-1} P[K > k] \leqq \chi_\gamma^2[1]\sigma^2(1 + \sigma^2/2).\end{aligned} \tag{3.25}$$

Letting $\epsilon \to 0$ we arrive at,

$$\lim_{\delta\to 0} \log^2(1 + \delta) \textstyle\sum_{k=1}^{k0-1} P[K > k] = \chi_\gamma^2[1]\sigma^2(1 + \tfrac{1}{2}\sigma^2). \tag{3.26}$$

From (3.21), (3.22) and (3.26) we obtain (3.19).

The result established in the present theorem proves that the sequential M.L. procedure is inefficient in the Chow-Robbins sense, for large values of σ^2. Indeed, the comparison of (3.19) with (2.3) yields

$$\lim_{\delta\to 0} E\{K\}/n_0 = 1 + \tfrac{1}{2}\sigma^2, \qquad 0 < \sigma^2 < \infty. \tag{3.27}$$

There is not available, as yet, a more efficient procedure. As demonstrated in the next section, the sequential estimation procedure based on the sample mean is asymptotically inferior to that based on the M.L. estimator, for all $0 < \sigma^2 < \infty$.

4. The sequential S.M. procedure. Let $X_1, X_2 \cdots$, be a sequence of independent random variables, identically distributed like $\exp\{N(\mu, \sigma^2)\}$. Let $\bar{X}_n$ be the sample mean of the first n random variables; i.e., $\bar{X}_n = \sum_{i=1}^n X_i/n$. According to the central limit theorem,

$$\mathcal{L}(n^{\frac{1}{2}}(\bar{X}_n - \xi)) \to N(0, \xi^2(e^{\sigma^2} - 1)), \quad \text{as} \quad n \to \infty. \tag{4.1}$$

Consider the following sequential estimation procedure: Given $X_1, \cdots, X_n$, compute $v_n = \sum_{i=1}^n (Y_i - \bar{Y}_n)^2/n$, where $Y_i = \log X_i$ $(i = 1, \cdots, n)$. Take N observations (at least 2), where

$$N = \text{smallest integer } n \geqq \max\{2, c_n\delta^{-2}(e^{v_n} - 1)\}. \tag{4.2}$$

As before, $\{c_n\}$ is an appropriate sequence of bounded positive constants. Define,

$$W_n' = (e^{v_n} - 1)/(e^{\sigma^2} - 1) \tag{4.3}$$

$$f'(n) = n/c_n \tag{4.4}$$

and

$$t' = (e^{\sigma^2} - 1)/\delta^2 \tag{4.5}$$

Then, according to Lemma 1.1, N is a well-defined, decreasing function of δ, satisfying: $\lim_{\delta\to0} N = \infty$ a.s., and $\lim_{\delta\to0} E\{N\} = \infty$ a.s.. Moreover,

$$\lim_{\delta\to\infty} N\delta^2/\chi_\gamma^2[1](e^{\sigma^2} - 1) = 1 \quad \text{a.s..} \tag{4.6}$$

We show now that the sequential S.M. procedure has asymptotically the prescribed proportional closeness property. For this purpose, let $Z_n = \bar{X}_n/\xi$, $\tau_n^2 = n^{-1}(e^{\sigma^2} - 1)$, and $\hat{\tau}_n^2 = c_n/n\chi_\gamma^2[1](e^{v_n} - 1)$. Furthermore, let $a^2(t) = (\chi_\gamma^2[1]t)^{-1}$, where $t = \delta^{-2}$. Then, $N =$ the smallest integer $n \geqq 2$, such that $\hat{\tau}_n \leqq a(t)$, and $n(t) =$ smallest integer n: $\tau_n \leqq a(t)$. According to (4.6), $\lim_{\tau\to\infty} N/n(t) = 1$ a.s.. Finally, the sequence $\{Z_n\}$ is uniformly continuous in probability, since Z_n is the mean of independent identically distributed random variables. Thus, according to Theorem 1.2,

$$\begin{aligned} \lim_{\delta\to0} P[\bar{X}_N - \xi < \delta\xi] \\ = \lim_{t\to\infty} P[Z_n - 1 < u_{\frac{1}{2}(1+\gamma)}a(t)] = \Phi(u_{\frac{1}{2}(1+\gamma)}) = \tfrac{1}{2}(1+\gamma). \end{aligned} \tag{4.7}$$

Similarly,

$$\lim_{\delta\to0} P[\bar{X}_N - \xi < -\delta\xi] = \Phi(-u_{\frac{1}{2}(1+\gamma)}) = \tfrac{1}{2}(1-\gamma). \tag{4.8}$$

Hence,

$$\lim_{\delta\to0} P[|\bar{X}_N - \xi| < \delta\xi] = \gamma, \qquad \text{for all } 0 < \xi < \infty, 0 < \sigma^2 < \infty. \tag{4.9}$$

This proves the asymptotic consistency.

A slight modification of the proof of Lemma 3.1 will yield that, $E\{N\} < \infty$ for all $0 < \delta < 1$. Moreover, one can prove, like in Theorem 3.3, that,

$$\lim_{\delta\to0} E\{N\}\delta^2/\chi_\gamma^2[1](e^{\sigma^2} - 1) = 1, \qquad 0 < \sigma^2 < \infty. \tag{4.10}$$

The asymptotic efficiency of the sequential S.M. procedure, relative to that of the sequential M.L. procedure, is given by the limit

$$\begin{aligned} \lim_{\delta\to0} E\{K\} \log^2 (1+\delta)/E\{N\}\delta^2 &= \sigma^2(1+\tfrac{1}{2}\sigma^2)/(e^{\sigma^2} - 1) \\ &= \sigma^2(1+\tfrac{1}{2}\sigma^2)/[\sigma^2(1+\tfrac{1}{2}\sigma^2) + \tfrac{1}{6}\sigma^6 + \cdots] < 1, \quad \text{for all } 0 < \sigma^2 < \infty. \end{aligned} \tag{4.11}$$

Hence, as $\sigma^2 \to \infty$, the relative efficiency (4.11) tends to 0. The relative efficiency (4.11) tends to 1 if $\sigma^2 \to 0$. The sequential estimation procedure based on the sample mean is asymptotically inferior to that based on the M.L. estimator, for all $0 < \sigma^2 < \infty$.

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REFERENCES

[1] AITCHISON, J. and BROWN, J. A. C. (1957). *The Lognormal Distribution*. Cambridge Univ. Press.

[2] Anscombe, F. J. (1952). Large sample theory of sequential estimation. *Proc. Cambridge Philos. Soc.* **48** 600–607.

[3] Chow, Y. S. and H. Robbins (1965). On the asymptotic theory of fixed-width sequential confidence intervals for the mean. *Ann. Math. Statist.* **36** 457–462.

[4] Dantzig, G. B. (1940). On the non-existence of tests of "Students" hypothesis having power functions independent of σ_2. *Ann. Math. Statist.* **11** 186–192.

[5] Ehrenfeld, S. and S. B. Littauer (1964). *Introduction to Statistical Method.* McGraw-Hill, New York.

[6] Gleser, L. J., Robbins, H., and Starr, N. (1964). Some asymptotic properties of fixed-width sequential confidence intervals for the mean of a normal population with unknown variance. Report on National Science Foundation Grant NSF-GP-2074, Department of Mathematical Statistics, Columbia University.

[7] Starr, N. (1966). The performance of a sequential procedure for the fixed-width interval estimation of the mean. *Ann. Math. Statist.* **37** 36–51.