ON THE EXPECTED VALUE OF A STOPPED MARTINGALE[1]

By Lester E. Dubins and David A. Freedman

University of California, Berkeley

Throughout this note, $X_1, X_2, \cdots$ is a martingale, and $K = \sup_n E|X_n|$. As is easily verified, $E|X_t| \leqq K$ for every stopping time t. This note studies the existence of t such that $E|X_t| = K$ when $K = \infty$, and finds necessary and sufficient conditions on the distribution of the martingale for $E(X_t)$ to be equal to $E(X_1)$ for all t.

Theorem 1. *If* $\sup_n E|X_n| = \infty$, *then there is a stopping time* t *for* $X_1, X_2, \cdots$ *such that* $E|X_t| = \infty$.

Let $\mathfrak{F}_j$ be the σ-field generated by $X_1, \cdots, X_j$. As usual, a *stopping time* t for $X_1, X_2, \cdots$ is a random variable whose range is the set of positive integers with $+\infty$ adjoined, such that for each n, the event $\{t = n\} \varepsilon \mathfrak{F}_n$. Say t is *finite* if it is finite almost surely. Whether or not t is finite, $E|X_t|$ is evaluated as $\int_{t<\infty}|X_t|$.

Of course, $E|X_t|$ may be finite for all finite stopping times t, and yet be infinite for some stopping time t. Here is an example which helped us find Theorems 1 and 2. Let $X_1 = 0$. On $X_n \neq 0$, let $X_{n+1} = X_n$ a.e. On $X_n = 0$, given $X_1, \cdots, X_n$, let $X_{n+1} = 0$ with conditional probability $1 - 2p_{n+1}$, while $X_{n+1} = x_{n+1}$ and $X_{n+1} = -x_{n+1}$ with conditional probability p_{n+1} each. Let $0 < p_n < \frac{1}{2}$, $\sum p_n < \infty$, $0 < x_n < \infty$, and $\sum p_n x_n = \infty$.

Let

$$V_j = \sup_{n \geq j} E\{|X_n| \,|\mathfrak{F}_j\}. \tag{1}$$

Lemma 1. *With the understanding that the* V*'s may be infinite on a set of positive measure,* $V_1, V_2, \cdots$ *is a martingale relative to* $\mathfrak{F}_1, \mathfrak{F}_2, \cdots$.

Proof. Plainly, V_j is $\mathfrak{F}_j$-measurable and

$$\begin{aligned} E\{V_{j+1} \mid \mathfrak{F}_j\} &= E\{\lim_n E[|X_n| \,|\mathfrak{F}_{j+1}]|\mathfrak{F}_j\} \\ &= \lim_n E\{E[|X_n| \,|\mathfrak{F}_{j+1}]|\mathfrak{F}_j\} \\ &= \lim_n E\{|X_n| \,|\mathfrak{F}_j\} \\ &= V_j. \end{aligned}$$

Lemma 2. *If* t *is a stopping time for an integrable stochastic process* $Y_1, Y_2, \cdots$, $\mathfrak{F}$ *is a* σ*-field of measurable sets,* n *is a positive integer, and the event* $\{t = n\} \varepsilon \mathfrak{F}$, *then almost everywhere on* $\{t = n\}$,

$$E\{Y_t \mid \mathfrak{F}\} = E\{Y_n \mid \mathfrak{F}\}. \tag{2}$$

Proof. Since both sides of (2) are plainly $\mathfrak{F}$-measurable, it is only necessary

Received 18 April 1966.

[1] This paper has been partially supported by the National Science Foundation, Grant GP-2593, and by a Sloan Foundation Grant to Freedman.

to check this for $A \subset \{t = n\}$, $A \varepsilon \mathfrak{F}$:

$$(3) \qquad \int_A E\{Y_t \mid \mathfrak{F}\} = \int_A Y_t = \int_A Y_n = \int_A E\{Y_n \mid \mathfrak{F}\}.$$

PROOF OF THEOREM 1. Under the hypothesis $\sup_n E|X_n| = \infty$, $\int V_j = \infty$ for all j, as is implied by Lemma 1. Suppose first that

$$(4) \qquad \int_{\{V_j<\infty\}} V_j = \infty \text{ for some } j.$$

For each ω with $V_j(\omega) < \infty$, let $t(\omega)$ be the least $n \geqq j$ such that $E\{|X_n| \,|\mathfrak{F}_j\}(\omega) \geqq V_j(\omega) - 1$; if $V_j(\omega) = \infty$, let $t(\omega) = j$. Plainly, $t \geqq j$ and t is $\mathfrak{F}_j$-measurable, so t is a stopping time. Moreover, according to Lemma 2, on the event $\{t = n\}$, $E\{|X_t| \,|\mathfrak{F}_j\} = E\{|X_n| \,|\mathfrak{F}_j\}$. So $E\{|X_t| \mid \mathfrak{F}_j\} \geqq V_j - 1$ wherever $V_j < \infty$. Therefore,

$$(5) \quad E|X_t| = E(E\{|X_t| \mid \mathfrak{F}_j\}) \geqq \int_{\{V_j<\infty\}} E\{|X_t| \mid \mathfrak{F}_j\} \geqq \int_{\{V_j<\infty\}} (V_j - 1) = \infty.$$

Suppose next that

$$(6) \qquad \int_{\{V_j<\infty\}} V_j < \infty \quad \text{for all} \quad j.$$

Let A_j be the event $\{V_j = \infty\}$. By Lemma 1:

(7) For all j, $A_j \varepsilon \mathfrak{F}_j$, and A_j includes almost all of A_{j+1}.

Consider first the case that for some j, there are infinitely many disjoint subsets $B_1, B_2, \cdots$ of A_j which have positive probability and are $\mathfrak{F}_j$-measurable. On B_i, let t be the least $n \geqq j$ such that $E\{|X_n| \mid \mathfrak{F}_j\} \geqq 1/P(B_i)$, and on the complement of A_j let $t = j$. The stopping time t is $\mathfrak{F}_j$-measurable and

$$(8) \qquad \begin{aligned} \int_{B_i} |X_t| &= \textstyle\sum_n \int_{B_i \cap \{t=n\}} |X_n| \\ &= \textstyle\sum_n \int_{B_i \cap \{t=n\}} E\{|X_n| \mid \mathfrak{F}_j\} \\ &\geqq [1/P(B_i)] \textstyle\sum_n P(B_i \cap \{t = n\}) \\ &= 1. \end{aligned}$$

Consequently, $E|X_t| = \infty$.

Finally, suppose in addition to (6):

(9) For each j, there are only a finite number of disjoint $\mathfrak{F}_j$-measurable subsets of A_j having positive probability.

An $\mathfrak{F}_j$-measurable subset of A_j of minimal positive probability is an *atom* of A_j. Since $P(A_j) > 0$, there is at least one atom of A_j. In fact, since

$$(10) \qquad \int_{\{V_{j+1}<\infty\}} V_{j+1} < \infty, \quad \text{and} \quad \int_B V_{j+1} = \int_B V_j = \infty,$$

for each atom B of A_j, the $\mathfrak{F}_{j+1}$-measurable set $B \cap A_{j+1}$ has positive probability. So

(11) Each atom of A_j includes at least one atom of A_{j+1}.

Consequently, there exists a sequence $B_1 \supset B_2 \supset \cdots$, where each B_j is an atom of A_j. Since $V_j = \infty$ on B_j, $E\{X_n^+ \mid \mathfrak{F}_j\}$ and $E\{X_n^- \mid \mathfrak{F}_j\}$, which are constant on B_j, converge on B_j to ∞ as $n \to \infty$. So there is a sequence $j_1 < j_2 < \cdots$ such that on B_{j_k}

$$E\{X_{j_{k+1}}^+ \mid \mathfrak{F}_{j_k}\} \geqq 1/P(B_{j_k}) \tag{12}$$

and

$$E\{X_{j_{k+1}}^- \mid \mathfrak{F}_{j_k}\} \geqq 1/P(B_{j_k}). \tag{13}$$

These inequalities plainly imply

$$\int_{B_{j_k}} X_{j_{k+1}}^+ \geqq 1 \quad \text{and} \quad \int_{B_{j_k}} X_{j_{k+1}}^- \geqq 1. \tag{14}$$

Let B be the intersection of $B_1, B_2, \cdots$ and define t thus. On B, let $t = \infty$; off B, let t be the least j_{k+1} such that $B_{j_{k+1}}$ fails to occur. Plainly, t is a stopping time and

$$\begin{aligned} \int_{\{t=j_{k+1}\}} |X_{j_{k+1}}| &= \int_{B_{j_k}-B_{j_{k+1}}} |X_{j_{k+1}}| \\ &\geqq \min\left(\int_{B_{j_k}} X_{j_{k+1}}^+, \int_{B_{j_k}} X_{j_{k+1}}^-\right) \geqq 1. \end{aligned} \tag{15}$$

The equality in (15) is obvious; the first inequality holds because $X_{j_{k+1}}$ is constant on $B_{j_{k+1}}$; the second inequality holds by (14). So $E|X_t| = \infty$, completing the proof of the theorem.

(In contrast to Theorem 1, if $\sup_n E|X_n|$ is finite, there may exist no stopping time that achieves the sup.)

THEOREM 2. *In order that $E|X_t|$ be finite for every finite stopping time t, it is necessary and sufficient that* (6), (9), *and this condition hold*:

(16) *For every sequence $B_1 \supset B_2 \supset \cdots$ such that each B_j is an atom of A_j, $\lim P(B_j) > 0$.*

A compactness argument or König's lemma [König, 1936, Theorem 6 on p. 81] can be used to prove

LEMMA 3. *Suppose that* (7), (9), (11), *and* (16) *hold and that t is a finite stopping time. Then there exists a positive integer n such that, for almost all $\omega \in A_n$, $t(\omega) \leqq n$.*

PROOF OF THEOREM 2. As the proof of Theorem 1 makes evident, if any one of the three conditions fails to hold, there is a finite stopping time t for which $E|X_t| = \infty$. Suppose now that all three conditions obtain and that t is a finite stopping time. Choose n as in Lemma 3, and let s be the sup of t and n. Plainly, s is a stopping time and

$$\begin{aligned} E|X_t| \leqq E|X_s| &= \int_{\{s=n\}} |X_s| + \int_{\{s>n\}} |X_s| \leqq \int |X_n| + \int_{\{t>n\}} |X_t| \\ &= \int |X_n| + \int_{\{t>n\}} E\{|X_t| \mid \mathfrak{F}_n\} \\ &\leqq \int |X_n| + \int_{\{V_n<\infty\}} E\{|X_t| \mid \mathfrak{F}_n\} \\ &\leqq \int |X_n| + \int_{\{V_n<\infty\}} V_n < \infty. \end{aligned} \tag{17}$$

This completes the proof of Theorem 2.

Of course, $E|X_t|$ may be finite for all stopping times t, and yet there is a finite stopping time s with $E(X_s) \neq E(X_1)$. For example, this occurs when the X_n are positive and converge to 0 a.e. As the proof of Theorem 3 shows, there is no example essentially different from this one.

As is well known [Doob, 1953, p. 319], if $\sup_n E|X_n| < \infty$, then X_n converges almost surely. This implies its own generalization:

LEMMA 4. *Almost everywhere on* $\bigcup_j\{V_j < \infty\}$, *$X_n$ converges to a finite limit.*

(Incidentally, $\lim_n V_n = \lim_n |X_n|$ a.e. on $\bigcup_j\{V_j < \infty\}$. To see this, prove it first for uniformly integrable $\{X_n\}$. Second, argue that $\lim_n X_n = 0$ a.e. and $P(\lim_n V_n > 0) > 0$ imply $V_1 = \infty$. The general case follows, because any martingale with $V_1 < \infty$ is the sum of a uniformly integrable martingale and a martingale converging to 0 a.e.)

THEOREM 3. *$E(X_t) = E(X_1)$ for all finite stopping times t if and only if both of these conditions hold*:

(a) *$E|X_t| < \infty$ for all finite stopping times t; and*

(b) *For all n, the restriction of $X_{n+1}, X_{n+2}, \cdots$ to the event $\{V_n < \infty\}$ is uniformly integrable.*

PROOF OF THEOREM 3. Suppose first that (a) and (b) hold, and let t be any finite stopping time. According to Theorem 2, (7), (9), (11) and (16) hold, so n can be chosen in accordance with Lemma 3. As is easily verified, almost everywhere that $t \leqq n$,

$$E\{X_t \mid \mathfrak{F}_n\} = X_t . \tag{18}$$

Almost everywhere that $t > n$, $V_n < \infty$, so (b) implies that $X_{n+1}, X_{n+2}, \cdots$ is uniformly integrable on the event $t > n$. Consequently, almost everywhere that $t > n$,

$$E\{X_t \mid \mathfrak{F}_n\} = X_n . \tag{19}$$

Together, (18) and (19) say

$$E\{X_t \mid \mathfrak{F}_n\} = X_{t\wedge n} , \tag{20}$$

which implies

$$E(X_t) = E(X_{t\wedge n}). \tag{21}$$

Since $t \wedge n$ is bounded, the right side of (21) equals $E(X_1)$.

If (a) fails for a finite stopping time t, then plainly $E(X_t)$ is not well defined, and certainly is not equal to $E(X_1)$. So there remains only to consider the case that (a) holds and (b) fails. Suppose therefore that there is a least i such that $P\{V_i < \infty\} > 0$ and $X_{i+1}, X_{i+2}, \cdots$ is not uniformly integrable on $\{V_i < \infty\}$. It is convenient to suppose $P\{V_i < \infty\} = 1$, the general case being similar. By Lemma 4, X_n converges almost surely to a finite limit X_∞. Moreover, $E|X_\infty| < \infty$, because $E|X_\infty| \leqq E(V_i)$ by Fatou's lemma, and $E(V_i) < \infty$ by (6). Let

$$X_n^* = E\{X_\infty \mid \mathfrak{F}_n\}. \tag{22}$$

As shown in [Doob, 1953, VII, 4], $X_{i+1}^*, X_{i+2}^*, \cdots$ is uniformly integrable and converges to X_∞. There must be a least $j > i$ such that $P(X_j \neq X_j^*) > 0$. Suppose without real loss of generality that $P(X_j > X_j^*) > 0$. Define a finite stopping time t thus. If $X_j \leqq X_j^*$, let $t = j$. If $X_j > X_j^*$, let t be the least $n > j$ such that $X_n - X_n^* < X_j - X_j^*$. Then $X_t - X_t^* \leqq X_j - X_j^*$, and strict inequality holds with positive probability. Therefore,

$$E(X_t - X_t^*) < E(X_j - X_j^*). \tag{23}$$

Since $X_{i+1}^*, X_{i+2}^*, \cdots$ is uniformly integrable, $EX_t^* = EX_j^*$, so $E(X_t) < E(X_j) = E(X_1)$, completing the proof.

COROLLARY. *Suppose* $\sup_n E|X_n| < \infty$. *In order that* $E(X_t) = E(X_1)$ *for all finite stopping times t, it is necessary and sufficient that* $X_1, X_2, \cdots$ *be uniformly integrable.*

REFERENCES

DOOB, J. L. (1953). *Stochastic processes*. Wiley, New York.

KÖNIG, DENES (1936). *Theorie der endlichen und unendlichen Graphen*. Akademische Verlags-Gesellschaft, Leipzig.