LIMIT THEOREMS FOR STOPPED RANDOM WALKS III[1]

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1. Introduction. This paper explores asymptotic properties of certain first passage problems in several dimensions. Throughout we consider a function h which is a homogeneous function of degree one in k variables defined throughout Euclidean k-space E_k. Under consideration are random walks constructed from a sequence of random k-dimensional column vectors $\{X_n, n \geqq 1\}$. Set $S_n = X_1 + \cdots + X_n$, $n \geqq 1$ and $S_0 = 0$. If $n \geqq 1$ let $H_n = h(S_n)$. Given $t > 0$ define a stopping variable $M(t)$ to be the least integer n such that $H_n \geqq t$, with $M(t) = \infty$ if for all $n \geqq 1$, $H_n < t$. Study of the asymptotic behavior of $M(t)$ as $t \to \infty$ in the multidimensional case was started in Farrell [8]. Related results have appeared in Bickel and Yahav [2]. A slightly different random variable, $M'(t)$, is used in Section 3. See (3.6).

As indicated above, although the random walk is k-dimensional the quantities of interest here are definable in terms of the one-dimensional point process $\{H_n, n \geqq 1\}$. It is the main purpose of this paper to show that results like an analogue of Blackwell's theorem in renewal theory still hold here. See [2], [3] and [6].

Some elementary results, given in Section 2, can be obtained whenever the random variable sequence $\{X_n, n \geqq 1\}$ obeys the strong law of large numbers (the limit need not be constant and this is noted in the statements of lemmas and theorems,) $\lim_{n\to\infty} S_n/n = \mu$. Usually in renewal theory one assumes $\mu = EX_1$. Unless explicitly stated in hypotheses this is not assumed here. To distinguish cases we write at the start of each theorem in parentheses the appropriate hypotheses.

In the cases where we assume $\{X_n, n \geqq 1\}$ are independently and identically distributed we will always assume the existence of finite first moments for the component random variables of X_1 and we will write $\mu = EX_1$. We will suppose throughout that the components of μ are positive (even in the case μ is allowed to be a random variable.) Problems for which $\mu \neq 0$ and this is not so can, by a rotation of coordinate axes, be brought to this form. We will call $\{x \mid \min_{1 \leqq i \leqq k} x_i > 0\} = Q$ the open first quadrant and will call the closure of Q the first quadrant (of E_k). We will always suppose that h is continuous as a function on Q but for special reasons discussed later we will suppose in Section 3 that if $x \not\in Q$ then $h(x) = 0$. This restriction is special to Section 3. Unless otherwise stated we will suppose that h is positive everywhere on Q. Some of our results use the assumption that h has continuous first partial derivatives throughout Q. We

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do not in general assume this. Throughout we will use α for the column vector of first partial derivatives $h'(x)$ evaluated at $x = \mu$. Using a superscript T for transpose we note $\mu^T\alpha = h(\mu)$ (Euler's identity). Further, in some applications of the law of large numbers we need the remark that the first partial derivatives of h are homogeneous functions of degree 0.

In Section 3 we prove a generalization of Blackwell's theorem for the sequence $\{H_n, n \geqq 1\}$ assuming that $\{X_n, n \geqq 1\}$ are independently and identically distributed and assuming continuous first partial derivatives for h. It will be observed that although the argument of Section 3 uses the existence of derivatives the statement of the value of the limit depends only on the value of $h(\mu)$. Using this observation along with an additional assumption of finite second moments we obtain generalizations in Section 4. Bickel and Yahav [2] consider the case $k = 2$ and homogeneous functions which are norms and for which $\{x \mid h(x) = 1\}$ is a (convex) polygon.

Section 5 represents a first attempt to find more terms in the asymptotic expansion of $EM(t)$ as $t \to \infty$. The results of Section 5 are basically about $H_t - t$, where $H_t = h(S_{M(t)})$.

For purposes of abbreviation we use "Ind" for "Independently" and "Id" for "Identically."

2. Elementary properties of $M(t)$.

THEOREM 2.1. (Strong law, $\mu \varepsilon Q$ a random variable). *Assume that with probability one, $h(\mu) > 0$. Then with probability one, $\lim_{t\to\infty} M(t)/t = 1/h(\mu)$.*

PROOF. For preciseness we suppose that all random variables considered are functions on a space Ω with points ω. By the law of large numbers we may find an integer valued random variable R, a real random variable sequence $\{\epsilon_n, n \geqq 1\}$, and a null set $\Omega_0 \subset \Omega$, satisfying if $\omega \not\varepsilon \Omega_0$ then $\lim_{n\to\infty} \epsilon_n(\omega) = 0$, and if $\omega \not\varepsilon \Omega_0$ and $n \geqq R(\omega)$ then

$$(1 - \epsilon_n(\omega))h(\mu(\omega)) \leqq n^{-1}h(S_n(\omega)) < (1 + \epsilon_n(\omega))h(\mu(\omega)).$$

Then $M(n(1 - \epsilon_n(\omega))h(\mu(\omega)))(\omega) \leqq n$ and $M(n(1 + \epsilon_n(\omega))h(\mu(\omega)))(\omega) > n$ provided $n \geqq R(\omega)$ and $\omega \not\varepsilon \Omega_0$. Set $s_n(\omega) = n(1 - \epsilon_n(\omega))h(\mu(\omega))$ and $t_n(\omega) = n(1 + \epsilon_n(\omega))h(\mu(\omega))$, $n \geqq 1$. Then it follows that if $\omega \not\varepsilon \Omega_0$ then

$$\limsup_{n\to\infty} M(s_n(\omega))(\omega)/s_n(\omega) \leqq 1/h(\mu(\omega));$$

$$\liminf_{n\to\infty} M(t_n(\omega))(\omega)/t_n(\omega) \geqq 1/h(\mu(\omega)).$$

For all $\omega \varepsilon \Omega$, $M(\cdot)(\omega)$ is a nondecreasing function, and since if $\omega \not\varepsilon \Omega_0$, $\lim_{n\to\infty} s_n(\omega)/t_n(\omega) = 1$, the theorem follows.

THEOREM 2.2 (Ind and Id). *Suppose there is a real number $\delta > 0$ such that if $x \varepsilon Q$ then $\delta h(x) \geqq \min_{1 \leqq i \leqq k} x_i$. Then if $t > 0$, $EM(t) < \infty$ and*

$$\lim_{t\to\infty} E\,|t^{-1}M(t) - 1/h(\mu)| = 0.$$

(A proof of this result is given in Farrell [8].)

In view of Theorem 2.2 it is natural to ask if the hypothesis of independence is needed in order that $EM(t) < \infty$ should follow. It is easy to construct stationary sequences $\{Y_n, n \geqq 1\}$ of random variables for which $EM(t) = \infty$ for some values of $t > 0$. We do not know, however, if an additional hypothesis of metric transitivity would be sufficient to guarantee $EM(t) < \infty$ for all t. We mention one example of potential interest. The random variables $\{Y_n, n \geqq 1\}$ take only the values 0 and 1. Let $\{\gamma_n, n \geqq 1\}$ be a real number sequence satisfying, $\gamma_0 = 1$, if $n \geqq 0$ then $1 \geqq \gamma_n \geqq 0$, $\gamma_{n+1} - \gamma_n \leqq 0$, and $\gamma_n - 2\gamma_{n+1} + \gamma_{n+2} \geqq 0$. Then probabilities for a stationary random variable sequence may be constructed in such a way that if $n \geqq 1$ then $P(Y_1 = 0, \cdots, Y_n = 0) = \gamma_n$. We omit the calculations which show that this is possible. In these examples

$$EM(\tfrac{1}{2}) = \textstyle\sum_{n=1}^{\infty} P(M(\tfrac{1}{2}) \geqq n)$$
$$= 1 + \textstyle\sum_{n=1}^{\infty} P(Y_1 = 0, \cdots, Y_n = 0) = \sum_{n=0}^{\infty} \gamma_n .$$

Consequently any example of a convex sequence satisfying the stated restrictions and for which $\sum_{n=0}^{\infty} \gamma_n = \infty$ gives an example for which $EM(\frac{1}{2}) = \infty$.

The following type of lemma is well known. See for example Hsu [10]. Theorem 2.3 follows Feller [9].

LEMMA 2.1. (Ind, Id, and continuous first partials). *Assume* $EX_1^TX_1 < \infty$ *and let* $\sigma^2 = \alpha^T(EX_1X_1^T)\alpha - \alpha^T\mu\mu^T\alpha$. *Then*

$$\lim_{n\to\infty} P(n^{-\frac{1}{2}}(h(S_n) - nh(\mu)) < t) = \textstyle\int_{-\infty}^{t} (1/\sigma(2\pi)^{\frac{1}{2}}) \exp(-x^2/2\sigma^2)\, dx.$$

THEOREM 2.3. (Ind, Id, and continuous first partials). *Assume* $X_1 \varepsilon Q$ *and that on* Q *the function* h *is a positive function with positive continuous first partial derivatives. Let* σ^2 *be as in Lemma* 2.1. *Then*

$$\lim_{s\to\infty} P(s^{-\frac{1}{2}}(M(s) - s/h(\mu)) \leqq \sigma t(h(\mu))^{-\frac{3}{2}}) = \textstyle\int_{-\infty}^{t} (2\pi)^{-\frac{1}{2}} \exp(-x^2/2)\, dx.$$

PROOF. $P(h(S_n) \geqq nh(\mu) + n^{\frac{1}{2}}\sigma t) = P(M(nh(\mu) + n^{\frac{1}{2}}\sigma t) \leqq n)$. Let $t_n = nh(\mu) + n^{\frac{1}{2}}\sigma t$ and solve for n. Elimination of n in favor of t_n together with use of Lemma 2.1 gives

$$\textstyle\int_t^{\infty} (2\pi)^{-\frac{1}{2}} \exp(-x^2/2)\, dx = \lim_{n\to\infty} P(t_n^{-\frac{1}{2}}(M(t_n) - t_n/h(\mu)) \leqq -\sigma t(h(\mu))^{-\frac{3}{2}}).$$

Using the facts that $\lim_{n\to\infty} t_{n+1}/t_n = 1$ and that $M(\cdot)$ is a nondecreasing function the argument is easily completed.

LEMMA 2.2. *Suppose the components of* X_n *are nonnegative*, $n \geqq 1$, *and on the set* Q *the functional* h *is a positive nondecreasing function of each of its variables. If* $b - a > \epsilon > 0$ *then*

$$\epsilon(M(b) - M(a + \epsilon)) \leqq \textstyle\int_a^b (M(t + \epsilon) - M(t))\, dt$$
$$\leqq \epsilon(M(b + \epsilon) - M(a)).$$

PROOF. $M(t + \epsilon) - M(t)$ is the number of times H_n falls in the interval $[t, t + \epsilon)$. As t goes from a to b, each point x in the interval $[a + \epsilon, b)$ lies in all intervals $[t, t + \epsilon)$ for a length of time $t + \epsilon > x$ to $t = x$, that is, a length of time

ϵ. Therefore the first inequality above follows. A similar argument establishes the second inequality.

THEOREM 2.4 (Strong law, μ a random variable). *Assume if $n \geqq 1$ then $X_n \varepsilon Q$ and let $\mu = \lim_{n\to\infty} S_n/n$. Suppose $h(\mu) > 0$ with probability one. Suppose h is a positive nondecreasing function of each of its variables on Q. Then with probability one,*

$$\lim_{t\to\infty} t^{-1} \int_0^t (M(s + \epsilon) - M(s))\, ds = \epsilon/h(\mu).$$

PROOF. Use Lemma 2.2 and Theorem 2.1.

3. Blackwell's theorem. In this section we use the norm $\|x\| = (x^T x)^{\frac{1}{2}}$ for vectors $x \varepsilon E_k$. We make the following specific assumptions:

(3.1) The function h is defined everywhere in E_k. On Q the function is positive and continuous. If $x \not\varepsilon Q$ then $h(x) = 0$.

(3.2) There exists a constant $K_1 > 0$ such that if $x \varepsilon Q$ and $y \varepsilon Q$ then $h(x + y) \leqq h(x) + K_1 \|y\|$.

(3.3) If $y \varepsilon Q$ then $\inf_{x \varepsilon Q} (h(x + y) - h(x)) = h^*(y) > 0$. There exists a constant $K_2 > 0$ such that if $x \varepsilon Q$ then $K_2 h^*(x) \geqq \min_{1 \leqq i \leqq k} x_i$.

(3.4) On Q the function h is continuously differentiable. We write $h'(x)$ for the column vector of partial derivatives of h evaluated at x. We suppose h' is a uniformly continuous function on $\{x \mid x \varepsilon Q$ and $h(x) = 1\}$.

(3.5) $\{X_n, n \geqq 1\}$ is a sequence of independently and identically distributed random vectors taking values in E_k. As noted in Section 1 we suppose $E \|X_1\| < \infty$, and that $\alpha = h'(\mu)$. We assume the random variable $\alpha^T X_1$ is not a lattice valued random variable.

In this section it is convenient to use a random variable M'.

(3.6) If $a > 0$ then $M'(a) - 1$ is the number of integers $n \geqq 1$ such that $h(S_n) < a$.

We begin by making some remarks about the assumptions. We wish to include the case where with positive probability $X_1 \not\varepsilon Q$. Without some assumption about second moments it follows from results of Erdös [7] that the expected number of sums S_n which fall outside Q may be infinite. Specifically for the purpose of eliminating from consideration in this section the problem of sums falling outside Q we have required in (3.1) that if $x \not\varepsilon Q$ then $h(x) = 0$. Thus if $a > 0$ and $S_n \not\varepsilon Q$ then $0 = h(S_n) < a$. Each such sum is counted by $M'(a) - 1$ which is the number of sums S_n not in Q plus the number of sums S_n in Q satisfying $0 < h(S_n) < a$. Since we suppose $\mu \varepsilon Q$ the law of large numbers requires that if $a > 0$ then $M'(a)$ is finite almost everywhere. It does not follow that $EM'(a)$ is finite. We will see below that if $a > 0$ and $b > 0$ then $E(M'(a + b) - M'(a)) < \infty$; that is, the expected number of sums falling in the set $\{x \mid x \varepsilon Q$ and $a \leqq h(x) < a + b\}$ is finite. Finally it should be noted that the random variable $M'(a)$ is not the same as $M(a)$ considered in Section 2 unless $X_1 \varepsilon Q$. Consequently later in this section we make the identification of $M'(a)$ and $M(a)$ after reducing the case $P(X_1 \not\varepsilon Q) > 0$ to the case $P(X_1 \varepsilon Q) = 1$.

Assumption (3.2) implies that the partial derivatives of h are in value $\leqq K_1$ everywhere in Q and thus that if $x \varepsilon Q$ then $\|h'(x)\| \leqq K_1 k^{\frac{1}{2}}$. On the other hand (3.3) requires the components of $h'(x)$ to be positive if $x \varepsilon Q$. Therefore on Q the function h is a strictly increasing function of each of its variables, and on Q the function h is uniformly continuous.

THEOREM 3 (Ind, Id, and continuous first partials). *Under assumptions* (3.1) *to* (3.5), *if* $a > 0$ *and* $b > 0$ *then*

$$E(M'(a + b) - M'(a)) < \infty$$

and

$$\lim_{a \to \infty} E(M'(a + b) - M'(a)) = b/h(\mu).$$

The remainder of this section contains a series of lemmas which lead to a proof of Theorem 3. Lemma 3.1 establishes a uniform integrability result needed for positive random variables. After proving Lemma 3.3 the remainder of the section is concerned with the case of nonnegative random variables. We will remind the reader of this by putting $X_1 \varepsilon Q$ in parentheses at the start of each lemma.

LEMMA 3.1. *Suppose* $\{Y_n, n \geqq 1\}$ *is a sequence of independently and identically distributed random variables, that* $Y_1 \varepsilon Q$, *and* $E\|Y_1\| < \infty$. *Let* $t \varepsilon E_k$ *and let* $R(t, a)$ *be the least integer* n *such that* $h(t + Y_1 + \cdots + Y_n) \geqq a$. *Let* $R^*(a)$ *be the least integer* n *such that* $h^*(Y_1 + \cdots + Y_n) \geqq a$. *Then*

$$(3.7) \quad \int_{\{R(t,a+b)-R(t,a) \geqq m\}} (R(t, a + b) - R(t, a))\, dP \leqq \int_{\{R^*(b) \geqq m\}} R^*(b)\, dP$$

PROOF. It follows from (3.3) and Theorem 2.2 that if $a > 0$ then $ER^*(a) < \infty$. Let $R^*(q, a)$ be the least integer n such that $h^*(Y_{q+1} + \cdots + Y_{q+n}) \geqq a$. Then if $t + Y_1 + \cdots + Y_n \varepsilon Q$ we find by use of (3.3) that

$$h^*(Y_{n+1} + \cdots + Y_{n+q}) \leqq h(Y_1 + \cdots + Y_{n+q} + t) - h(Y_1 + \cdots + Y_n + t).$$

Therefore if $q = R^*(n, b)$ and if $n = R(t, a)$ it follows that $a + b \leqq h(Y_1 + \cdots + Y_{n+q} + t)$ so that $n + q \geqq R(t, a + b)$. Therefore $R^*(R(t, a), b) \geqq R(t, a + b) - R(t, a)$. From this we obtain $ER^*(0, b) \geqq E(R(t, a + b) - R(t, a))$.

For brevity let $R(t, a, b) = R(t, a + b) - R(t, a)$. Let $a > 0$ and $b > 0$. Then

$$\{R(t, a) = n, R(t, a, b) \geqq m\} \subset \{R(t, a) = n, R^*(n, b) \geqq m\}.$$

Further, the event $\{R(t, a) = n\}$ is independent of the random variable $R^*(n, b)$. Therefore

$$(3.8) \quad \int_{\{R(t,a)=n, R(t,a,b) \geqq m\}} R(t, a, b)\, dP \leqq \int_{\{R(t,a)=n, R^*(n,b) \geqq m\}} R^*(n, b)\, dP$$
$$= P(R(t, a) = n) \int_{\{R^*(0,b) \geqq m\}} R^*(0, b)\, dP.$$

Sum both sides of inequality (3.8) to obtain (3.7).

LEMMA 3.2. *Suppose Theorem* 3 *holds if the additional hypothesis* $X_1 \varepsilon Q$ *is made. Then if* $t \varepsilon E_k$ *and* $b > 0$,

$$\lim_{a \to \infty} E(R(t, a + b) - R(t, a)) = b/h(\mu).$$

(R *as in Lemma* 3.1 *with the identification* $X_n = Y_n$, $n \geqq 1$.)

PROOF. Let $S_n \varepsilon Q$ and $S_n + t \varepsilon Q$. Then the line segment joining these two points is also in Q. If we set $f(a) = h(at + S_n)$, $0 \leqq a \leqq 1$, then by the mean value theorem $h(t + S_n) - h(S_n) = f(1) - f(0) = h'(a_0 t + S_n)^T t$ where $0 < a_0 < 1$. Since $h'(a_0 t + S_n) = h'((a_0 t + S_n)/n)$, by the law of large numbers, with probability one, $c = h'(\mu)^T t = \lim_{n\to\infty} h'((a_0 t + S_n)/n)^T t = \lim_{n\to\infty} (h(t + S_n) - h(S_n))$. If the random variables are functions on Ω then there exist a null set Ω_0 and a function $\delta(\cdot, \epsilon)$ such that if $\omega \not\varepsilon \Omega_0$ and if $a \geqq \delta(\omega, \epsilon)$ then

$$R(0, a + c - \epsilon)(\omega) \leqq R(t, a)(\omega) \leqq R(0, a + c + \epsilon)(\omega).$$

Using the uniform integrability established in Lemma 3.1 and the hypothesis Lemma 3.2 we have

$$(b - 2\epsilon)/h(\mu) \leqq \liminf_{a\to\infty} \int (R(t, a + b) - R(t, a))\, dP$$
$$\leqq \limsup_{a\to\infty} \int (R(t, a + b) - R(t, a))\, dP \leqq (b + 2\epsilon)/h(\mu).$$

Since $\epsilon > 0$ is now arbitrary, the conclusion of the lemma follows.

We now prove that the case of signed random variables follows as a consequence.

LEMMA 3.3. *If* $a > 0$ *and* $b > 0$ *then* $E(M'(a + b) - M'(a)) < \infty$. *If Theorem* 3 *holds whenever* (3.1) *to* (3.5) *hold and* $X_1 \varepsilon Q$ *then Theorem* 3 *follows from* (3.1) *to* (3.5).

PROOF. We use an argument very much like that of Blackwell [4]. The argument is a random variable argument. We now define the necessary notations.

Continue the usage $S_0 = 0$ and define $N_0 = 0$. If $k \geqq 0$ and $n \geqq 0$ define $Z_{n,k} = S_{N_0+\cdots+N_k+n} - S_{N_0+\cdots+N_k}$, and define N_{k+1} to be the least integer n such that $Z_{n,k} \varepsilon Q$, $N_{k+1} = \infty$ if for all $n \geqq 0$, $Z_{n,k} \not\varepsilon Q$. Since $\mu \varepsilon Q$, by the law of large numbers, with probability one if $k \geqq 0$ then $N_k < \infty$. $\{N_k, k \geqq 1\}$ is a sequence of independently and identically distributed random variables and $EN_1 < \infty$. See for example Farrell [8].

Let $\{X_n^*, n \geqq 1\}$ be a sequence of independently and identically distributed k-dimensional random vectors such that X_1 and X_1^* have the same distribution. Let $Z_0^* = 0$ and if $n \geqq 1$, $Z_n^* = X_1^* + \cdots + X_n^*$. Define N^* to be the least integer $n \geqq 1$ such that $Z_n^* \varepsilon Q$, $N^* = \infty$ if for all $n \geqq 0$, $Z_n^* \not\varepsilon Q$. If A is a k-dimensional Borel set then we define for $n \geqq 0$, $F_n(A) = P(Z_n^* \varepsilon A, n < N^*)$. Then $\sum_{n=0}^{\infty} F_n(\cdot)$ is a finite measure of total mass EN^*.

If $k \geqq 1$, let $Y_k = S_{N_0+\cdots+N_k} - S_{N_0+\cdots+N_{k-1}}$. Then if $a > 0$ and $b > 0$,

$$E(M'(a + b) - M'(a))$$

$$(3.9) \quad = \sum_{n=1}^{\infty} P(a \leqq h(S_n) < a + b)$$
$$= \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} P(a \leqq h(Y_1 + \cdots + Y_k + Z_{n,k}) < a + b,\ n < N_{k+1})$$
$$= \sum_{k=1}^{\infty} \int P(a \leqq h(Y_1 + \cdots + Y_k + t) < a + b)\left(\sum_{n=0}^{\infty} F_n(dt)\right).$$

Let $R(t, a)$ and $R^*(a)$ be as in Lemma 3.1. Then $E(R(t, a + b) - R(t, a)) =$

$\sum_{k=1}^{\infty} P(a \leqq h(Y_1 + \cdots + Y_k + t) < a + b) \leqq ER^*(b)$. It follows that $E(M'(a + b) - M'(a)) < \infty$.

Using Lemma 3.2, the boundedness just established, and the bounded convergence theorem, we obtain from (3.9) that

$$(3.10) \qquad \lim_{a\to\infty} E(M'(a + b) - M'(a)) = bE(N^*)/E(N^*)h(\mu).$$

This is the assertion of Lemma 3.3. We have used in the last step an identity due to Wald [11] in the same way that it was used by Blackwell [4].

Throughout the remainder of this section we suppose $X_1 \varepsilon Q$, and hence make the identification $M'(a) = M(a)$, $a > 0$.

LEMMA 3.4. *There exists a nonincreasing function* $f_1 : [0, \infty) \to [0, \infty)$ *such that* $\lim_{a\to\infty} f_1(a) = 0$ *and with probability one*

$$\lim_{n\to\infty} (f_1(n))^{-1} \|(S_n/h(S_n)) - (\mu/h(\mu))\| = 0.$$

PROOF. By the law of large numbers, with probability one, $\lim_{n\to\infty} \|(S_n/h(S_n)) - (\mu/h(\mu))\| = 0$. Choose a real number sequence $\epsilon_n \downarrow 0$ such that

$$P(\sup_{n\geqq m} \|(S_n/h(S_n)) - (\mu/h(\mu))\| \geqq \epsilon_m) \leqq 1/m.$$

Then let f_1 be a nonincreasing function such that if $n \geqq 1$ then $f_1(n) = (\epsilon_n)^{\frac{1}{2}}$. Then f_1 satisfies the conditions of the lemma.

By hypothesis (3.4) the function h' is uniformly continuous on $\{x \mid x \varepsilon Q$ and $h(x) = 1\}$. Therefore there exists a bounded nondecreasing function $f_2 : [0, \infty) \to [0, \infty)$ such that if $x \varepsilon Q$ and $y \varepsilon Q$ then

$$|h'(x) - h'(y)| \leqq f_2(\|(x/h(x)) - (y/h(y))\|).$$

We use here the facts that h' is homogeneous of degree zero and that the components of h' are bounded by K_1. By the assumed uniform continuity we may suppose $\lim_{a\to 0+} f_2(a) = 0$.

Using the functions f_1 and f_2 we define a function δ: $[0, \infty) \to [0, \infty)$ by

$$(3.11) \qquad \delta(a) = \min (a^{1/6}, (f_2(a^{-\frac{1}{2}}))^{-\frac{1}{2}}, (f_2(f_1(a^{\frac{1}{2}})))^{-\frac{1}{2}}).$$

We define an integer valued random variable by

$$(3.12) \qquad N(a) = M(a - \delta(a)).$$

LEMMA 3.5. *The following relations hold:*

δ is a nondecreasing function such that

$$(3.13) \qquad \begin{aligned} &\lim_{a\to\infty} \delta(a) = \infty; \\ &\lim_{a\to\infty} \delta(a)/a = 0; \qquad \lim_{a\to\infty} \delta(a) f_2(f_1(\epsilon a)) = 0 \text{ if } \epsilon > 0; \\ &\lim_{a\to\infty} \delta(a) f_2(\epsilon\delta(a)/a) = 0 \quad \text{if} \quad \epsilon > 0. \end{aligned}$$

PROOF. Each of the three functions entering into the definition of δ are nondecreasing functions with limit ∞ as $a \to \infty$. Therefore the first part of the lemma follows. Since $0 \leqq \delta(a)/a \leqq a^{-5/6}$, $\lim_{a\to\infty} \delta(a)/a = 0$ follows. Similarly $\lim_{a\to\infty} \epsilon\delta(a)/a^{\frac{1}{2}} = 0$. If $\epsilon\delta(a) \leqq a^{\frac{1}{2}}$ then $\delta(a)f_2(\epsilon\delta(a)/a) \leqq \delta(a)f_2(a^{-\frac{1}{2}}) \leqq (f_2(a^{-\frac{1}{2}}))^{\frac{1}{2}}$ so that $\lim_{a\to\infty} \delta(a)f_2(\epsilon\delta(a)/a) = 0$. Last, if $\epsilon a \geqq a^{\frac{1}{2}}$ then $f_1(\epsilon a) \leqq f_1(a^{\frac{1}{2}})$ and $\delta(a)f_2(f_1(\epsilon a)) \leqq \delta(a)f_2(f_1(a^{\frac{1}{2}})) \leqq (f_2(f_1(a^{\frac{1}{2}})))^{\frac{1}{2}}$ which implies $\lim_{a\to\infty} \delta(a)f_2(f_1(\epsilon a)) = 0$.

LEMMA 3.6. ($X_1 \varepsilon Q$). *Assume that $X_1 \varepsilon Q$ and that* (3.1) *to* (3.5) *hold. For any real number a let $[a]$ be the greatest integer $\leqq a$. There exists a constant $K_3 > 0$ such that*

$$(3.14) \qquad P(h(S_{M(a)}) - a \geqq \gamma) \leqq K_3 \int_{[\gamma]-2}^{\infty} P(K_1\|X_1\| \geqq s)\, ds < \infty.$$

PROOF.

$$\begin{aligned} P(h(S_{M(a)}) \geqq a + \gamma) &= \textstyle\sum_{n=0}^{\infty} P(h(S_{n+1}) \geqq a + \gamma, h(S_n) < a) \\ &\leqq \textstyle\sum_{n=0}^{\infty} P(K_1\|X_1\| \geqq a + \gamma - h(S_n), h(S_n) < a) \\ &= \textstyle\int_0^{a-} P(K_1\|X_1\| \geqq a + \gamma - t)\, d\sum_{n=0}^{\infty} P(h(S_n) \leqq t). \end{aligned}$$

Note that $EM(t) = \sum_{n=0}^{\infty} P(h(S_n) < t)$. From Lemma 3.1 we find that $E(M(t + 1) - M(t)) \leqq ER^*(1) = K_3$. Since $P(K_1\|X_1\| \geqq a + \gamma - t)$ is a nondecreasing function of t, we find

$$\begin{aligned} P(h(S_{M(a)}) \geqq a + \gamma) &\leqq K_3 \textstyle\sum_{n=1}^{[a]+1} P(K_1\|X_1\| \geqq a + \gamma - n) \\ &\leqq K_3 \textstyle\sum_{n=1}^{[a]+1} P(K_1\|X_1\| \geqq [a] + [\gamma] - n) \\ &= K_3 \textstyle\sum_{n=[\gamma]-1}^{[a]+[\gamma]-1} P(K_1\|X_1\| \geqq n) \\ &\leqq K_3 \textstyle\int_{[\gamma]-2}^{\infty} P(K_1\|X_1\| \geqq s)\, ds. \end{aligned}$$

This completes the proof of Lemma 3.6.

LEMMA 3.7. ($X_1 \varepsilon Q$). *Let $\epsilon > 0$ and let $\{a_i, i \geqq 1\}$ be a nondecreasing positive real number sequence such that $\lim_{i\to\infty} a_i = \infty$. Then with probability one,*

$$0 = \lim_{i\to\infty} P(\max_{N(a_i)+1 \leqq j \leqq M(a_i+b)} |h(S_j) - h'(S_{N(a_i)})^T S_j| > \epsilon \mid S_{N(a_i)}, N(a_i)).$$

PROOF. We will use the random variables $R^*(b + \delta(a))$ and $R^*(q, b + \delta(a))$ defined in Lemma 3.1 and its proof. Let

$$T_{a,b} = X_1 + \cdots + X_{R^*(b+\delta(a))} \quad \text{and} \quad T_{a,b}^* = X_{N(a)+1} + \cdots + X_{N(a)+R^*(N(a),b+\delta(a))}.$$

Since $M(a + b) \leqq N(a) + R^*(N(a), b + \delta(a))$ it follows that $\|X_{N(a)+1} + \cdots + X_{M(a+b)}\| \leqq \|T_{a,b}^*\|$. Further,

$$\begin{aligned} E\|T_{a,b}^*\| = E\|T_{a,b}\| &\leqq E(\|X_1\| + \cdots + \|X_{R^*(b+\delta(a))}\|) \\ &= (E\|X_1\|)(ER^*(b + \delta(a))). \end{aligned}$$

We have used Wald's identity. See [11]. By virtue of Theorem 2.2 there exists a constant $K_4 > 0$ such that $E\|T_{a,b}^*\| \leqq K_4\delta(a)$ if $a \geqq 1$. Therefore $P(\|T_{a,b}^*\| \geqq \gamma) \leqq K_4\delta(a)/\gamma$.

We now use the fact that $X_1 \varepsilon Q$ to compute certain inequalities. Since $h(S_{N(a)})$

$= h'(S_{N(a)})^T S_{N(a)}$, since for some θ_j with $0 < \theta_j < 1$, $h(S_j) - h(S_{N(a)}) = h'(\theta_j(S_j - S_{N(a)}) + S_{N(a)})^T(S_j - S_{N(a)})$, and since $\|S_j - S_{N(a)}\| \leqq \|S_{M(a+b)} - S_{N(a)}\|$,

$$\begin{aligned}
|h(S_j) &- h'(S_{N(a)})^T S_j| \\
&= |h(S_j) - h(S_{N(a)}) + h'(S_{N(a)})^T(S_{N(a)} - S_j)| \\
&\leqq \|h'(\theta_j(S_j - S_{N(a)}) + S_{N(a)}) - h'(S_{N(a)})\| \, \|S_j - S_{N(a)}\| \\
&\leqq f_2\left(\left\|\frac{\theta_j(S_j - S_{N(a)}) + S_{N(a)}}{h(\theta_j(S_j - S_{N(a)}) + S_{N(a)})} - \frac{S_{N(a)}}{h(S_{N(a)})}\right\|\right) \|S_{M(a+b)} - S_{N(a)}\| \\
&\leqq f_2\left(\left\|\frac{S_j - S_{N(a)}}{h(S_{N(a)})}\right\| + \left\|\frac{S_{N(a)}(h(S_{N(a)}) - h(\theta_j(S_j - S_{N(a)}) + S_{N(a)}))}{(h(S_{N(a)}))^2}\right\|\right) \\
&\qquad\qquad \cdot \|S_{M(a+b)} - S_{N(a)}\| \\
&\leqq f_2\left(\left\|\frac{S_{M(a+b)} - S_{N(a)}}{h(S_{N(a)})}\right\| \left(1 + K_1 \frac{\|S_{N(a)}\|}{h(S_{N(a)})}\right)\right) \|S_{M(a+b)} - S_{N(a)}\| \\
&\leqq f_2\left(\left(\frac{\|T^*_{a,b}\|}{h(S_{N(a)})}\right)\left(1 + K_1 \frac{\|S_{N(a)}\|}{h(S_{N(a)})}\right)\right) \|T^*_{a,b}\|.
\end{aligned}$$

Let $\gamma > 0$ be given and let A_γ be a real number, B_γ be the event

$$\inf_{j \geqq A_\gamma} h(S_{N(a_j)}/a_j) \geqq \tfrac{1}{2} \quad \text{and} \quad \sup_{j \geqq A_\gamma} \|S_{N(a_j)}/a_j\| \leqq 2\|\mu\|/h(\mu).$$

Suppose A_γ satisfies $P(B_\gamma) \geqq 1 - \gamma$. Such a choice is possible by virtue of the law of large numbers and Theorem 2.1. Then we calculate

$$\begin{aligned}
\int \limsup_{i\to\infty} &P(\max_{N(a_i) \leqq j \leqq M(a_i+b)} |h(S_j) - h'(S_{N(a_i)})^T S_j| > \epsilon \mid S_{N(a_i)}, N(a_i))\, dP \\
&\leqq \gamma + \lim_{i\to\infty} P(f_2((\|T^*_{a_i,b}\|/a_i)(2 + 8K_1\|\mu\|/h(\mu)))\|T^*_{a_i,b}\| > \epsilon).
\end{aligned}$$

From Lemma 3.5 together with the first paragraph of the proof of this lemma it follows that

$$0 = \lim_{i\to\infty} P(f_2((\|T_{a_i,b}\|/a_i)(2 + 8K_1\|\mu\|/h(\mu)))\|T_{a_i,b}\| > \epsilon).$$

Therefore with probability one

$$0 = \limsup_{i\to\infty} P(\max_{N(a_i) \leqq j \leqq M(a_i+b)} |h(S_j) - h'(S_{N(a_i)})^T S_j| > \epsilon \mid S_{N(a_i)}, N(a_i)).$$

This is the conclusion of the lemma.

LEMMA 3.8. $(X_1 \varepsilon \theta)$. *Let* $\epsilon > 0$ *and let* $\{a_i, i \geqq 1\}$ *be a nondecreasing positive real number sequence. Let* $A_{i,\epsilon}$ *be the event*

$$\{\max_{1 \leqq j \leqq M(a_i+b)-N(a_i)} |(h'(S_{N(a_i)}) - h'(\mu))^T(S_{N(a_i)+j} - S_{N(a_i)})| > \epsilon\}$$

Then with probability one,

$$0 = \lim_{i\to\infty} P(A_{i,\epsilon} \mid S_{N(a_i)}, N(a_i)).$$

PROOF.

$$\max_{1 \leqq j \leqq M(a_i+b)-N(a_i)} |(h'(S_{N(a_i)}) - h'(\mu))^T(X_{N(a_i)+1} + \cdots + X_{N(a_i)+j})|$$
$$\leqq \|h'(S_{N(a_i)}) - h'(\mu)\| \, \|X_{N(a_i)+1} + \cdots + X_{N(a_i)+R^*(N(a_i),b+\delta(a_i))}\|$$
$$\leqq f_2(\|S_{N(a_i)}/h(S_{N(a_i)}) - \mu/h(\mu)\|)(\|\textstyle\sum_{p=N(a_i)+1}^{R^*(N(a_i),b+\delta(a_i))} X_p\|).$$

Since the function f_2 is bounded the expression above is integrable and we find by the Tchebychev inequality that

$$P(A_{i,\epsilon} \mid S_{N(a_i)}, N(a_i))$$
$$\leqq \epsilon^{-1} f_2(\|S_{N(a_i)}/h(S_{N(a_i)}) - \mu/h(\mu)\|)(ER^*(b + \delta(a_i)))(E\|X_1\|).$$

Since $(E\|X_1\|)(ER^*(b+\delta(a_i))) \leqq K_4\delta(a_i)$, using the definition of f_1 and using Lemma 3.5 we find that with probability one for all large values of i that $\|S_{N(a_i)}/h(S_{N(a_i)}) - \mu/h(\mu)\| \leqq f_1(a_i/2h(\mu))$ so that with probability one for all large values of i,

$$P(A_{i,\epsilon} \mid S_{N(a_i)}, N(a_i)) \leqq \epsilon^{-1} K_4 f_2(f_1(a_i/2h(\mu)))\delta(a_i).$$

As the right side tends to zero as $i \to \infty$ the conclusion of the lemma follows.

LEMMA 3.9. $(X_1 \varepsilon Q)$.

$$E(R^*(b))^2 = E(R^*(N(a), b))^2 < \infty \quad \text{if} \quad b > 0.$$

PROOF. Order the elements of Q by $x > y$ if and only if $x - y \varepsilon Q$. Let $x_0 \varepsilon Q$ be a vector such that $P(X_1 > x_0) > 0$. Let $Y_n = x_0$ if $X_n > x_0$ and let $Y_n = 0$ if $X_n \not> x_0$, $n \geqq 1$. Then if exactly j of $Y_1 = x_0, \cdots, Y_n = x_0$ hold, $h^*(Y_1 + \cdots + Y_n) = jh^*(x_0)$. Thus we have the special renewal problem considered by Doob [6], page 425. Lemma 3.9 follows at once from Doob's results.

Let $M(q, t, a, b)$ be the number of integers $n \geqq 1$ such that $h(t + X_{q+1} + \cdots + X_{q+n})$ falls in the interval $[a, a + b)$. Let $L(q, a, b)$ be the number of integers $n \geqq 1$ such that $h'(\mu)^T(X_{q+1} + \cdots + X_{q+n})$ falls in the interval $[a, a + b)$. Let $A(a)$ be the event that

$$\begin{aligned} L(N(a), a - h(S_{N(a)}) + \epsilon, b - 2\epsilon) & \\ \leqq M(N(a), S_{N(a)}, a, b) & \\ \leqq L(N(a), a - h(S_{N(a)}) - \epsilon, b + 2\epsilon). & \end{aligned} \tag{3.15}$$

LEMMA 3.10. $(X_1 \varepsilon Q)$. *Let* $\mathfrak{X}_{A(a)}$ *be the indicator function of the set* $A(a)$. *If* $\{a_i, i \geqq 1\}$ *is a nondecreasing positive real number sequence such that* $\lim_{i\to\infty} a_i = \infty$ *then with probability one*

$$\lim_{i\to\infty} E(1 - \mathfrak{X}_{A(a_i)} \mid S_{N(a_i)}, N(a_i)) = 0.$$

PROOF. This is an immediate consequence of Lemmas 3.7 and 3.8.

LEMMA 3.11. $(X_1 \varepsilon Q)$. *Given the hypotheses of Lemma* 3.10, *with probability* *one,*

$$0 = \lim_{i\to\infty} E((1 - \mathfrak{X}_{A(a_i)})M(N(a_i), S_{N(a_i)}, a_i, b) \mid S_{N(a_i)}, N(a_i));$$

$$0 = \lim_{i\to\infty} E((1 - \mathfrak{X}_{A(a_i)})L(N(a_i), a - h(S_{N(a_i)}) + \epsilon, b - 2\epsilon) \mid S_{N(a_i)}, N(a_i));$$

$$0 = \lim_{i\to\infty} E((1 - \mathfrak{X}_{A(a_i)})L(N(a_i), a - h(S_{N(a_i)}) - \epsilon, b + 2\epsilon) \mid S_{N(a_i)}, N(a_i)).$$

PROOF. We give a proof of the first of these three relations, the other two following by similar arguments. Observe that $M(N(a_i), S_{N(a_i)}, a_i, b) \leqq R^*(N(a_i), b)$. Therefore, using this inequality and the Cauchy-Schwarz inequality we find that

$$E((1 - \mathfrak{X}_{A(a_i)})M(N(a_i), S_{N(a_i)}, a_i, b) \mid S_{N(a_i)}, N(a_i)) \leqq (E((1 - \mathfrak{X}_{A(a_i)}) \mid S_{N(a_i)}, N(a_i)))^{\frac{1}{2}}(ER^*(b)^2)^{\frac{1}{2}}.$$

The assertion of Lemma 3.11 now follows by an application of Lemma 3.10.

LEMMA 3.12. $(X_1 \varepsilon Q)$. *If* $\{a_i, i \geqq 1\}$ *is a nondecreasing positive real number sequence such that* $\lim_{i\to\infty} a_i = \infty$ *then* $\lim_{i\to\infty}$ *in probability* $E(M(N(a_i), S_{N(a_i)}, a_i, b) \mid S_{N(a_i)}, N(a_i)) = b/h(\mu)$.

PROOF. Take conditional expectations of the three terms in (3.15) and apply the one-dimensional renewal theorem. The one-dimensional theorem says that there exists a function $f: [0, \infty) \to [-1, \infty)$ such that $EL(q, a, b) = b(1 + f(a))/h(\mu)$. f satisfies the condition $\lim_{a\to\infty} f(a) = 0$. Therefore

$$(3.16)\quad EL(N(a), a - h(S_{N(a)}) + \epsilon, b - 2\epsilon) \mid S_{N(a)}, N(a)) = (b - 2\epsilon)(1 + f(\epsilon + a - h(S_{N(a)})))/h(\mu)$$

and

$$(3.17)\quad EL(N(a), a - h(S_{N(a)}) - \epsilon, b + 2\epsilon) \mid S_{N(a)}, N(a)) = (b + 2\epsilon)(1 + f(a - h(S_{N(a)}) - \epsilon))/h(\mu).$$

By Lemma 3.6, since $N(a) = M(a - \delta(a))$ and $\lim_{a\to\infty} \delta(a) = \infty$, it follows that the functions in (3.16) and (3.17) have limits in probability respectively $(b - 2\epsilon)/h(\mu)$ and $(b + 2\epsilon)/h(\mu)$. If we now apply Lemma 3.11, Lemma 3.12 will follow since $\epsilon > 0$ becomes arbitrary.

PROOF OF THE THEOREM OF SECTION 3 ON THE HYPOTHESIS $X_1 \varepsilon Q$. By Lemma 3.3 it is sufficient to suppose $X_1 \varepsilon Q$. Since $\{S_n \leqq a, N(b) = n\}$ is independent of the random variable $R^*(n, b')$ we obtain with probability one $E(M(N(a), S_{N(a)}, a, b) \mid S_{N(a)}, N(a)) \leqq ER^*(0, b)$. Therefore we have a uniformly bounded set of random variables such that if $\{a_i, i \geqq 1\}$ is a nondecreasing positive real number sequence satisfying $\lim_{i\to\infty} a_i = \infty$ then $E(M(N(a_i), S_{N(a_i)}, a_i, b) \mid S_{N(a_i)}, N(a_i))$ converges in probability. By the bounded convergence theorem

$$\lim_{i\to\infty} E(M(a_i + b) - M(a_i)) = \lim_{i\to\infty} E(E(M(N(a_i), S_{N(a_i)}, a_i, b) \mid S_{N(a_i)}, N(a_i))) = b/h(\mu).$$

This holds for every choice of a nondecreasing positive real number sequence. The proof is complete.

4. Blackwell's theorem with corners. In this section we will continue to assume $\{X_n, n \geqq 1\}$ is a sequence of independently and identically distributed random k-dimensional vectors. We will assume throughout Section 4 that the distribution of X_1 is non-degenerate and that $E\|X_1\|^2 < \infty$. Using these hypotheses we may prove an analogue of Blackwell's theorem for functions like, for example, $h(x) = \max_{1 \leqq i \leqq k} x_i$. That is, we will consider functions h that do not have continuous derivatives everywhere, and especially, not having a continuous derivative at μ.

The assumption that $X_1 \varepsilon Q$ is made and used throughout this section. It is an unessential assumption and may be removed by an argument like that used to prove Lemmas 3.1 to 3.3. We omit a discussion of the details. Under the assumption of finite second moments, if C is a cone in k-dimensional space which contains μ in its interior then the expected number of sums S_n, $n \geqq 1$, which fall outside C is finite. This follows from a direct application of results in Erdös [7]. Therefore in obtaining an analogue of Blackwell's theorem one need only consider a cone C generated by some set like $\{x \mid \|x - \mu\| < \epsilon\}$.

In many examples the function h can be pieced together from parts of smooth functions $h_1, \cdots, h_q$. Thus for example, if $h_j(x) = x_j$ then $\max_{1 \leqq i \leqq k} x_i = h_j(x)$ on the appropriate cone. One place in which this section will lack generality is in the description of such decompositions. We consider only the simplest decompositions.

In k dimensions we will describe a cone C by linear functionals $\omega_1, \cdots, \omega_q$ such that

$$\omega_i(\mu) = 0, \qquad 1 \leqq i \leqq q, \tag{4.1}$$

and define the cone C by

$$C = \{x \mid \text{for all } i,\ 1 \leqq i \leqq q,\ \omega_i(x) > 0\}. \tag{4.2}$$

On the set C we consider the restriction of h and suppose that there exists a function h_1 satisfying (3.1) to (3.5) such that

$$\text{if} \quad x \,\varepsilon\, C \quad \text{then} \quad h(x) = h_1(x). \tag{4.3}$$

In the sequel we will consider only this one piece of a decomposition.

We will want to assume that (3.2) and (3.3) hold for the function h. For example, for all x, y, $\max_{1 \leqq i \leqq k} y_i \geqq \max_{1 \leqq i \leqq k}(x_i + y_i) - \max_{1 \leqq i \leqq k}(x_i) \geqq \min_{1 \leqq i \leqq k} y_i$. Therefore (3.2) and (3.3) hold for $h(x) = \max_{1 \leqq i \leqq k} x_i$.

In order to properly state the result, define a set $B^*(a)$ as follows:

$$\text{If } a > 0 \text{ then } B^*(a) \text{ is the event that } S_{M(a)} \,\varepsilon\, C. \tag{4.4}$$

THEOREM 4. (Ind, Id, and second moments). *Suppose* (4.1) *to* (4.3) *hold and the function h satisfies* (3.2) *and* (3.3). *Then* $\lim_{a\to\infty} P(B^*(a))$ *exists and*

$$\lim_{a\to\infty} \int_{B^*(a)} (M(a + b) - M(a))\, dP = (b/h(\mu)) \lim_{a\to\infty} P(B^*(a)). \tag{4.5}$$

The remainder of this section consists of a sequence of lemmas leading to a proof of Theorem 4. If the cone C is empty then Theorem 4 is trivial. Therefore we assume for the remainder of this section that C is not empty. We observe that C is an open set.

LEMMA 4.1. *$\lim_{a\to\infty} P(B^*(a))$ exists.*

PROOF. Since if $1 \leqq i \leqq q$ then $\omega_i(\mu) = 0$, we find that $S_{M(a)} \varepsilon C$ if and only if $T_a = (S_{M(a)} - M(a)\mu)/M(a)^{\frac{1}{2}} \epsilon C$, as follows by substitution in the definition (4.2). Further, by Theorem 2.1, with probability one, $\lim_{a\to\infty} M(a)/a = 1/h(\mu)$. It follows from the multivariate analogue of results of Anscombe [1] that as $a \to \infty$, T_a is asymptotically normally distributed. Since we assume that X_1 has a nondegenerate covariance matrix, the limiting normal distribution is nondegenerate. Since C is an open set whose boundary has k-dimensional Lebesgue measure zero the indicator function $\mathfrak{X}_C$ is lower semi-continuous and $\lim_{a\to\infty} P(B^*(a)) = \lim_{a\to\infty} E\mathfrak{X}_C(T_a)$ exists. The proof of Lemma 4.1 is complete.

Let δ: $[0, \infty)$ be the function defined in (3.11). Since $0 < \delta(a) < a^{1/6}$, it follows that $0 < \delta^2(a)/a^{\frac{1}{2}} \leqq a^{-1/6}$. Therefore

$$\lim_{a\to\infty} \delta^2(a)/a^{\frac{1}{2}} = 0. \tag{4.6}$$

Let $N(a) = M(a - \delta(a))$. The basic idea of the remainder of the proof is that if $S_{N(a)} \varepsilon C$ then with high probability $S_j \varepsilon C$ for $j = N(a), \cdots, M(a + b)$, provided only that a is large and $b + \delta(a)$ is small compared with $a^{\frac{1}{2}}$. Therefore given that the random walk is in C, conditionally we may consider the problem relative to the smooth function h_1.

To make this precise, let $M(q, t, a, b)$ be as defined in the last part of Section 3 and let $M_1(q, t, a, b)$ be the corresponding quantity for the function h_1.

LEMMA 4.2. *If $a > 0$ then*

$$M(N(a), S_{N(a)}, a, b) \leqq M(0, 0, a, b); \tag{4.7}$$

if in (4.7) inequality holds then $h(S_{N(a)}) \geqq a$.

We have assumed that h satisfies (3.2). A consequence of Lemma 3.6 is:

LEMMA 4.3.

$$\lim_{a\to\infty} P(h(S_{N(a)}) \geqq a) = 0. \tag{4.8}$$

The next lemma says that without loss of generality we may consider the process as starting at $S_{N(a)}$. It is a consequence of applying (4.7) and (4.8) together with Lemma 3.1.

LEMMA 4.4. *If $b > 0$ then*

$$\lim_{a\to\infty} E|M(0, 0, a, b) - M(N(a), S_{N(a)}, a, b)| = 0.$$

For later reference, if $a > 0$ we define sets as follows:

$A(a)$ is the event that $S_j \varepsilon C$ for all j such that

$N(a) \leqq j \leqq M(a + b)$; $B(a)$ is the event that $S_{N(a)} \varepsilon C$.

LEMMA 4.5.

$$\lim_{a\to\infty} P(B(a) - A(a)) = 0. \tag{4.9}$$

PROOF. Since we suppose h is continuous on Q by Theorem 2.1, with probability one, $\lim_{a\to\infty} N(a)/(a - \delta(a)) = 1/h(\mu)$. Since by Lemma 3.5 $\lim_{a\to\infty} \delta(a)/a = 0$, with probability one $\lim_{a\to\infty} N(a)/a = 1/h(\mu)$.

As observed above, as $a \to \infty$ T_a is asymptotically normal and therefore also $\omega_i(T_a)$ is asymptotically normal. Therefore if $\epsilon > 0$,

$$\begin{aligned}\lim_{a\to\infty} P(|\omega_i(S_{N(a)})| \geqq \epsilon\delta^2(a)) \\ = \lim_{a\to\infty} P(|\omega_i(T_a)| \geqq \epsilon\delta^2(a)/(N(a)^{\frac{1}{2}})) = 1,\end{aligned} \tag{4.10}$$

since by (4.6) and the remarks above, with probability one, $\lim_{a\to\infty} (\delta^2(a)/a^{\frac{1}{2}}) \cdot (a/N(a))^{\frac{1}{2}} = 0$.

Let $R^*(q, a)$ be as in the proof of Lemma 3.1. Then we recall that $M(a + b) \leqq N(a) + R^*(N(a), b + \delta(a))$. Then

$$\begin{aligned}&\lim_{a\to\infty} P(\max_{1\leqq j\leqq M(a+b)-N(a)} \delta^{-2}(a)|\omega_i(S_{N(a)+j} - S_{N(a)})| < \epsilon) \\ &\geqq \lim_{a\to\infty} P(\max_{1\leqq j\leqq R^*(N(a),b+\delta(a))} \delta^{-2}(a)|\omega_i(S_{N(a)+j} - S_{N(a)})| < \epsilon) \\ &= \lim_{a\to\infty} \textstyle\sum_{n=1}^{\infty} P(N(a) = n, \\ &\quad \max_{1\leqq j\leqq R^*(n,b+\delta(a))} \delta^{-2}(a)|\omega_i(S_{n+j} - S_n)| < \epsilon) \\ &= \lim_{a\to\infty} \textstyle\sum_{n=1}^{\infty} P(N(a) = n)P(\max_{1\leqq j\leqq R^*(0,b+\delta(a))} \delta^{-2}(a)|\omega_i(S_j)| < \epsilon) \\ &= 1.\end{aligned} \tag{4.11}$$

The last step follows since with probability one,

$$\lim_{a\to\infty} R^*(0, b + \delta(a))/\delta(a) = 1/h^*(\mu).$$

(4.10) and (4.11) together show that (4.9) must hold.

From Lemma 4.4 we may consider the process as starting with $S_{N(a)}$. From Lemma 4.5, if this starting point is in C then we may consider that the process continues in C until $M(a + b)$. Under the event $A(a)$, if $n = N(a)$ then $M(n, S_n, a, b) = M_1(n, S_n, a, b)$. Finally Lemma 4.4 holds also for the function M_1. Therefore

$$\begin{aligned}&\lim_{a\to\infty} \textstyle\int_{B(a)} M(0, 0, a, b)\, dP = \lim_{a\to\infty} \int_{A(a)} M(0, 0, a, b)\, dP \\ &= \lim_{a\to\infty} \textstyle\int_{A(a)} M_1(N(a), S_{N(a)}, a, b)\, dP \\ &= \lim_{a\to\infty} \textstyle\int_{B(a)} M_1(N(a), S_{N(a)}, a, b)\, dP \\ &= \lim_{a\to\infty} \textstyle\int_{B(a)} E(M_1(N(a), S_{N(a)}, a, b) \mid S_{N(a)}, N(a))\, dP \\ &= (b/h(\mu)) \lim_{a\to\infty} P(B(a)).\end{aligned} \tag{4.12}$$

This limit exists since by Lemma 4.1, $\lim_{a\to\infty} P(B(a)) = \lim_{a\to\infty} P(B^*(a))$ exists and by Lemma 4.5, $\lim_{a\to\infty} P(A(a)) = \lim_{a\to\infty} P(B(a))$. In order to complete the proof of Theorem 4 we need the following observations:

Repetition of the first part of the proof of Lemma 4.5 will show that

$$\lim_{a\to\infty} P(|\omega_i(S_{M(a)})| \geqq \epsilon\delta^2(a)) = 1. \tag{4.13}$$

This together with (4.11) suffices to establish that

$$\lim_{a\to\infty} P(B^*(a) - A(a)) = 0. \tag{4.14}$$

This in turn implies that

$$\lim_{a\to\infty} P((B^*(a) - B(a)) \cup (B(a) - B^*(a))) = 0. \tag{4.15}$$

(4.15) together with Lemma 3.1 suffice to show that

$$\lim_{a\to\infty} \int_{B(a)} M(0, 0, a, b)\, dP = \lim_{a\to\infty} \int_{B^*(a)} M(0, 0, a, b)\, dP.$$

Therefore the conclusion of Theorem 4 holds and the proof is complete.

The author is indebted to H. Kesten for suggesting the broad outline of the argument of this section.

5. Asymptotic average life. In this section we will always suppose that (3.1) to (3.7) hold and that $X_1 \varepsilon Q$. In addition we will assume h has continuous second partial derivatives defined everywhere on Q. Let

$$H_t = h(S_{M(t)}), \quad \text{and} \quad EM(t) = (t + \delta(t))/h(\mu), \qquad t > 0. \tag{5.1}$$

(This is a new usage of δ.) Define constants by

$$\begin{aligned} \mu_{ij} &= E(X_{1i} - \mu_i)(X_{1j} - \mu_j), && 1 \leqq i, j \leqq k; \\ \beta_{ij}(\theta) &= (\partial^2 h/\partial x_i \partial x_j)(\theta), && 1 \leqq i, j \leqq k; \\ \beta_{ij} &= \beta_{ij}(\mu), && 1 \leqq i, j \leqq k. \end{aligned} \tag{5.2}$$

We show in this section that under suitable restrictions,

$$\lim_{t\to\infty} \delta(t) = E((\alpha^T X_1)^2/2h(\mu)) - \textstyle\sum_{i=1}^k \sum_{j=1}^k \beta_{ij}\mu_{ij}. \tag{5.3}$$

In the sequel we use the notation (β_{ij}) for the $k \times k$ matrix with the indicated entries, so we may write $\sum_{i=1}^k \sum_{j=1}^k \beta_{ij}\mu_{ij} = \operatorname{tr}(\beta_{ij})(\mu_{ij})$.

Were we to replace h by its tangent plane and calculate the limit corresponding to (5.3) then, of course, the β_{ij} for the tangent plane are all zero, and we obtain the well known answer $E(\alpha^T X_1)^2/2h(\mu)$. The other term in (5.3), when divided by $h(\mu)$, therefore represents the signed expected number of observations falling between the surface $h(x) = t$ and the plane $\alpha^T x = t$.

In order to make our calculations go through we have found the following assumption convenient:

(5.4) There is a constant $\delta > 0$ such that $\delta \leqq X_{1i} \leqq 1/\delta$, $1 \leqq i \leqq k$.

This assumption implies that $M(t)/t$ is uniformly bounded away from zero and infinity as $t \to \infty$, which is the main reason we make this assumption. In addition we assume the functions $\beta_{ij}(\theta)$ are bounded on the set $\{\theta \mid h(\theta) = 1\}$. We assume throughout this section that the hypotheses (3.1) to (3.5) hold for h and that h has continuous second partial derivatives throughout Q.

In order to abbreviate we introduce the new random variables $Y(t)$, $t > 0$ by means of the following equations.

$$(5.5) \qquad \text{If} \quad t > 0 \quad \text{then} \quad (M(t))^{\frac{1}{2}}Y(t) = S_{M(t)} - M(t)\mu.$$

Then using two terms of a Taylor series expansion,

$$(5.6) \quad H_t/M(t) = h(\mu) + \alpha^T Y(t)/(M(t))^{\frac{1}{2}} + Y(t)^T(\beta_{ij}(\theta_{ij}))Y(t)/M(t).$$

The vectors θ_{ij} lie on the line segment between μ and $\mu + Y(t)/(M(t))^{\frac{1}{2}}$. Therefore, from (5.5) and the law of large numbers, as $t \to \infty$, θ_{ij} tends to μ with probability one.

Using the identity $M(t)h(\mu) + (M(t))^{\frac{1}{2}}\alpha^T Y(t) = \alpha^T S_{M(t)}$, we obtain from (5.6) the formula

$$(5.7) \qquad H_t - t = \alpha^T S_{M(t)} - t + Y(t)^T(\beta_{ij}(\theta_{ij}))Y(t).$$

Formula (5.7) is the formula from which we derive our result, Theorem 5, stated below. By results due to Anscombe [1], as $t \to \infty$, $Y(t)$ is asymptotically normally distributed. By showing that the family of random variables $\{Y(t)^T(\beta_{ij}(\theta_{ij}))Y(t), t > 0\}$ is uniformly integrable we will obtain

$$(5.8) \qquad \lim_{t\to\infty} EY(t)^T(\beta_{ij}(\theta_{ij}))Y(t) = \operatorname{tr}(\beta_{ij})(\mu_{ij}).$$

On the otherhand, by showing the family $\{H_t - t, t > 0\}$ to be uniformly integrable and using results of Section 3 we will obtain

$$(5.9) \qquad \lim_{t\to\infty} E(H_t - t) = E(\alpha^T X_1)^2/2h(\mu).$$

Taken together these results will prove

THEOREM 5. (Ind, Id and continuous second partial derivatives). *Given assumptions* (3.1) *to* (3.5), *given* (5.4), *and that the functions* β_{ij} *are bounded on* $\{x \mid h(x) = 1, x \varepsilon Q\}$, *the family* $\{\alpha^T S_{M(t)} - t, t > 0\}$ *is uniformly integrable, and*

$$(5.10) \quad \lim_{t\to\infty} E(\alpha^T S_{M(t)} - t) = E(\alpha^T X_1)^2/2h(\mu) - \operatorname{tr}(\beta_{ij})(\mu_{ij}).$$

The remainder of this section consists of a proof of Theorem 5. We observe first that by Wald's identity, see [11], $E(\alpha^T S_{M(t)}) = EM(t)\alpha^T\mu = h(\mu)EM(t)$. This gives part of the formulas necessary to verify (5.10).

In the proof of the theorem we will need to know that the $(\beta_{ij}(\theta_{ij}))$ are bounded as $t \to \infty$. Since the second partial derivatives of h are homogeneous functions of degree minus one which are assumed continuous, it is sufficient to show the θ_{ij} are bounded away from the origin. Let e be the vector having all components equal to one. From (5.4), if $t > h(e/\delta)$, then all components of $S_{M(t)}/M(t)$ are $\geqq \delta$. Since θ_{ij} is on the line segment between μ and $S_{M(t)}/M(t)$, it follows that the components of the vector θ_{ij} all exceed $\min(\delta, \min_{1\leqq i\leqq k}\mu_i) = \delta' > 0$. Therefore all the vectors θ_{ij} lie in the set $\{x \mid h(x) \geqq \delta' h(e)\}$. By our hypothesis on the second partial derivatives of h it follows that there exists a constant K_5 such that

$$\text{(5.11)} \qquad \text{if } t > h(e/\delta) \text{ then } |\beta_{ij}(\theta_{ij})| < K_5, \quad 1 \leqq i, j \leqq k.$$

We now prove $\{H_t - t, t > 0\}$ is a uniformly integrable family. If we integrate the inequality in (3.14) with respect to γ then we obtain at once

$$\int_a^\infty P(H_t - t > \gamma)\, d\gamma \leqq K_3 \int_a^\infty d\gamma \int_{[\gamma]-2}^\infty P(K_1\|X_1\| \geqq s)\, ds,$$

which is finite since $E\| X_1 \|^2 < \infty$. Therefore the family $\{H_t - t, t > 1\}$ is uniformly integrable. A direct calculation using Theorem 3 shows that

$$\text{(5.12)} \qquad \lim_{t\to\infty} E(H_t - t) = E(\alpha^T X_1)^2/2h(\mu).$$

In order to treat the last term of (5.7) we now prove several lemmas. A close reading will show that the proof of Lemma 5.2 uses Lemma 5.1. The case $n = 1$ of Lemma 5.1 may be written $ES_M^2 = \sigma^2 E(M)$, a result of interest in applications of the Cramér-Rao inequality to sequential analysis. Since submission of the original version of this paper a proof of Lemma 5.1 together with other results has appeared in Chow, Robbins, and Teicher [5] so we omit a proof.

LEMMA 5.1. *Let $\{Z_n, n \geqq 1\}$ be a sequence of independently and identically distributed random variables. Let N be a stopping variable* (i.e., $P(N \geqq 0) = 1$, *N is integer valued, and if $n \geqq 1$ the event $\{N \leqq n\}$ is independent of the random variables $\{Z_i, i \geqq n + 1\}$.) If $EZ_1 = 0$ and $EZ_1^2 = \sigma^2$ and $EN < \infty$ then*

$$\sum_{m=n}^\infty \int_{\{N=n\}} \left(\sum_{i=n}^m Z_i\right)^2 dP = \sigma^2 \sum_{i=n}^\infty P(n \geqq i).$$

LEMMA 5.2. *Given the hypotheses of Lemma 5.1, if $n \geqq 1$ is an integer then $E(\sum_{i=1}^N Z_i - \sum_{i=1}^n Z_i)^2 = \sigma^2 E|N - n|$.*

PROOF.

$$\begin{aligned} E\left(\sum_{i=1}^N Z_i - \sum_{i=1}^n Z_i\right)^2 &= \int_{\{N<n\}} \left(\sum_{i=N+1}^n Z_i\right)^2 dP + \int_{\{N\geqq n+1\}} \left(\sum_{i=n+1}^N Z_i\right)^2 dP \\ &= \sum_{m=1}^{n-1} \int_{\{N=m\}} \left(\sum_{i=m+1}^n Z_i\right)^2 + \sigma^2 \sum_{m=n+1}^\infty P(N \geqq m) \\ &= \sum_{m=1}^{n-1} (n - m)\sigma^2 P(N = m) + \sigma^2 \sum_{m=n+1}^\infty P(N \geqq m) \\ &= n\sigma^2 P(N \leqq n - 1) - \sigma^2 \sum_{m=1}^{n-1} mP(N = m) \\ &\quad + \sigma^2 \sum_{m=1}^\infty mP(N = m + n) \\ &= \sigma^2 E|N - n|, \qquad \text{which is the desired result.} \end{aligned}$$

LEMMA 5.3. *Suppose $\{N_n, n \geqq 1\}$ is a sequence of stopping variables such that if $n \geqq 1$ then $EN_n < \infty$. Suppose $\lim_{n\to\infty} E|(N_n/n) - 1| = 0$. Then*

$$0 = \lim_{n\to\infty} E\left(\left(\sum_{i=1}^{N_n} Z_i - \sum_{i=1}^n Z_i\right)/n^{\frac{1}{2}}\right)^2.$$

PROOF. Apply Lemma 5.2.

We now complete the proof of Theorem 5. Let $Z(t) = Y(t)M(t)^{\frac{1}{2}}$. By (5.4) it follows that there is a constant $K_6 > 0$ such that if $t > h(e/\delta)$ then $t/M(t) \leqq K_6$. As noted in (5.11), $|\beta_{ij}(\theta_{ij})| \leqq K_5$, $1 \leqq i, j \leqq k$. Hence to prove uniform integrability of $\{Y(t)^T(\beta_{ij}(\theta_{ij}))Y(t),\ t \geqq 1\}$ it is sufficient to prove uniform integrability of the components of $\{t^{-1}Z(t)Z(t)^T,\ t \geqq 1\}$, or equivalently, the uniform integrability of $\{t^{-1}Z(t)^TZ(t),\ t \geqq 1\}$. By Lemma 5.1,

$$\sup_{t\geqq 1} t^{-1}EZ(t)^TZ(t) \leqq \sup_{t\geqq 1} (EM(t)/t) \text{ tr } (\mu_{ij}) < \infty .$$

If we define γ_t to be the least integer n such that $n \geqq t/h(\mu)$ then with probability one, $\lim_{t\to\infty} M(t)/\gamma_t = 1$. From Theorem 2.2 we obtain the convergence in L_1 required for the application of Lemma 5.3. Thus we find

$$\lim_{\gamma\to\infty} E\gamma_t^{-1}(Z(t) - \textstyle\sum_{i=1}^{\gamma_t} (X_i - \mu))^T(Z(t) - \textstyle\sum_{i=1}^{\gamma_t} (X_i - \mu)) = 0.$$

Therefore to prove uniform integrability it is sufficient to prove $\{t^{-1}(S_{\gamma_t} - \gamma_t\mu)^T \cdot (S_{\gamma_t} - \gamma_t\mu),\ t \geqq 1\}$ is a uniformly integrable family. But since we suppose that X_1 has compact support this follows at once from the Cauchy-Schwarz inequality.

Finally, $\lim_{t\to\infty} (\gamma_t/M(t))^{\frac{1}{2}}(\beta_{ij}(\theta_{ij})) = (\beta_{ij})$ with probability one, so that from the asymptotic normality of $Y(t)^T(\beta_{ij}(\theta_{ij}))Y(t)$ the result now follows.

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