MARTINGALE TRANSFORMS[1]

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1. Introduction. Let $f = (f_1, f_2, \cdots)$ be a martingale on a probability space $(\Omega, \mathfrak{A}, P)$. Let $d_1 = f_1$, $d_2 = f_2 - f_1, \cdots$ so that $f_n = \sum_{k=1}^n d_k$, $n \geqq 1$. It is convenient to say that $g = (g_1, g_2, \cdots)$ is a *transform* of f if $g_n = \sum_{k=1}^n v_k d_k$, where v_n is a real $\mathfrak{A}_{n-1}$-measurable function, $n \geqq 1$, and $\mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \cdots \subset \mathfrak{A}$ are σ-fields such that $\{f_n, \mathfrak{A}_n, n \geqq 1\}$ is a martingale. Note that g need not be a martingale. It is easy to see that g is a martingale if and only if $E|g_n|$ is finite for all n. This condition is satisfied, for example, if each v_n is bounded. Transforms of real (but not of extended real) submartingales may be defined similarly.

Such transforms, particularly in the case in which the v_n may take only 0 and 1 as possible values, have a long history and an interesting gambling interpretation. See Halmos [5], Doob [3], and some of the earlier work referred to in [5]. The emphasis is sometimes not on the transform g itself but on related sequences $\{g_{m_n}, n \geqq 1\}$ where the m_n are stopping times. Halmos's skipping theorem and Doob's optional stopping and sampling theorems, which give conditions assuring that $\{g_{m_n}\}$ is a martingale or a submartingale, are examples (g may equal f).

We prove here that, under mild conditions, martingale transforms converge almost everywhere. We also prove several related almost everywhere convergence theorems for martingales and establish a number of inequalities that follow from this convergence.

Inequalities and almost everywhere convergence results for the sequences $\{g_{m_n}\}$ mentioned above, whether or not they are martingales or submartingales, follow immediately.

Our first result (Theorem 1) is that a transform g of an L_1 bounded martingale f converges almost everywhere on the set where the maximal function v^* of the multiplier sequence $v = (v_1, v_2, \cdots)$ is finite. Here $v^*(\omega) = \sup_n |v_n(\omega)|$, the boundedness condition on f is that $\sup_n E|f_n| < \infty$, and g converging almost everywhere means that $\lim_{n\to\infty} g_n(\omega)$ exists and is finite for almost all ω.

Consider a gambler faced with the prospect of winning d_n dollars playing game n in an infinite sequence of games. Under $\sup_n E|f_n| < \infty$ and the usual condition of fairness (the expectation of d_n given $\mathfrak{A}_{n-1}$, his experience before playing game n, is 0), his sequence f of fortunes $f_n = \sum_{k=1}^n d_k$ converges almost surely to a finite limit f_∞. Is there anything that he can do to make his fate more interesting? Suppose that, under new rules, he can win $v_n d_n$ rather than d_n dollars, where he is allowed to choose v_n just before the nth game on the basis of his past experience $\mathfrak{A}_{n-1}$. Can he choose $v = (v_1, v_2, \cdots)$ subject to $v^* < \infty$ so that his new

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sequence g of fortunes $g_n = \sum_{k=1}^n v_k d_k$ converges to ∞, or, if this is not possible, so that g oscillates pleasantly? According to Theorem 1, he cannot; his new sequence will also converge almost surely to a finite limit g_∞. However, even if $v^* \leqq 1$, g can be more interesting than f in at least one respect: $E|f_\infty|$ must be finite since, by Fatou, $E|f_\infty| \leqq \liminf_n E|f_n|$; on the other hand $E|g_\infty|$ can be infinite as the following simple example shows.

Let Ω be the set of positive integers and P satisfy $P(\{k\}) = 1/k - 1/(k+1)$, $k \in \Omega$. Let $f_n(k) = n$ if $n < k$, $= -1$, if $n \geqq k$. It is easily seen that $f = (f_1, f_2, \cdots)$ is an L_1 bounded martingale with limit function $f_\infty = -1$. Let g be the transform of f under the constant multiplier sequence $v = (1, -1, 1, -1, \cdots)$. Since d, the difference sequence of f, satisfies $d_n(k) = 1$ if $n < k$, $= -k$ if $n = k$, $=0$ if $n > k$, we have that $g_n(k) = i_n$ if $n < k$, $= i_{k-1} + (-1)^k k$ if $n \geqq k$, where $i_{2n} = 0, n \geqq 0, i_{2n-1} = 1, n \geqq 1$. Thus, g_∞, the limit function of g, satisfies $P(|g_\infty| \geqq k) \geqq P(\{k, k+1, \cdots\}) = 1/k$ implying that $E|g_\infty| = \infty$ and g is not L_1 bounded.

Using Theorem 1 and a stopping time argument, we prove (Theorem 3) that if f and g are martingales relative to the same sequence of σ-fields (g is not necessarily a transform of f), f is L_1 bounded, and $S_n(g) \leqq S_n(f)$, $n \geqq 1$, then g converges almost everywhere. Here, and throughout the paper, $S_n(f) = [\sum_{k=1}^n d_k^2]^{\frac{1}{2}}$ where d is the difference sequence of f. These inequalities are satisfied, of course, if $|e_n| \leqq |d_n|$, $n \geqq 1$, where e is the difference sequence of g. This is the case if g is the transform of f under a multiplier sequence v satisfying $v^* \leqq 1$. However, simple examples show that this relationship may hold between two martingales f and g without g being a transform of f. Using a result of [2], we show (Theorem 6) that, under the same conditions,

$$\lambda P(g^* > \lambda) \leqq M \sup_n E|f_n|, \qquad \lambda > 0,$$

where $g^*(\omega) = \sup_n |g_n(\omega)|$ and M is a real number that satisfies the inequality for all probability spaces, f, g, and λ. We also prove an upcrossing inequality for g (Theorem 7).

Let $S(f) = [\sum_{n=1}^\infty d_n^2]^{\frac{1}{2}}$. Austin [1] has shown that if f is an L_1 bounded martingale, then $S(f)$ is finite almost everywhere. Here we prove (Theorem 8) that

$$\lambda P(S(f) > \lambda) \leqq M \sup_n E|f_n|, \qquad \lambda > 0,$$

where M is the same number as before. Furthermore, (Theorem 9) if $1 < p < \infty$ and f is a martingale, then

$$M_p E S_n(f)^p \leqq E|f_n|^p \leqq N_p E S_n(f)^p$$

where M_p and N_p are positive real numbers, depending on p but not on the probability space, f, or n. Some special cases of these inequalities have been known for a long time (see Section 3), for example, in the case that d is an independent sequence satisfying $Ed_n = 0$, $n \geqq 1$.

In this paper, we consider only real martingales indexed by the set of positive integers. With slight modifications, most of the results, the upcrossing theorem being an obvious exception, carry over to complex martingales. Results for other index sets, such as the set of integers or the set of nonpositive integers, also easily follow. Some of the results, but not all, carry over to the submartingale case. Roughly, those that refer to transforms usually do carry over to the submartingale case; those that refer to the $S_n(f)$ usually do not. Whether any particular martingale result presented here does or does not carry over is usually fairly easy to check.

If $f = (f_1, f_2, \cdots)$ is any sequence of real functions, let f^*, the maximal function of the sequence f, be defined by $f^*(\omega) = \sup_n |f_n(\omega)|$. This notation will be used throughout the paper.

2. Almost everywhere convergence.

THEOREM 1. *Suppose that g is a transform of an L_1 bounded martingale f. Then g converges almost everywhere on the set where the maximal function v^* of the multiplier sequence v is finite.*

This result implies the corresponding result for submartingales and, more generally, for all sequences f of the form $f = f' + f''$, where f' is an L_1 bounded martingale and f'' is a nondecreasing almost everywhere convergent sequence of real functions.

PROOF. We proceed in steps.

(i) *If g is a transform of an L_2 bounded martingale f and $v^* \leqq 1$, then $Eg_n^2 \leqq Ef_n^2$, $n \geqq 1$, and g converges almost everywhere.* Here g is a martingale and $Ef_n^2 = E(\sum_{k=1}^n d_k)^2 = \sum_{k=1}^n E\,d_k^2 \geqq \sum_{k=1}^n Ee_k^2 = Eg_n^2$, using the orthogonality of the difference sequences d and e of f and g, respectively. Therefore, g is also an L_2 bounded martingale, hence converges almost everywhere.

(ii) *If g is a transform of a uniformly bounded submartingale f and $v^* \leqq 1$, then g converges almost everywhere.* Since adding the same number to each term of f does not change d_n for $n \geqq 2$, in the proof we may and do assume that $f \geqq 0$. Therefore, $\sum_{k=1}^n Ed_k^2 \leqq Ef_n^2$, $n \geqq 1$, since, for $n \geqq 2$, $Ef_{n-1}\,d_n = E[f_{n-1}E(d_n \mid \mathcal{A}_{n-1})] \geqq 0$ implying that $Ef_n^2 = E(f_{n-1} + d_n)^2 \geqq Ef_{n-1}^2 + Ed_n^2$. Let $\hat{f} = (\hat{f}_1, \hat{f}_2, \cdots)$ and $\hat{g} = (\hat{g}_1, \hat{g}_2, \cdots)$ be defined by $\hat{f}_n = \sum_{k=1}^n \hat{d}_k$, $\hat{g}_n = \sum_{k=1}^n v_k\,\hat{d}_k$, $n \geqq 1$, where $\hat{d}_1 = d_1$ and $\hat{d}_n = d_n - E(d_n \mid \mathcal{A}_{n-1})$, $n \geqq 2$. Then $\hat{f}$ is a martingale and $\hat{g}$ is a transform of $\hat{f}$. Since $E\hat{d}_n^2 = E[d_n - E\,(d_n \mid \mathcal{A}_{n-1})]^2 \leqq Ed_n^2$, $n \geqq 2$, we have that $E\hat{f}_n^2 = \sum_{k=1}^n E\hat{d}_k^2 \leqq \sum_{k=1}^n Ed_k^2 \leqq Ef_n^2$, $n \geqq 1$, so that $\hat{f}$ is L_2 bounded. By (i), $\hat{g}$ converges almost everywhere; f and $\hat{f}$ converge almost everywhere by the classical submartingale convergence theorem. Therefore, $\sum_{k=2}^n E(d_k \mid \mathcal{A}_{k-1}) = f_n - \hat{f}_n$ converges almost everywhere as $n \to \infty$, and, since each term in the sum is nonnegative almost everywhere, $\sum_{k=2}^n v_kE(d_k \mid \mathcal{A}_{k-1})$ does also. Since $g_n = \hat{g}_n + \sum_{k=2}^n v_kE(d_k \mid \mathcal{A}_{k-1})$, $n \geqq 2$, the desired result follows.

(iii) *If g is a transform of an L_1 bounded martingale f and $v^* \leqq 1$, then g converges almost everywhere.* By a result due to Krickeberg [7], there are nonnegative martingales f' and f'' relative to the original sequence of σ-fields such that $f = f' - f''$. Clearly, $g = g' - g''$, where g' and g'' are the transforms of f' and f'', respectively, under the original multiplier sequence v. Therefore, in what follows, we may and do assume that $f \geqq 0$. Let $c > 0$. Then $\hat{f}_n = -\min(f_n, c)$, $n \geqq 1$, defines a uniformly bounded submartingale $\hat{f}$. Let $\hat{g}$ be the transform of $\hat{f}$ under $-v$. By (ii), $\hat{g}$ converges almost everywhere and, since $g(\omega) = \hat{g}(\omega)$ if $f^*(\omega) = \sup_n |f_n(\omega)| < c$, we have that g converges almost everywhere on the set $\{f^* < c\}$. Now $P(f^* < \infty) = 1$, which follows from the fact that f converges almost everywhere. Therefore, letting $c \to \infty$, we obtain the desired result.

The proof of Theorem 1 may now be completed as follows. Let $c > 0$. Let $\hat{v}_n(\omega) = v_n(\omega)$ if $|v_n(\omega)| < c$, $=0$ otherwise, $n \geqq 1$. Let $\hat{g}$ be the transform of f under the uniformly bounded multiplier sequence $\hat{v} = (\hat{v}_1, \hat{v}_2, \cdots)$. Clearly, (iii) implies that $\hat{g}$ converges almost everywhere. Since $g(\omega) = \hat{g}(\omega)$ if $v^*(\omega) < c$, we have that g converges almost everywhere on $\{v^* < c\}$. Therefore, g converges almost everywhere on $\{v^* < \infty\}$ and the proof is complete.

Remark. One consequence of Theorem 1 is that if f is an L_1 bounded martingale, then, for all possible choices of $+$ and $-$, the series $\sum_{n=1}^{\infty} \pm d_n$ converges almost everywhere. Here, as usual, d is the difference sequence of f. This implies, using a standard Fubini argument, that $S(f)^2 = \sum_{n=1}^{\infty} d_n^2 < \infty$ almost everywhere. Thus, Theorem 1 provides another proof of Austin's result [1]. Austin proves that $S(f)$ is finite by showing that the integral of $S(f)^2$ over the set $\{f^* < c\}$ is finite for all $c > 0$. (J. L. Doob has noticed that Theorem 1 provides still another proof of Austin's result: Let g be the transform of an L_1 bounded martingale f under the multiplier sequence $(0, f_1, f_2, \cdots)$. Then $S_n(f)^2 = f_n^2 - 2g_n$, $n \geqq 1$, and both f and g converge almost everywhere. Austin's result follows.)

Theorem 2. *If f is a martingale such that $ES(f) < \infty$, then f converges almost everywhere.*

Proof. Let $r_1, r_2, \cdots$ be the Rademacher functions on the unit interval. Their properties relevant here are that they take only 1 and -1 as possible values, they are independent relative to Lebesgue measure, and $\int_0^1 r_n(t)\, dt = 0$, $n \geqq 1$. For each t in the unit interval, $\{\sum_{k=1}^{n} r_k(t) d_k, n \geqq 1\}$ is a martingale from which follows that $E|\sum_{k=1}^{n} r_k(t) d_k|$ is nondecreasing in n. Since

$$\int_0^1 E\,|\textstyle\sum_{k=1}^{n} r_k(t) d_k|\, dt \leqq E[\int_0^1 |\sum_{k=1}^{n} r_k(t) d_k|^2\, dt]^{\frac{1}{2}} = ES_n(f),$$

we have, by the monotone convergence theorem, that

$$\int_0^1 \sup_n E\,|\textstyle\sum_{k=1}^{n} r_k(t) d_k|\, dt \leqq ES(f)$$

implying that, for some t,

$$\sup_n E\,|\textstyle\sum_{k=1}^{n} r_k(t) d_k| \leqq ES(f) < \infty.$$

For this t, $g_n = \sum_{k=1}^{n} r_k(t) d_k$ defines an L_1 bounded martingale $g = (g_1, g_2, \cdots)$ and f is the transform of g under the (constant) multiplier sequence $(r_1(t), r_2(t), \cdots)$. Therefore, by Theorem 1, f converges almost everywhere.

Theorem 3. *Suppose that f and g are martingales relative to the same sequence of*

σ-fields. If f is L_1 bounded and $S_n(g) \leqq S_n(f)$, $n \geqq 1$, then g converges almost everywhere.

PROOF. Let $c > 0$ and $m = \inf \{n: |f_n| \geqq c$ or $S_n(f) \geqq c\}$ where $\inf \phi = \infty$. Then $ES_m(f) < \infty$ where $S_\infty = S$. For $S_m(f) \leqq c + |d_m| \leqq 2c + |f_m|$ on $\{m < \infty\}$, $S_m(f) \leqq c$ on $\{m = \infty\}$, and letting $m_n = \min(m, n)$, we have that $\int_{\{m<\infty\}} |f_m| \leqq \liminf_n \int_{\{m<\infty\}} |f_{m_n}| \leqq \sup_n E|f_{m_n}| \leqq \sup_n E|f_n| < \infty$, in which $E|f_{m_n}| \leqq E|f_n|$ follows from the fact that $\{|f_n|, n \geqq 1\}$ is a submartingale. Let $\hat{g}_n = g_{m_n}$. Then $\hat{g} = (\hat{g}_1, \hat{g}_2, \cdots)$ is a martingale by the optional stopping theorem [3]. Here we have used our assumption that f and g are martingales relative to the same sequence of σ-fields. Clearly, $S(\hat{g}) = S_m(g) \leqq S_m(f)$. Therefore, $ES(\hat{g}) < \infty$ and, by Theorem 2, $\hat{g}$ converges almost everywhere. On the set $\{f^* < c, S(f) < c\}$, $m = \infty$ and $g = \hat{g}$. Since both f^* and $S(f)$ are finite almost everywhere (see Austin [1] or the above remark), the almost everywhere convergence of g follows.

Considering $f_n = \sum_{k=1}^n r_k/k$ and $g_n = \sum_{k=1}^n 1/k$ where $r_1, r_2, \cdots$ are the Rademacher functions, we see that Theorem 3 does not carry over to the submartingale case.

THEOREM 4. *Suppose that f is a martingale with difference sequence d satisfying $Ed^* < \infty$. Then f converges almost everywhere on the set where $S(f) < \infty$, and $S(f) < \infty$ almost everywhere on the set where $\sup_n f_n < \infty$. More generally, if g is a transform of f under a multiplier sequence v, then g converges almost everywhere on the set where $S(g) < \infty$ and $v^* < \infty$, and $S(g) < \infty$ almost everywhere on the set where $\sup_n g_n < \infty$ and $v^* < \infty$.*

By a result of Doob ([3], page 320), the condition $Ed^* < \infty$ on a martingale f implies that the two sets $\{f$ converges$\}$ and $\{\sup_n f_n < \infty\}$ are equivalent (symmetric difference has measure 0). By Theorem 4, under the same condition, $\{S(f) < \infty\}$ is a third set equivalent to each, and, if $v^* < \infty$ almost everywhere, the same three sets for the transform g are equivalent.

PROOF. Let $c > 0$. Let $m = \inf \{n: S_n(f) \geqq c\}$, $m_n = \min(m, n)$, $\hat{f}_n = f_{m_n}$. Then $\hat{f} = (\hat{f}_1, \hat{f}_2, \cdots)$ is a martingale and $S(\hat{f}) \leqq c + d^*$. Therefore, $ES(\hat{f}) < \infty$ and, by Theorem 2, $\hat{f}$ converges almost everywhere. On the set $\{S(f) < c\}$, $m = \infty$ and $f = \hat{f}$ so that f converges almost everywhere on this set, hence almost everywhere on $\{S(f) < \infty\}$.

To prove the second part of the first statement, let $c > 0$, $m = \inf \{n: |f_n| \geqq c\}$, $m_n = \min(m, n)$, $\hat{f}_n = f_{m_n}$. Since $|\hat{f}_n| \leqq c + d^*$, $\hat{f} = (\hat{f}_1, \hat{f}_2, \cdots)$ is an L_1 bounded martingale and, by Austin's result, $S(\hat{f})$ is finite almost everywhere. On $\{f^* < c\}$, $m = \infty$ and $S(f) = S(\hat{f})$. Therefore, $S(f) < \infty$ almost everywhere on $\{f^* < \infty\}$. But, by Doob's result mentioned above, this set is equivalent to $\{\sup_n f_n < \infty\}$.

Now suppose that g is a transform of f under a multiplier sequence v. Let $c > 0$, $\hat{v}_n(\omega) = v_n(\omega)$ if $|v_n(\omega)| < c$, $=0$ otherwise, and $\hat{g}_n = \sum_{k=1}^n \hat{v}_k d_k$. Then $\hat{g} = (\hat{g}_1, \hat{g}_2, \cdots)$ is a martingale with difference sequence $\hat{e}$ satisfying $E\hat{e}^* \leqq cE\, d^* < \infty$. Therefore, by the first statement of the theorem, $\hat{g}$ converges almost everywhere on the set where $S(\hat{g}) < \infty$. Since $g(\omega) = \hat{g}(\omega)$ if $v^*(\omega) < c$ and $S(\hat{g}) \leqq S(g)$ we have that g converges almost everywhere on the set where

$S(g) < \infty$ and $v^* < c$. Also, $S(\hat{g}) < \infty$ almost everywhere on the set where $\sup_n \hat{g}_n < \infty$. Therefore, $S(g) = S(\hat{g}) < \infty$ almost everywhere on $\{\sup_n \hat{g}_n < \infty, v^* < c\} = \{\sup_n g_n < \infty, v^* < c\}$. Let $c \to \infty$ to complete the proof.

The final part of Theorem 4 was inspired by the possibility of the following application. By a result due to Gundy [4], if g is the sequence of 2^nth partial sums of a Walsh series, then $\{g$ converges$\}$ and $\{S(g) < \infty\}$ are equivalent. As Gundy has noticed, commenting on the present work, such a g is the transform of a martingale f with difference sequence d satisfying $|d_n(\omega)| = 1, \omega \varepsilon \Omega, n \geqq 1$ ($d_1 = 1$ and $d_2, d_3, \cdots$ are Rademacher functions). Using the fact that here $S(g) = (\sum_{n=1}^{\infty} v_n^2)^{\frac{1}{2}} \geqq v^*$ and that $v_n \to 0$, hence $v^* < \infty$, if g converges, where v is the multiplier sequence, we see that Gundy's result follows from Theorem 4.

THEOREM 5. *Suppose that f is an L_1 bounded martingale with difference sequence d. If A is an atom, then $\sum_{n=1}^{\infty} |d_n| < \infty$ almost everywhere on A.*

PROOF. There is a real number sequence $a = (a_1, a_2, \cdots)$ such that $d = a$ almost everywhere on A, otherwise there would be a subset B of A in $\mathfrak{A}$ such that $0 < P(B) < P(A)$, contradicting the assumption that A is an atom. Let $v_n(\omega) = 1$ if $a_n \geqq 0$, $= -1$ if $a_n < 0$, $\omega \varepsilon \Omega$. By Theorem 1, $g_n = \sum_{k=1}^{n} v_k d_k$ converges almost everywhere as $n \to \infty$, and since $g_n = \sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n} |d_k|$ almost everywhere on A, the result follows.

3. Inequalities.

THEOREM 6. *There is a real number M such that if f and g are martingales relative to the same sequence of σ-fields and $S_n(g) \leqq S_n(f)$, $n \geqq 1$, then*

$$\lambda P(g^* > \lambda) \leqq M \sup_n E|f_n|, \qquad \lambda > 0.$$

PROOF. If a real number M satisfies this inequality for the Lebesgue unit interval, then M satisfies the inequality for every probability space. Therefore, in the proof we may and do assume that our probability space is the Lebesgue unit interval. Let C be the collection of martingales g on this space such that g is in C if and only if there is a martingale f with $\sup_n E|f_n| \leqq 1$, such that f and g are martingales relative to the same sequence of σ-fields and $S_n(g) \leqq S_n(f)$, $n \geqq 1$. The collection C has the following two properties: (i) If g is in C, then its maximal function g^* is finite on a set of positive measure. (ii) If $g_1, g_2, \cdots$ are in C, then there are independent $\hat{g}_1, \hat{g}_2, \cdots$ in C such that g_k^* and $\hat{g}_k^*$ have the same distribution, $k \geqq 1$, and if $a_1, a_2, \cdots$ are real numbers such that $\sum_{k=1}^{\infty} |a_k| = 1$, then $\sum_{k=1}^{\infty} a_k \hat{g}_k$ is in C. That is, letting $\hat{g}_k = (\hat{g}_{k1}, \hat{g}_{k2}, \cdots)$, there is a $\bar{g}$ in C such that $\sum_{j=1}^{k} a_j \hat{g}_{jn} \to \bar{g}_n$ almost everywhere as $k \to \infty$, $n \geqq 1$. Property (i) is a consequence of Theorem 3 and property (ii) will be verified below. Given any collection C of sequences satisfying (i) and (ii), there is a real number M such that

$$\lambda P(g^* > \lambda) \leqq M, \qquad \lambda > 0, \qquad g \varepsilon C.$$

This is a special case of Theorem 2 of [2], and establishes the desired inequality if $\sup_n E|f_n| = 1$. For $0 < \sup_n E|f_n| < \infty$, the inequality immediately follows. If $\sup_n E|f_n| = 0$ or ∞, the inequality is trivially true.

We now show that C has property (ii). Suppose that $g_1, g_2, \cdots$ are in C and that f_k is a martingale related to g_k as f is to g in the definition of C. Let $(f_1', g_1'), (f_2', g_2'), \cdots$ be independent pairs of related martingales such that (f_k, g_k) and (f_k', g_k') have the same distribution, $k \geqq 1$. Choose distinct positive integers p_{kj} satisfying $p_{k1} < p_{k2} < \cdots$ and let $\hat{d}_{kn} = d_{kj}'$ if $n = p_{kj}$ for some j, $=0$ otherwise, where $d_k' = (d_{k1}', d_{k2}', \cdots)$ is the difference sequence of f_k'. Let $\hat{f}_{kn} = \sum_{j=1}^{n} \hat{d}_{kj}$ and define $\hat{e}_{kn}$ and $\hat{g}_{kn}$ similarly using g_k' and the p_{kj}. Then $\hat{f}_k = (\hat{f}_{k1}, \hat{f}_{k2}, \cdots)$ and $\hat{g}_k = (\hat{g}_{k1}, \hat{g}_{k2}, \cdots)$ are martingales relative to $\{\mathcal{A}_{kn}, n \geqq 1\}$ where $\mathcal{A}_{kn}$ is the σ-field generated by $\hat{f}_{k1}, \cdots, \hat{f}_{kn}$ and $\hat{g}_{k1}, \cdots, \hat{g}_{kn}$. Also, $\sup_n E|\hat{f}_{kn}| \leqq 1$ and $S_n(\hat{g}_k) \leqq S_n(\hat{f}_k)$, $n \geqq 1$. Therefore, $\hat{g}_1, \hat{g}_2, \cdots$ are in C. Clearly, they are independent and g_k^* and $\hat{g}_k^*$ have the same distribution. Suppose that $a_1, a_2, \cdots$ are real numbers such that $\sum_{k=1}^{\infty} |a_k| = 1$. Let $\bar{f}_n = \sum_{k=1}^{\infty} a_k \hat{f}_{kn}$ and $\bar{g}_n = \sum_{k=1}^{\infty} a_k \hat{g}_{kn}$. Each series contains only a finite number of nonzero terms since $\{p_{ij} : j \geqq 1\}$ and $\{p_{kj} : j \geqq 1\}$ are disjoint if $i \neq k$. Let $\mathcal{A}_n$ be the smallest σ-field containing $\bigcup_{k=1}^{\infty} \mathcal{A}_{kn}$. Then $\bar{f} = (\bar{f}_1, \bar{f}_2, \cdots)$ and $\bar{g} = (\bar{g}_1, \bar{g}_2, \cdots)$ are martingales relative to $\{\mathcal{A}_n, n \geqq 1\}$ since almost everywhere

$$\begin{aligned} E(\bar{f}_{n+1} \mid \mathcal{A}_n) &= \textstyle\sum_{k=1}^{\infty} a_k E(\hat{f}_{k,n+1} \mid \mathcal{A}_n) \\ &= \textstyle\sum_{k=1}^{\infty} a_k E(\hat{f}_{k,n+1} \mid \mathcal{A}_{kn}) \\ &= \textstyle\sum_{k=1}^{\infty} a_k \hat{f}_{kn} \\ &= \bar{f}_n \end{aligned}$$

with a similar calculation for $\bar{g}$. Also, $E|\bar{f}_n| \leqq \sum_{k=1}^{\infty} |a_k| E|\hat{f}_{kn}| \leqq 1$ and

$$\begin{aligned} S_n(\bar{f})^2 &= \textstyle\sum_{j=1}^{n} (\sum_{k=1}^{\infty} a_k \hat{d}_{kj})^2 = \sum_{j=1}^{n} \sum_{k=1}^{\infty} a_k^2 \hat{d}_{kj}^2 \\ &= \textstyle\sum_{k=1}^{\infty} a_k^2 S_n(\hat{f}_k)^2 \geqq \sum_{k=1}^{\infty} a_k^2 S_n(\hat{g}_k)^2 = S_n(\bar{g})^2, \qquad n \geqq 1. \end{aligned}$$

Therefore, $\bar{g} = \sum_{k=1}^{\infty} a_k \hat{g}_k$ is in C and the proof is complete.

We now derive an upcrossing inequality for the martingales g of Theorems 3 and 6. The inequality follows from Theorem 6 with the use of a slight modification of the combinatorial part of Snell's proof of the upcrossing theorem for submartingales [3].

Let $a < b$ be real numbers. If $x = (x_1, x_2, \cdots)$ is a real number sequence, let $u_1(x) = 1$, $u_{n+1}(x) = 1$ if $x_n \geqq b$, $= u_n(x)$ if $a < x_n < b$, $=0$ if $x_n \leqq a$, $U_n^{ab}(x) = \sum_{k=1}^{n} [u_{k+1}(x) - u_k(x)]^+$, and $U^{ab}(x) = \sup_n U_n^{ab}(x)$. Here, $t^+ = \max(0, t)$ for t real. Since $(b - a)[u_{n+1}(x) - u_n(x)]^+ \leqq (x_n - a)[u_{n+1}(x) - u_n(x)]$, we have that

$$\begin{aligned} (b - a)U_n^{ab}(x) &\leqq \textstyle\sum_{k=1}^{n} (x_k - a)[u_{k+1}(x) - u_k(x)] \\ &= u_{n+1}(x)x_n - \textstyle\sum_{k=1}^{n} u_k(x)(x_k - x_{k-1}) + a[1 - u_{n+1}(x)] \\ &\leqq |x_n| + |\textstyle\sum_{k=1}^{n} u_k(x)(x_k - x_{k-1})| + a^+ \end{aligned}$$

where $x_0 = 0$.

THEOREM 7. *There is a real number M such that if f and g are martingales rela-*

tive to the same sequence of σ-fields and $S_n(g) \leqq S_n(f)$, $n \geqq 1$, *then*

$$\lambda P((b - a)U^{ab}(g) - a^+ > \lambda) \leqq M \sup_n E|f_n|, \qquad \lambda > 0, \qquad a < b.$$

PROOF. Let $a < b$ and $h_n = \sum_{k=1}^n w_k e_k$ where $g_n = \sum_{k=1}^n e_k$, $w_n = u_n(g)$, and u_n is the function (depending on a and b) defined above. Then $h = (h_1, h_2, \cdots)$ is the transform of g under the multiplier sequence $w = (w_1, w_2, \cdots)$ and $w^* \leqq 1$, hence h is a martingale relative to the same sequence of σ-fields as g and f, and $S_n(h) \leqq S_n(g) \leqq S_n(f)$, $n \geqq 1$. Therefore, by Theorem 6, $\lambda P(h^* > \lambda) \leqq M \sup_n E|f_n|$, $\lambda > 0$, and the same inequality holds for g^*. Now $(b - a)U_n^{ab}(g) - a^+ \leqq |g_n| + |h_n| \leqq g^* + h^*$, hence

$$\begin{aligned}\lambda P((b - a)U^{ab}(g) - a^+ > \lambda) &\leqq \lambda P(g^* > \lambda/2) + \lambda P(h^* > \lambda/2)\\ &\leqq 4M \sup_n E|f_n|, \qquad \lambda > 0,\end{aligned}$$

the desired result.

REMARKS. Suppose that f, as above, is L_1 bounded. Then $EU^{ab}(f)$ is finite by the standard upcrossing theorem. On the other hand $EU^{ab}(g)$ can be infinite (although $EU^{ab}(g)^p$ is finite for $0 < p < 1$ as the above inequality easily implies). For the g defined in the example described in Section 1, we have that $P(U^{01}(g) \geqq k) = P(g_n = i_n, n < 2k + 2) = 1/(2k + 2)$, implying that $EU^{01}(g) = \infty$.

If the statements of Theorems 6 and 7 are modified by relaxing the martingale assumption on g to the assumption that g is a submartingale, then they are no longer true. However, if g is a transform of a submartingale f under a multiplier sequence v satisfying $v^* \leqq 1$, then the inequalities of Theorems 6 and 7 do hold for g (with possibly a different M). This follows at once from Theorem 6 and the proof of Theorem 7.

THEOREM 8. *If f is a martingale then*

$$\lambda P(S(f) > \lambda) \leqq M \sup_n E|f_n|, \qquad \lambda > 0,$$

and

$$\lambda P(f^* > \lambda) \leqq MES(f), \qquad \lambda > 0,$$

where M is the same number as in Theorem 6.

PROOF. In the proof we may and do suppose that there is a function r with values in $\{-1, 1\}$ such that r and f are independent and $P(r = 1) = \frac{1}{2}$. For if no such r exists we can always consider another space (for example, the obvious product space) on which is defined a martingale with the same distribution as f and for which such a function does exist. Let k be a positive integer. Let $g_n = 0$ if $n < k$, $= rS_k(f)$ if $n \geqq k$. Then both f and $g = (g_1, g_2, \cdots)$ are martingales relative to $\{\mathcal{A}_n, n \geqq 1\}$ where $\mathcal{A}_n$ is generated by $f_1, \cdots, f_n$ if $n < k$, by $r, f_1, \cdots, f_n$ if $n \geqq k$, $S_n(g) \leqq S_n(f)$, $n \geqq 1$, and $g^* = S_k(f)$. Thus, by Theorem 6, $\lambda P(S_k(f) > \lambda) = \lambda P(g^* > \lambda) \leqq M \sup_n E|f_n|$, $\lambda > 0$. Letting $k \to \infty$ gives the desired result.

To prove the second inequality, we use the fact, established in the proof of

Theorem 2, that f is a transform of a martingale g under a multiplier sequence uniformly bounded by 1 such that $\sup_n E|g_n| \leqq ES(f)$. Therefore, applying Theorem 6 to f, we have that $\lambda P(f^* > \lambda) \leqq M \sup_n E|g_n| \leqq MES(f)$, completing the proof.

In the special case that f is the sequence of 2^nth partial sums of a Walsh series (clearly such an f is a martingale; see [6], [2], [4]), the first inequality (essentially) of Theorem 8 has been obtained by Yano [10].

THEOREM 9. *Let $1 < p < \infty$. There are positive real numbers M_p and N_p such that if f is a martingale then*

$$M_p ES_n(f)^p \leqq E|f_n|^p \leqq N_p ES_n(f)^p, \qquad n \geqq 1.$$

This contains the inequality obtained by Paley [9] in the case that f is the sequence of 2^nth partial sums of a Walsh series, and the inequality obtained by Marcinkiewicz and Zygmund [8] in the case that d, the difference sequence of f, is independent and satisfies $Ed_n = 0$, $n \geqq 1$. (In the latter case, the inequality is also true for $p = 1$.) Actually, Paley's proof of his result can be carried over to the general martingale case with only slight modifications. (Richard F. Gundy is perhaps the first person to have noticed this.) Therefore, our proof below may be viewed as an alternative to Paley's proof. Although using some of the same devices, it rests on two results unavailable to Paley: Marcinkiewicz's interpolation theorem and the fact that, under mild conditions, martingale transforms g satisfy the inequality of Theorem 6.

If $1 < p < \infty$ and f is L_p bounded, by letting $n \to \infty$, one obtains from Theorem 9 the obvious inequality for $S(f)$ and f_∞, where f_∞ is the almost everywhere (and L_p) limit of f. Note that $ES(f)$ can be infinite even if f is L_1 bounded: in the example of Section 1, $P(S(f) \geqq k) \geqq P(\{k, k+1, \cdots\}) = 1/k$, $k \geqq 1$.

PROOF. Let $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$ be σ-fields and v_n an $\mathcal{A}_{n-1}$-measurable function into the interval $[-1, 1]$, $n \geqq 1$. Let $D_1 = E(\cdot \mid \mathcal{A}_1)$, $D_n = E(\cdot \mid \mathcal{A}_n) - E(\cdot \mid \mathcal{A}_{n-1})$, $n \geqq 2$, and $T_n = \sum_{k=1}^n v_k D_k$. Note that if f_∞ is integrable, then $f_n = E(f_\infty \mid \mathcal{A}_n)$ defines a martingale $f = (f_1, f_2, \cdots)$ and its transform g under v satisfies $g_n = T_n f_\infty$ almost everywhere, $n \geqq 1$. By part (i) of the proof of Theorem 1, $E|T_n f_\infty|^2 = Eg_n^2 \leqq Ef_n^2 \leqq Ef_\infty^2$, if the last expectation is finite, and, by Theorem 6, $\lambda P(|T_n f_\infty| > \lambda) = \lambda P(|g_n| > \lambda) \leqq \lambda P(g^* > \lambda) \leqq M \sup_n E|f_n| \leqq ME|f_\infty|$, $\lambda > 0$. Therefore, by Marcinkiewicz's interpolation theorem ([11], Volume II), if $1 < p \leqq 2$, there is a positive real number M_p such that if f_∞ is an integrable function then $E|T_n f_\infty|^p \leqq M_p E|f_\infty|^p$. Using the basic properties of conditional expectations, it is easy to check that the restriction of T_n to L_2 is self-adjoint. This implies that the inequality also holds for $2 \leqq p < \infty$ since one may take $M_p = M_q$ where $1/p + 1/q = 1$. Moreover, M_p depends only on p and may be chosen not to depend on the probability space, the sequence of σ-fields, the multiplier sequence v, the function f_∞, or the integer n. Note that if f is any martingale relative to the above sequence of σ-fields and g is its transform under the above v, then $T_n f_n = g_n$ almost everywhere. Therefore,

$$E|g_n|^p \leqq M_p E|f_n|^p, \qquad n \geqq 1.$$

Let $r_1, r_2, \cdots$ be the Rademacher functions (see the proof of Theorem 2). We use Khintchine's inequality: if $0 < p < \infty$, there are positive real numbers A_p and B_p such that if $a = (a_1, a_2, \cdots)$ is a real number sequence then

$$A_p(\sum_{k=1}^n a_k^2)^{p/2} \leqq \int_0^1 |\sum_{k=1}^n a_k r_k(t)|^p \, dt \leqq B_p(\sum_{k=1}^n a_k^2)^{p/2}, \qquad n \geqq 1.$$

Now let f be a martingale with difference sequence d. For each t, $\{\sum_{k=1}^n r_k(t)d_k, n \geqq 1\}$ is a transform of f and conversely. Therefore, for $1 < p < \infty$,

$$\begin{aligned} A_p M_p^{-1} E S_n(f)^p &\leqq M_p^{-1} E \int_0^1 |\sum_{k=1}^n r_k(t) d_k|^p \, dt \\ &= \int_0^1 M_p^{-1} E \, |\sum_{k=1}^n r_k(t) d_k|^p \, dt \\ &\leqq E\,|f_n|^p \\ &\leqq B_p M_p E S_n(f)^p, \qquad n \geqq 1, \end{aligned}$$

the desired result.

THEOREM 10. *There is a real number M such that if f is a martingale then*

$$E S_n(f) \leqq M E\,|f_n| \log^+ |f_n| + M, \qquad n \geqq 1,$$

and

$$E\,|f_n| \leqq M E S_n(f) \log^+ S_n(f) + M, \qquad n \geqq 1.$$

For real t, $\log^+ t = 0$ if $t < 1$, $= \log t$ if $t \geqq 1$.

PROOF. This, as the previous theorem, follows from Theorem 6 by interpolation. Using an argument similar to that of the previous proof together with an interpolation theorem ([11], Volume II, page 118), we have that there exists a real number M such that if f is a martingale and g is a transform of f under a multiplier sequence v satisfying $v^* \leqq 1$ then

$$E\,|g_n| \leqq M E\,|f_n| \log^+ |f_n| + M, \qquad n \geqq 1.$$

Therefore, if $r_1, r_2, \cdots$ are the Rademacher functions and t is in the unit interval, then

$$E\,|\sum_{k=1}^n r_k(t) d_k| \leqq M E\,|f_n| \log^+ |f_n| + M,$$

and, again using Khintchine's inequality,

$$A_1 E S_n(f) \leqq E \int_0^1 |\sum_{k=1}^n r_k(t)\, d_k| dt \leqq M E\,|f_n| \log^+ |f_n| + M.$$

To prove the second inequality of Theorem 10, we use the fact ([11], Volume II, page 235) that there is a real number B such that if a is a real number sequence then

$$\int_0^1 |\sum_{k=1}^n a_k r_k(t)| \log^+ |\sum_{k=1}^n a_k r_k(t)| \, dt \leqq B(\sum_{k=1}^n a_k^2)^{\frac{1}{2}} \log^+ (\sum_{k=1}^n a_k^2)^{\frac{1}{2}} + B.$$

Since f is a transform of $\{\sum_{k=1}^n r_k(t) d_k, n \geqq 1\}$,

$$E\,|f_n| \leqq M E\,|\sum_{k=1}^n r_k(t) d_k| \log^+ |\sum_{k=1}^n r_k(t) d_k| + M.$$

Integrating both sides with respect to t leads to the desired result.

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