THE EXISTENCE AND UNIQUENESS OF STATIONARY MEASURES FOR MARKOV RENEWAL PROCESSES[1]

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0. Summary. In [4], Doob shows that $F^*(x) = \mu^{-1} \int_0^x [1 - F(u)]\, du$ is a stationary probability measure for a renewal process when the common distribution function F has a finite mean μ. In [2], Derman shows that an irreducible, null recurrent Markov chain (MC) has a unique positive stationary measure. In this paper, similar results are obtained for a class of irreducible recurrent Markov renewal processes (MRP). Since MRP's are generalizations of MC's and renewal processes these results generalize those mentioned above. Stationary measures are also derived for a class of MRP's with auxiliary paths.

1. Introduction. Let $\{X_n: n \geqq 1\}$ be a positive renewal process; that is, a sequence of independent non-negative random variables (rv's) with a common non-degenerate distribution function (df), F. Set $S_n = X_1 + \cdots + X_n$, $(n \geqq 1)$, $S_0 = X_0 = 0$ and $N(t) = \sup\{n \geqq 0: S_n \leqq t\}$. Two related processes of interest in renewal theory are the "age" and "excess" processes defined respectively by $U_t = t - S_{N(t)}$ and $V_t = S_{N(t)+1} - t$. Both of these processes are Markovian. If $\mu = E(X_1) < +\infty$, Doob [4] showed in 1948 that the df, $F^*(x) = \mu^{-1} \int_0^x [1 - F(y)]\, dy$, gives the unique stationary *probability* measure for the V-process. That is, F^* satisfies

$$F^*(x) = \int_{0-}^{+\infty} P[V_{t+s} \leqq x \mid V_s = y]\, dF^*(y) \tag{1.1}$$

for all $x \geqq 0$. If $\mu = +\infty$, it still remains true that the measure determined by the mass function

$$\tilde{F}(x) = \int_0^x [1 - F(y)]\, dy \tag{1.2}$$

is a stationary measure since (1.1) is a well defined identity in $\tilde{F}$ even when $\tilde{F}(+\infty) = +\infty$. This result, together with the result that $\tilde{F}$ is the unique σ-finite stationary measure, will be a consequence of the main theorems of this paper. (Whenever we speak of the uniqueness of a measure, it is understood that only uniqueness up to multiplication by a constant is intended.)

Let $\{J_n : n \geqq 0\}$ be an irreducible recurrent MC defined on a countable state space, taken for convenience to be the non-negative integers $I^+ = \{0, 1, 2, \cdots\}$, with transition matrix $P = (p_{ij})$. If this MC is positive then it is well known that a unique stationary probability measure exists and may be expressed in

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terms of the mean recurrence times of the states. If the chain is null-recurrent, then Derman [2] shows that a unique positive stationary measure $\{m_i^*: i \varepsilon I^+\}$ still exists; that is, $m_j^* > 0$ and $m_j^* = \sum_i m_i^* p_{ij}$ for all $j \varepsilon I^+$. In this case, m_i^* may be defined explicitly as the expected number of visits to state i which occur between two successive visits to some fixed state, say state 0.

As should be expected, the stationary measure of a recurrent MRP is, under certain assumptions, a combination of those for a Markov chain and a renewal process, *even though the Markov chain involved may be transient.* The proof of the existence and uniqueness of a stationary probability measure for a positive recurrent, irreducible MRP with a finite number of states is very straightforward and was given for example in [10]. The proof given there also applies to a strongly regular MRP with countably many states. (See Definition 2.2 of [11] and Section 5 below.) The proof of existence and uniqueness is much more involved, however, for general MRP's.

Although the problem of stationarity of measures is solely concerned with the transition measures of a process and not with any properties of the sample paths, it is essential to the proofs given below to make assumptions about the sample behavior of the processes concerned. The class of MRP's to be considered in this paper are those which are referred to in [11] (see page 1749) as satisfying *hypothesis A*. Although the reader must be referred to [11] for a complete definition of these processes, the following discussion will serve to outline the definition as well as to give an explanation of the notation used. The reader should also consider the general definition given by Feller [5].

We are concerned with stochastic processes $\{Z_t : t \geqq 0\}$ defined on a measurable space $(\Omega, \mathfrak{F})$ and obtained as follows. Ω is the collection of all functions having domain $[0, +\infty)$, having range $\bar{I}^+ = I^+ \cup \{\infty\}$, and satisfying the following conditions. Each $\omega \varepsilon \Omega$ is right continuous, has left limits, and is such that if $\omega(t) \to \infty$ as $t \to s$ from one side, then $\omega(t) \to \infty$ as $t \to s$ from both sides. Z_t is defined by $Z_t(\omega) = \omega(t)$ and $\mathfrak{F}$ is the natural σ-field associated with such a definition. (The state, ∞, will always be a fictitious state, added so that the right continuity of the sample paths may be assumed.) Associated with the Z-process are the quantities

$$V_t = \sup\{u \geqq t: Z_s = Z_t \quad \text{for all} \quad s \varepsilon [t, u]\} - t,$$

$$Z_t^+ = Z_{t+V_t},$$

$$N_j(t) = \text{card}\{0 < u \leqq t: Z_{u-} \neq Z_u = j\}, \qquad j \varepsilon I^+,$$

where Z_{s-} denotes the left limit of $Z_{(\cdot)}$ at s. The interpretation of these quantities is as follows: Z_t denotes the state of a process at time t, V_t denotes the remaining time until the next transition after time t, Z_t^+ denotes the next state visited after t and $N_j(t)$ records the number of entrances into state j during $(0, t]$. Another quantity of interest, which is to be denoted by U_t, is the amount of time since the last transition of the Z-process. In order to define U_t conveniently it is necessary when working with stationary measures to have the process "in progress" at time $t = 0$; that is, to allow U_0 to be an arbitrary initial rv whose distribution would have to be specified as part of the initial measure for the Z-process. We then define

$$U_t = U_0 + t \quad \text{if} \quad t < V_0$$
$$= t - \inf\{u \leqq t: Z_s = Z_t \quad \text{for all} \quad s \,\varepsilon\, [u, t]\} \quad \text{if} \quad t \geqq V_0 .$$

When $U_0 = 0$ a.s., this definition is equivalent to the one given in [11]. In this paper we will be working with the processes $\{(Z_t, U_t): t \geqq 0\}$ and $\{(Z_t, V_t): t \geqq 0\}$. These two types of processes are closely related, the one being essentially a time-reversal of the other.

The assumptions made above on Ω, the sample space of the Z-process, insure that if the Z-process has explosions, there can be no instantaneous jumps to, or returns from, infinity. They also imply that any finite state has a successor and every finite state but the first has an immediate predecessor.

To obtain a probability structure, it is assumed that $\{(Z_t, U_t): t \geqq 0\}$ is a Markov process having the strong Markov property for all stopping times, T, for which $Z_T \,\varepsilon\, I^+$ (a.s.) and having a transition function denoted by $P_t(i, x; j, y) = P[Z_{t+s} = j, U_{t+s} \leqq y \mid Z_s = i, U_s = x]$.

The evolution of an MRP is determined by a matrix of transition functions $Q_{ij}(\cdot, \cdot)$, $i, j \,\varepsilon\, I^+$, defined on $[0, +\infty) \times (-\infty, +\infty)$ such that $Q_{ij}(u, \cdot)$ is a mass function, $Q_{ij}(u; x) = 0$ for $x \leqq 0$ and

$$(1.3) \quad P[Z_t^+ = j, V_t \leqq x \mid Z_t = i, U_t = u; (Z_s, U_s), 0 \leqq s < t] = Q_{ij}(u; x)$$

with probability one for all $t \geqq 0$. If we set $H_i(u, x) = \sum_j Q_{ij}(u; x)$, it is further assumed that $H_i(u, +\infty) = 1$ for all $u \geqq 0$ and $i \,\varepsilon\, I^+$. The properties of this transition function and its "backward" relationship with the Q_{ij} are given in [11]. We shall refer to any of the Z- or (Z, U)- or (Z, V)-processes as an MRP.

Throughout this paper we shall consider only MRP's of the type just defined (those satisfying hypothesis A in [11]) so that further specific mention of this will be omitted. Also, throughout the remainder of this paper we shall work only with non-lattice MRP's. That is, the recurrence-time df's G_{ii} (see (1.5.3) below) are non-lattice. This represents no significant loss of generality since the lattice case can be treated in a simpler, though similar, manner.

It is convenient to consider, as in [11], the special class of MRP's for which $Q_{ij}(u; x) = p_{ij}H_i(u; x)$ where $p_{ij} = Q_{ij}(0, +\infty)$.

That this may be done without loss of generality may be seen as follows: if $Y_t = (Z_t, U_t)$ is the given Markov process with mass functions $\{Q_{ij}(\cdot; \cdot)\}$, set $Y_t' = (Z_t', U_t)$ where $Z_t' = (Z_t, Z_t^+)$ and set $Q'_{ij,km}(x; t) = \delta_{jk}p_{km}Q_{ij}(x; t)/Q_{ij}(x; +\infty)$. The reader may compute that the Y'-process is a regular MRP of the type considered in this paper whenever the Y-process is, and moreover, its transition matrix has the desired factorization. Once one has found the unique stationary measure for this special class, it is an easy job to use this to determine the stationary measure for a more general MRP (see Remark 3 of Section 4 below).

In Section 2, the existence of a stationary measure is proved. Section 3 contains a proof of the uniqueness of this measure. In Section 4, several remarks are made. In particular, it is shown that the stationary measure is a *probability* measure if and only if the MRP is positive recurrent. Also, the relationship between the results of this paper with those known for discrete and continuous parameter Markov chains is explained in Section 5. Section 6 extends the results of Sections 2 and 3 to MRP's with auxiliary paths.

Before proceeding to the question of stationarity, we state a lemma and list the definitions and notation which will be used in the remainder of this paper.

By summing over j in (1.3), one obtains
$H_i(u; x) = P[V_t \leqq x \mid Z_t = i, U_t = u; (Z_s, U_s), 0 \leqq s < t]$ for all $t \geqq 0$. Upon setting $H_i(\cdot) = H_i(0, \cdot)$, it is possible to evaluate $H_i(u, x)$ in terms of H_i as given in Lemma 1.1 below. The proof of the lemma follows from a consideration of the indicated conditional probabilities and the deterministic behavior of the U-process.

LEMMA 1.1. *For all $i \varepsilon I^+$, $x \geqq 0$ and all $u \geqq 0$ for which $H_i(u) < 1$, one has*

$$H_i(u; x) = [H_i(u + x) - H_i(u)][1 - H_i(u)]^{-1}. \tag{1.4}$$

Let $P_{i,x}$ denote the probability measure for the (Z, U)-process, determined by the initial condition that $Z_0 = i$ and $U_0 = x$. Let $E_{i,x}$ denote expectation with respect to $P_{i,x}$. Define

(1.5.1) $\quad Q_{ij}(t) = Q_{ij}(0; t), \qquad p_{ij} = Q_{ij}(+\infty);$

(1.5.2) $\quad P_{ij}(t) = P_{i,0}[Z_t = j] = P_t(i, 0; j, +\infty);$

(1.5.3) $\quad G_{ij}(i) = P_{i,0}[N_j(t) > 0];$

(1.5.4) $\quad M_{ij}(t) = E_{i,0}[N_j(t)] + \delta_{ij};$

(1.5.5) $\quad {}_kP_{ij}(t) = P_{i,0}[Z_t = j, N_k(t) = 0];$

(1.5.6) $\quad {}_kG_{ij}(t) = P_{i,0}[\text{for some } u \leqq t, N_j(u) > 0, N_k(u) = 0];$

(1.5.7) $\quad {}_kM_{ij}(t) = (1 - \delta_{jk})E_{i,0}[N_j(S_k)] + \delta_{ij}$ where $s_k = \min(t, T_k)$
$\qquad$ and $T_k = \inf\{t > V_0 : Z_t = k\};$

(1.5.8) $\quad S_i(j, x; t) = P_{i,0}[Z_t = j, U_t \leqq x];$

(1.5.9) $\quad R_i(j, x; t) = P_{i,0}[Z_t = j, V_t \leqq x].$

Let η_j and μ_{ij} be the first moments of H_j and G_{ij} respectively.

For any function $K(\cdot)$, $\bar{K}(\cdot)$ and $\tilde{K}(\cdot)$ are defined by

$$\bar{K}(s) = \int_0^s K(t)\, dt, \qquad \tilde{K}(s) = \int_0^s [K(+\infty) - K(t)]\, dt$$

whenever these quantities are well-defined. (This is consistent with the notation used in (1.2) above.) Define the convolution operation by $K * L(t) = \int_{0-}^t K(t - u)\, dL(u)$ whenever the integration is defined. Also, set $K^{(0)}(t) = 1$

or 0 according as $t \geqq 0$ or $t < 0$ and define $K^{(n)}(t) = K^{(n-1)} * K(t)$ for $n \geqq 1$. Since we are only interested in $t \geqq 0$ in this paper we shall often simply write 1 for $K^{(0)}(t)$.

In terms of the above notation it is possible to evaluate the transition function $P_t(i, x; j, y)$ of the (Z, U)-process to be

$$(1.6) \quad \begin{aligned} P_t(i, x; j, y) &= \textstyle\sum_k S_k(j, y; \cdot) * Q_{ik}(x; \cdot)(t) \quad \text{if} \quad t + x > y \\ &= \textstyle\sum_k P_{kj}(\cdot) * Q_{ik}(x; \cdot)(t) + \delta_{ij}[1 - H_i(x; t)] \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad t + x \leqq y. \end{aligned}$$

The (Z, V)-process is also a continuous parameter Markov process. If we let $Q_t(i, x; j, y) = P[Z_t = j, V_t \leqq y \mid Z_0 = i, V_0 = u]$ denote its transition function, then

$$(1.7) \quad \begin{aligned} Q_t(i, u; j, y) &= \textstyle\sum_k p_{ik} R_k(j, y; t - u) && \text{if} \quad u \leqq t \\ &= \delta_{ij} && \text{if} \quad t < u \leqq t + y \\ &= 0 && \text{if} \quad t + y < u. \end{aligned}$$

For each $j \,\varepsilon\, I^+$ and $x \geqq 0$, set

$$(1.8) \qquad m_j = {}_0M_{0j}(+\infty) \qquad \text{and} \qquad \pi(j, x) = m_j \tilde{H}_j(x).$$

Clearly $\pi(j, x)$ is finite for all $j \,\varepsilon\, I^+$ and $x \geqq 0$, so that π defines a σ-finite measure on $\mathcal{B}(\mathfrak{X})$, the Borel sets of $\mathfrak{X} = I^+ \times [0, \infty)$.

Observe further that $\pi(j, +\infty) < +\infty$ if and only if $\eta_j < +\infty$. In Section 2 it is shown that π is a stationary measure for both the (Z, U)-process and the (Z, V)-process. That π is a natural choice for a stationary measure is intuitively clear on the basis of the known result in [10] for the finite state case. However, it is motivated most strongly by the following Doeblin ratio theorem, proved in [11].

THEOREM 1.1. *For a recurrent, irreducible, non-lattice regular* MRP,

$$(1.9) \qquad \lim_{t \to +\infty} M_{ij}(t)/M_{kr}(t) = {}_rM_{rj}(+\infty);$$

$$(1.10) \qquad \lim_{t \to +\infty} \tilde{S}_k(i, x; t)/M_{00}(t) = \pi(i, x);$$

$$(1.11) \qquad \lim_{t \to +\infty} \bar{R}_k(j, y; t)/M_{00}(t) = \pi(j, y).$$

2. Stationarity of π. For any initial measure ν, one obtains a measure Q_ν for the (Z, V)-process. We will write

$$(2.1) \qquad Q_\nu(j, x; t) = \textstyle\sum_i \int_{0-}^{\infty} Q_t(i, u; j, x)\nu(i, du)$$

to denote the resulting measure of the event $[Z_t = j, V_t \leqq x]$. Let E_ν denote expectation with respect to the measure Q_ν. In this section we wish to show that the particular initial measure π determined by (1.8) is a stationary measure for the (Z, V)-process. That is, we wish to show that $Q_\pi(j, x; t) = \pi(j, x)$ for all $j \,\varepsilon\, I^+$, $x \geqq 0$ and $t \geqq 0$. This result is proved as an immediate consequence of the following basic lemma. Since for any initial measure ν,

$$E_\nu[N_j(t)] = \sum_i \sum_k p_{ik}\nu(i, \cdot) * M_{kj}(t), \tag{2.2}$$

and since the right hand side of (2.3) below is simply the derivative of this expression when $\nu = \pi$, this lemma will imply that $E_\pi[N_j(t)]$ is proportional to t, as should be the case if π is to be a stationary measure.

LEMMA 2.1. *For all $j \varepsilon I^+$ and all $t \geqq 0$*

$$m_j = \sum_i m_i \sum_k (p_{ik} - Q_{ik}) * M_{kj}(t) \tag{2.3}$$

and

$$m_j = \sum_i m_i p_{ij}. \tag{2.4}$$

PROOF. Assume $Z_0 = 0$ and $U_0 = 0$ and fix $j \varepsilon I^+$ and $t \geqq 0$. Let $T = \inf\{s > V_0 : Z_s = 0\}$ be the time of the first return to state 0. Let N' count the number of visits to j by the Z-process in $(t, t + T]$. Because of the strong Markov property and recurrence, $E_{0,0}[N_i(t + T) - N_i(T)] = E_{0,0}[N_i(t)]$. Hence, $E_{0,0}[N'] \equiv E_{0,0}[N_i(t + T) - N_i(t)] = E_{0,0}[N_j(T)] = m_j$, the left side of (2.3). To show the $E_{0,0}[N']$ is also equal to the right side of (2.3), for each $i \varepsilon I^+$, let $\{T_{ir}; r \geqq 1\}$ be the times of successive visits of the Z-process to state i, let $\{X_{ir}; r \geqq 1\}$ be the successive holding times in state i, and let $\{Y_{ij}(n, r); r \geqq 1\}$ be the time from the rth entrance into i until the nth visit ($n \geqq 1$) to j which occurs after T_{ir}. Observe that here $T_{01} = 0$, $T_{02} = T$, and $X_{ir} = V_{T_{ir}}$. Each visit to j in $(t, t + T]$ will be identified with a visit to some state in $[0, T)$, the identification being made by looking back $t+$ units of time from the instant j was entered, seeing what state the process was in and counting the number of visits of j since then. Formally, the rth visit to state i is said to be of *type-n* if $T_{ir} < T$ and $Y_{ij}(n, r) > t \geqq Y_{ij}(n, r) - X_{ir}$. Then $N' = \sum_i \sum_r \sum_n W_{ij}(n, r)$ where $W_{ij}(n, r) = 1$ or 0 according as the rth visit to state i is or is not of type-n. Then

$$E_{0,0}[W_{ij}(n, r)]$$
$$= \int_{0-}^{+\infty} P[Y_{ij}(n, r) > t \geqq Y_{ij}(n, r) - X_{ir} \mid Z_0 = 0, U_0 = 0, T_{ir} = u]$$
$${}_0G_{0i} * {}_0G_{ii}^{(r-1)}\,(du).$$

The integrand in this expression is straightforwardly computed to be

$$\sum_{k \neq j} (p_{ik} - Q_{ik}) * G_{kj} * G_{jj}^{(n-1)}(t) + (p_{ij} - Q_{ij}) * G_{jj}^{(n-1)}(t)$$

which does not depend on u or r. Hence, since

$$\sum_{r=1}^{+\infty} {}_0G_{0i} * {}_0G_{ii}^{(r-1)}(u) = {}_0M_{0i}(u)$$

for all u and $i \neq 0$, and since

$$\sum_{n=1}^{+\infty} G_{kj} * G_{jj}^{(n-1)}(t) = M_{kj}(t) - \delta_{kj},$$

one obtains

$$E_{0,0}[N'] = \sum_i m_i \sum_k (p_{ik} - Q_{ik}) * M_{kj}(t).$$

Equation (2.4) follows from (2.3) by noting that $(p_{ik} - Q_{ik}) * M_{kj}(0) = p_{ik}\delta_{kj}$ so that the right side of (2.3) reduces to $\sum_i m_i p_{ij}$.

It should be remarked that (2.4) does not follow from Derman's theorem [2] for Markov chains unless the MRP is strongly regular. If a recurrent MRP is regular but not strongly regular, the imbedded Markov chain with transition matrix (p_{ij}) is necessarily transient.

We now prove the existence of a stationary measure for an MRP.

THEOREM 2.1. *π is a stationary measure for both the (Z, U)-process and the (Z, V)-process.*

PROOF. As mentioned at the start of this section, stationarity of π for the (Z, V)-process will be established if it is shown that $Q_\pi = \pi$. From (2.1) and (1.7), one obtains

$$(2.5) \qquad Q_\pi(j, x; t) = \sum_{i,k} p_{ik}R_k(j, x; \cdot) * \pi(i, \cdot) + \pi(i, t + x) - \pi(i, t).$$

However, it is easily seen (a detailed derivation is included in [12]) that

$$(2.6) \qquad R_k(j, x; t) = [H_j(x + \cdot) - H_j(\cdot)] * M_{kj}(t).$$

Therefore, the first term on the right hand side of (2.5) becomes, in view of (2.2),

$$E_\pi[N_j(\cdot)] * [H_j(x + \cdot) - H_j(\cdot)](t).$$

Since $E_\pi[N_j(t)] = m_j t$, by Lemma 2.1 and the comment preceding it, one then obtains

$$\begin{aligned} Q_\pi (j, x; t) &= m_j \int_0^t [H_j (x + u) - H_j (u)]du + \pi(j,t + x) - \pi(j,t) \\ &= m_j \int_0^x [1 - H_j(u)]\, du = \pi(j, x) \end{aligned}$$

as required.

To prove the stationarity of π for the (Z, U)-process, one must show that $P_\pi = \pi$ where

$$(2.7) \qquad P_\pi(j, x; t) = \sum_i \int_{0-}^\infty P_t(i, u; j, x)\pi(i, du).$$

If $x < t$, substitution of (1.6) into this expression gives

$$\begin{aligned} (2.8) \qquad P_\pi(j, x; t) &= \sum_{i,k} \int_{0-}^\infty S_k(j, x; \cdot) * Q_{ik}(u; \cdot)(t)\pi(i, du) \\ &= \sum_{i,k} p_{ik}S_k(j, x; \cdot) * \int_{0-}^\infty H_i(u; \cdot)\pi(i, du)(t). \end{aligned}$$

But, by Lemma 1.1 and (1.8),

$$\begin{aligned} \int_{0-}^\infty H_i(u; v)\pi(i, du) &= m_i \int_0^\infty [H_i(u + v) - H_i(u)]\, du \\ &= m_i \int_0^v [1 - H_i(u)]\, du = \pi(i, v). \end{aligned}$$

Since $S_k(j, x; t) = K_j(\cdot, x) * M_{kj}(t)$ where $K_j(u, x) = 1 - H_j(u)$ if $u \leqq x$

and zero otherwise, one obtains

$$
\begin{aligned}
P_\pi(j, x; t) &= \textstyle\sum_{i,k} p_{ik} K_j(\cdot, x) * M_{kj} * \pi(i, \cdot)(t) \\
&= K_j(\cdot, x) * E_\pi[N_j(\cdot)](t) \qquad (2.9)\\
&= m_j \int_0^t K_j(v, x)\, dv = \pi(j, x).
\end{aligned}
$$

If $x \geqq t$, there is an additional term to evaluate, but a similar argument suffices.

3. Uniqueness of π. Let α be any σ-finite stationary measure for the (Z, V)-process which is not identically zero. Write $\alpha(i, A)$ for the α-measure of $\{i\} \times A$ and write $\alpha(i, x)$ for the α-measure of $\{i\} \times [0, x]$. Since α is stationary, it satisfies

$$Q_\alpha[V_t \varepsilon A, Z_t = i] = \alpha(i, A) \qquad (3.1)$$

where Q_α is defined by (2.1). The uniqueness of π will be established if it can be shown that (3.1) implies that α is proportional to π. The proof of this fact is in two parts. The first and simpler part is to show that there exist constants $\{c_i\}$ such that $\alpha(i, x) = c_i \tilde{H}_i(x)$. The second part is to show that these constants are proportional to $\{m_i\}$. These results are given below in Theorems 3.1 and 3.2 respectively. Before proving these theorems, some preliminary properties of α are recorded.

From the relationship

$$
\begin{aligned}
\alpha(i, A) &= Q_\alpha[V_y \varepsilon A, Z_y = i] \geqq Q_\alpha[V_0 > y, V_y \varepsilon A, Z_y = i] \\
&= Q_\alpha[V_0 \varepsilon A + y, Z_0 = i] = \alpha(i, A + y),
\end{aligned}
$$

it follows that

$$\alpha(i, A) \geqq \alpha(i, A + y) \qquad (3.2)$$

for all $i \varepsilon I^+$, $y \geqq 0$, and all Borel sets $A \subset (0, +\infty)$ where $A + y = \{x : x - y \varepsilon A\}$. Therefore, the following lemma applies to each measure $\alpha(i, \cdot)$.

LEMMA 3.1. *Let β be any σ-finite measure defined on the Borel subsets of* $[0, \infty)$ *which satisfies $\beta(A) \geqq \beta(A + y)$ for all $y \geqq 0$, and Borel sets $A \subset (0, +\infty)$. Then*

(i) $\beta([x, x + t]) < +\infty$ *for all* $x > 0$ *and* $0 \leqq t < +\infty$.

(ii) $\beta \ll \mu_L$ *on* $(0, +\infty)$ *so that* $\beta(A) = \int_A f(u)\mu_L(du)$ *for all Borel sets $A \subset (0, +\infty)$, where μ_L denotes Lebesgue measure, and f is equal* a.e.-μ_L *to a non-negative, non-increasing Lebesgue measurable function.*

PROOF. (i) Let A be a Borel set, $A \subset (0, +\infty)$, for which $\beta(A) < +\infty$. By hypothesis, $\beta(A + y)$ is a monotone decreasing function in y, and hence integrable. Upon integrating both sides of the inequality $\beta(A) \geqq \beta(A + y)$ with respect to y over $(0, t)$, one obtains

$$
\begin{aligned}
t\beta(A) &\geqq \textstyle\int_0^t \beta(A + y)\mu_L(dy) \\
&= \textstyle\int_0^t \int_{0-}^{+\infty} I_A(u - y)\beta(du)\mu_L(dy) \qquad (3.3)\\
&= \textstyle\int_{0-}^{+\infty} \int_0^t I_A(u - y)\mu_L(dy)\beta(du).
\end{aligned}
$$

Making the change of variable $u - y = v$, this becomes

$$
\begin{aligned}
t\beta(A) &\geqq \int_{0-}^{+\infty} \int_{u-t}^{u} I_A(v)\mu_L(dv)\beta(du) \\
&= \int_{-t}^{0} I_A(v) \int_{0-}^{v+t} \beta(du)\mu_L(dv) \\
&\quad + \int_{0}^{\infty} I_A(v) \int_{v-}^{v+t} \beta(du)\mu_L(dv) \\
&= \int_A \beta([0, t] + v)\mu_L(dv).
\end{aligned} \tag{3.4}
$$

Therefore, if there exists $x > 0$ and $t > 0$ such that $\beta([x, x + t]) = +\infty$, then $\beta([0, t] + v) = +\infty$ for $0 \leqq v \leqq x$. Choose $A \subset (0, +\infty)$ such that $\beta(A) < +\infty$ and $\mu_L(A \cap (0, x)) > 0$. This is possible because of the σ-finiteness of β. Then the left side of (3.4) is finite while the integrand on the right side is infinite on a set of positive Lebesgue measure. This contradiction establishes the validity of (i).

(ii) For $\epsilon > 0$, let $A \subset (\epsilon, +\infty)$ be a Borel set for which $\mu_L(A) = 0$. Then $\beta(A) \leqq \beta(A - y)$ for $0 \leqq y \leqq \epsilon$ and an argument similar to that used in (i) shows that $\beta(A) = 0$. This shows that β, when restricted to $(\epsilon, +\infty)$ for any $\epsilon > 0$, is absolutely continuous with respect to μ_L. From this and the monotonicity of $\beta(A + \cdot)$, it is a straightforward matter to complete the proof of (ii).

Because of (3.2), the above lemma applies to each of the measures $\alpha(i, \cdot)$ for $i \varepsilon I^+$. For these specific measures, however, the results of Lemma 3.1 may be strengthened as follows.

LEMMA 3.2. *For each* $i \varepsilon I^+$ *and* $x \geqq 0$, $\alpha(i, x) < \infty$, $\alpha(i, 0) = 0$, $\alpha(i, \cdot) \ll \mu_L$ *over* $[0, \infty)$ *and*

$$\alpha(i, x) - \alpha(i, (y, y + x]) = c_i \int_0^y [H_i(u + x) - H_i(u)]\, du \tag{3.5}$$

for some constants $c_i \geqq 0$.

PROOF. By definition, $\alpha(i, x) = \alpha(i, [0, x])$. For any $x \geqq 0$, $y > 0$, $i, j \varepsilon I^+$ and $0 < t < y$, consider

$$
\begin{aligned}
\alpha(j, (x, x + y]) &= Q_\alpha[x < V_t \leqq x + y, Z_t = j] \\
&\geqq Q_\alpha[V_0 \leqq t, Z_0 = i, x + t < V_{\tau_0} + V_0 \leqq x + t + y, Z_0^+ = j].
\end{aligned}
$$

The left hand side is finite by Lemma 3.1, whereas the right hand side equals

$$
\begin{aligned}
p_{ij} \int_{0-}^{t} &[H_i(x + t + y - u) - H_i(x + t - u)]\alpha(i, du) \\
&\geqq p_{ij}\alpha(i, t)[H_i(x + y) - H_i(x + t)].
\end{aligned}
$$

Since this holds for all j, x, y and t, it is possible to choose values such that the first and last factors do not vanish, thereby establishing the finiteness of $\alpha(i, t)$ for at least one, and hence all, positive values of t.

To establish that $\alpha(i, \cdot) \ll \mu_L$, it suffices, in view of Lemma 3.1 (ii), to show that $\alpha(i, \{0\}) = \alpha(i, 0) = 0$. However, this will be immediate once we have derived (3.5). To derive (3.5), observe that, by stationarity, we have $E_\alpha[N_i(t)] = E_\alpha[N_i(t + s) - N_i(s)]$ for all $s, t \geqq 0$. Thus, if $E_\alpha[N_i(s)] < +\infty$, then

$$E_\alpha[N_i(t+s)] = E_\alpha[N_i(t)] + E_\alpha[N_i(s)], \tag{3.6}$$

and since $N_i(\cdot)$ is non-decreasing a.s., it would follow that $E_\alpha[N_i(t)] = c_i t$ for all $t \geqq 0$ and some constant $c_i \geqq 0$. That $E_\alpha[N_i(s)]$ is finite follows by Fubini's theorem applied to the equation of stationarity (3.1), namely,

$$\begin{aligned} +\infty &> \alpha(i, x) \\ &= \textstyle\sum_{j,k} p_{jk} \int_{0-}^{t} M_{ki} * [H_i(x+\cdot) - H_i(\cdot)](t-u)\alpha(j, du) \\ &\quad + \alpha(i, (t, t+x]) \\ &= \textstyle\int_{0-}^{t} [H_i(x+t-u) - H_i(t-u)]\, dE_\alpha[N_i(u)] + \alpha(i, (t, t+x]). \end{aligned} \tag{3.7}$$

If one now substitutes $c_i u$ for $E_\alpha[N_i(u)]$, one obtains (3.5).

Corollary 3.1. *For $t \geqq x \geqq 0$, $y \geqq 0$ and $i \varepsilon I^+$,*

$$Q_\alpha[U_t \leqq x, V_t \leqq y, Z_t = i] = \alpha(i, y) + \alpha(i, x) - \alpha(i, x+y). \tag{3.8}$$

Proof. For any measurable set $A \subset [0, \infty)$ it is true that

$$\begin{aligned} \alpha(i, A) &= Q_\alpha[V_t \varepsilon A, Z_t = i] \\ &= Q_\alpha[V_{t-x} \leqq x, V_t \varepsilon A, Z_t = i] + Q_\alpha[V_{t-x} > x, V_t \varepsilon A, Z_t = i] \\ &= Q_\alpha[V_{t-x} \leqq x, V_t \varepsilon A, Z_t = i] + Q_\alpha[V_{t-x} \varepsilon (A - \{0\}) + x, Z_{t-x} = i] \end{aligned}$$

for any $t \geqq x$. This is simply an elaboration on the relationship given prior to equation (3.2). If A is such that $\alpha(i, A) < +\infty$, we therefore have that

$$Q_\alpha[V_{t-x} \leqq x, V_t \varepsilon A, Z_t = i] = \alpha(i, A) - \alpha(i, (A - \{0\}) + x). \tag{3.9}$$

By taking $A = [0, y]$, which has finite $\alpha(i, \cdot)$ measure by Lemma 3.2, and observing that the sets $[U_t \leqq x]$ and $[V_{t-x} \leqq x]$ differ by a set of measure zero, the proof of the corollary is complete.

We now state and prove the first part of the uniqueness argument.

Theorem 3.1. *If α is any σ-finite stationary measure for the (Z, V)-process which is not identically zero, then for all $j \varepsilon I^+$ and $x \geqq 0$,*

$$\alpha(j, x) = c_j \tilde{H}_j(x) = c_j \int_0^x [1 - H_j(u)]\, du \tag{3.10}$$

for positive constants $\{c_j\}$.

Proof. For any $i \varepsilon I^+$, define the measure $\beta(i, A) = c_i \int_A [1 - H_i]\, d\mu_L$ for any Borel set A, where c_i is the constant that appears in (3.5). Write $\beta(i, x) = \beta(i, (0, x])$. By (3.5), one obtains that $\alpha(i, A) - \alpha(i, A + y) = \beta(i, A) - \beta(i, A + y)$ for all $y \geqq 0$. This implies that $\alpha(i, \cdot) = \beta(i, \cdot) + b_i\mu_L$ for some constant b_i. To show that $b_i = 0$, observe first, that $b_i \geqq 0$ since by construction

$$0 \leqq \lim_{y\to\infty} \alpha(i, A + y) = \lim_{y\to\infty} \beta(i, A + y) + b_i\mu_L(A) = b_i\mu_L(A).$$

Therefore, by substituting $\beta(i, \cdot) + b_i\mu_L$ for $\alpha(i, \cdot)$ into the right side of (3.7), one obtains

$$\beta(i, x) - \beta(i, (t, t + x]) = \alpha(i, x) - \alpha(i, (t, t + x])$$
$$\geqq \sum_{jk} b_j p_{jk} M_{ki} * \overline{H_i(x + \cdot) - H_i(\cdot)}(t).$$

By letting $t \to \infty$ and using Fatou's lemma, one obtains

$$\beta(i, x) \geqq \sum_{j,k} b_j p_{jk} \tilde{H}_i(x) M_{ki}(\infty)$$

which implies that each $b_j = 0$ since $M_{ki}(\infty) = +\infty$ for all $k \varepsilon I^+$, due to the recurrence and irreducibility of the MRP.

The fact that each c_i is strictly positive follows similarly from (3.7). For if $c_i = 0$, then (3.7) implies that each of the non-negative summands $c_j p_{jk} \tilde{H}_j * M_{ki} * [H_i(\cdot + x) - H_i(\cdot)](t)$, is zero for all $t > 0$. This could happen only if $c_j = 0$ for all j, or equivalently, if $\alpha \equiv 0$. The trivial case, however, has been excluded.

Corollary 3.2. *For all $j \varepsilon I^+$ and almost all (with respect to μ_L) $x \geqq 0$,*

$$c_j = \sum_{i,k} c_i(p_{ik} - Q_{ik}) * M_{kj}(x). \tag{3.11}$$

Proof. By the proof of Lemma 3.2, one has

$$c_j t = E_\alpha[N_j(t)] = \sum_{i,k} p_{ik} M_{kj} * \alpha(i, \cdot)(t). \tag{3.12}$$

By Theorem 3.1, $\alpha(i, x) = c_i \tilde{H}_i(x) = c_i \int_0^x [1 - H_i(u)]\, du$. Thus, since $Q_{ik} = p_{ik} H_i$, (3.12) may be rewritten as

$$\int_0^t c_j\, dx = c_j t = \int_0^t \{\sum_{i,k} c_i(p_{ik} - Q_{ik}) * M_{kj}(x)]\, dx.$$

A comparison of the integrands completes the proof.

It may actually be shown that (3.11) holds for all x. However, this fact is not needed to complete the proof of uniqueness, and once the uniqueness is established, it will follow as a consequence of Lemma 2.1. The special case of (3.11) for $x = 0$ is given below in Corollary 3.3.

The first part of the uniqueness proof is now established. It remains to prove the second part; namely, that the constants $\{c_i\}$ are proportional to $\{m_i\}$. This part may be proved in two ways. The first is to apply a uniqueness theorem of Isaac [7] for discrete parameter Markov processes to skeletons of the MRP. The second involves reversing the (Z, V)-process and applying an appropriate Doeblin-ratio theorem as was done for Markov chains by Derman [2]. Rather than give one proof in complete detail, we shall outline both methods of proof, with slightly more emphasis on the latter since it gives more explicit information about the processes.

The first method proceeds as follows. For a fixed $h > 0$, set $X_n = (Z_{nh}, V_{nh})$ for $n = 0, 1, 2, \cdots$. Since α and π are stationary measures for the (Z, V)-process, they also are for the X-process, $\{X_n : n = 0, 1, \cdots\}$. It may be checked that π is such that whenever $\pi(A) > 0$ for an event A in the state space of

$X_0 = (Z_0, V_0)$, then $Q_\pi[X_n \varepsilon A$ infinitely often $| X_0 = x] = 1$ for almost all (π) x. We will say that π satisfies condition (I). (If we let condition (H), for Harris [6], denote the same condition except that "for almost all (π)" is replaced by "for all," it is interesting to note that in the original version of our paper, which appeared prior to Isaac's paper, it was pointed out by means of a counterexample that the skeletons of an MRP do *not* satisfy condition (H).) Since $\pi(i, x) = m_i\tilde{H}_i(x)$ by definition (1.8), and since it has been proved, Theorem 3.1, that $\alpha(i, x) = c_i\tilde{H}_i(x)$ with all coefficients positive, it follows that $\alpha \ll \pi$. We now apply Isaac's Theorem 1 and Corollary 1 of Theorem 2 from [7] to conclude that $\alpha = c\pi$ for some constant $c > 0$, thereby establishing the uniqueness of π. The major step in the above method of proof is to verify that π satisfies condition I.

The second method involves reversing the (Z, V)-process. The transitional mechanics of the reversed process are indicated in the following lemma, in which $Z_t^- \equiv Z_{(t-U_t)-}$ denotes the state occupied by the process immediately before the present state. Z_t^- is therefore only defined on $[V_0 \leqq t]$.

LEMMA 3.3. *For* $x, z, t \geqq 0$ *and* $i, j \varepsilon I^+$

$$Q_\alpha[Z_t^- = i, Z_t = j, U_t \leqq x, V_t \leqq z]$$

$$(3.13) \qquad = c_ip_{ij}c_j^{-1}[\alpha(j, x) + \alpha(j, z) - \alpha(j, x + z)] \quad \text{for} \quad x \leqq t$$

$$= c_ip_{ij}c_j^{-1}[\alpha(j, z) + \alpha(j, t) - \alpha(j, z + t)] \quad \text{for} \quad x > t.$$

PROOF. For $x \leqq t$, one obtains by the usual "backward" probabilistic arguments that

$$Q_\alpha[Z_t^- = i, Z_t = j, U_t \leqq x, V_t \leqq z] = p_{ij} \int_{t-x}^t [H_j(t + z - u) - H_j(t - u)]\alpha(i, du) + \int_0^t \int_{t-u-x}^{t-u} [H_j(z + t - u - v) - H_j(t - u - v)]\, dQ_{ij}(v)\, dE_\alpha[N_i(u)].$$

Since $E_\alpha[N_i(u)] = c_iu$ by (3.6), the above becomes, after the change of variable $s = t - u$,

$$c_ip_{ij}[\int_0^x [1 - H_i(t - u)][H_j(z + u) - H_j(u)]\, du + \int_0^t \int_{s-x}^s [H_j(z + s - v) - H_j(s - v)]\, dH_i(v)\, ds],$$

which in turn may be reduced to

$$c_ip_{ij} \int_0^x [1 - H_i(t - u) + H_i(t - x) + H_i(t - u) - H_i(t - x)][H_j(z + u) - H_j(u)]\, du = c_ip_{ij}c_j^{-1}[\alpha(j, x) + \alpha(j, z) - \alpha(j, x + z)]$$

by (3.5). A similar argument handles the case where $x > t$.

COROLLARY 3.3. *For all* $j \varepsilon I^+$,

$$(3.14) \qquad c_j = \sum_i c_ip_{ij}.$$

PROOF. By (3.8) and (3.10) $Q_\alpha[U_t \leqq x, Z_t = j] = \alpha(j, x)$ for $t \geqq x$. Thus, using (3.13), one obtains

$$\begin{aligned}\alpha(j, x) &= \textstyle\sum_i \lim_{z\to+\infty} Q_\alpha[Z_t^- = i, U_t \leqq x, Z_t = j, V_t \leqq z] \\ &= \textstyle\sum_i c_i p_{ij} c_j^{-1} \alpha(j, x).\end{aligned}$$

Since $\alpha(j, x) > 0$ for some $x > 0$, (3.14) is verified.

Equations (3.11) and (3.14) state that the constants c_i satisfy, for almost all x, the same equations, (2.3) and (2.4), which the m_i satisfy. In particular, (3.14) states that the constants c_i must be a solution of the stationarity equations, $x_j = \sum_i x_i p_{ij}$, for the embedded Markov chain with transition matrix $\mathbf{P} = (p_{ij})$. Since by (2.4) the m_i also form a solution to these equations, the uniqueness of α would be established *if it were known that these equations had a unique solution.* Such would be the case for example if the MRP were strongly regular so that the recurrence of the MRP would imply the recurrence of the embedded Markov chain with transition matrix $\mathbf{P}$. However, if the MRP is not strongly regular, the imbedded Markov chain associated with $\mathbf{P}$ must necessarily be transient so that the equations need not have a unique solution. (An example in which uniqueness does not hold is given in the next section.) A further argument is therefore needed to verify the proportionality of the c_i's to the m_i's. The verification is completed by applying an appropriate Doeblin-ratio theorem to the reversed MRP.

To define the reversed MRP, we exhibit, on the space of time-reversed sample paths, a transition function,

$$Q_t^*(i, x; j, y) \equiv Q_\alpha[V_0 \leqq y, Z_0 = j \mid V_t = x, Z_t = i; (V_s, Z_s), t \leqq s] \tag{3.15}$$

and a matrix of mass functions $Q_{ij}^*(u; x)$ which satisfy

$$Q_{ij}^*(u; x) = Q_\alpha[Z_t^- = j, U_t \leqq x \mid V_t = x, Z_t = i; (V_s, Z_s), t \leqq s]. \tag{3.16}$$

The notation Q_α indicates that these transition functions must depend upon the initial measure α, the given stationary measure. By means of (3.13), the desired conditional probability Q_{ij}^* is seen to be

$$Q_{ij}^*(u; x) = c_i^{-1} H_i(u; x) p_{ji} c_j\,;\; Q_{ij}^*(x) = c_i^{-1} H_i(x) p_{ji} c_j\,. \tag{3.17}$$

To verify this, one notes that for $t \geqq x$, $Q_\alpha[Z_t^- = i, Z_t = j, U_t \leqq x, V_t \leqq z]$ is equal to $\int_0^z Q_{ij}^*(u, x)\alpha(i, du)$ by definition, and equal to $c_i p_{ij} c_j^{-1} \int_0^z H_i(u, x) \cdot \alpha(i, du)$ by (3.5) and (3.13). Hence

$$\int_0^z Q_{ij}^*(u; x)\alpha(i, du) = \int_0^z c_i p_{ij} c_j^{-1} H_i(u, x)\alpha(i, du)$$

for all $z \geqq 0$, so that (3.17) is a version of the desired conditional probability.

Now put

$$M_{ij}^* = c_i^{-1} H_i * \left(\textstyle\sum_k p_{jk} M_{ki}\right) c_j + \delta_{ij} \tag{3.18}$$

and set

$$
\begin{aligned}
Q_t^*(i, x; j, y) &= \textstyle\sum_k p_{ik}^* S_k^*(j, y; \cdot) * H_i(x, \cdot)(t), & t + x > y, \\
&= \textstyle\sum_k p_{ik}^* p_{kj}^*(\cdot) * H_i(x, \cdot)(t) & \\
&\qquad + \delta_{ij}[1 - H_i(x, t)], & t + x \leqq y,
\end{aligned} \tag{3.19}
$$

where

$$S_k^*(j, y; t) = \textstyle\int_{(t-y)-}^{t} [1 - H_j(t - u)]\, dM_{kj}^*(u) \tag{3.20}$$

and $p_{jk}^* = Q_{jk}^*(\infty) = c_j^{-1} p_{kj} c_k$ are the analogues to $S_k(j, y; t)$ and p_{jk} for the reversed MRP. To see that Q_t^* satisfies (3.15), it is necessary to check that it satisfies

$$\textstyle\int_{0-}^{y} Q_s^*(j, u; i, x)\alpha(j, du) = \int_{0-}^{x} Q_s(i, u; j, y)\alpha(i, du), \tag{3.21}$$

both sides of which are evaluations of $Q_\alpha[Z_0 = i, V_0 \leqq x, Z_s = j, V_s \leqq y]$. The details are omitted, but are carried out by means of (3.11), (3.18), (3.19) and (3.20). The reader should also observe from (3.18) and 2.7.4. of [11], the intuitive fact that $M_{jj}^*(t) \equiv M_{jj}(t)$. That the reversed MRP has the sample function properties assumed in this paper is a consequence of defining it appropriately on the space of reversed sample functions of the original MRP.

THEOREM 3.2. *If α is a σ-finite stationary measure for the (Z, V)-process, then $\alpha = c\pi$ for some positive constant c.*

PROOF. Let $K_{ij}(t) = \sum_{k \neq i} p_{jk} G_{ki}(t) + p_{ji}$. Since $M_{ki}(t) = G_{ki} * M_{ii}(t)$ if $k \neq i$ (cf. (2.7.5) of [11]), (3.18) becomes

$$M_{ij}^*(t) = c_j c_i^{-1} H_i * M_{ii} * K_{ij}(t) + \delta_{ij}. \tag{3.22}$$

Dividing both sides by $M_{jj}^*(t) = M_{jj}(t)$, letting $t \to +\infty$, and using Doeblin-ratio theorem (1.9) together with Lemma 3.1 of [11], one obtains $G_{ij}^*(+\infty) = c_j c_i^{-1}\, {}_jM_{ji}(+\infty)$. But $M_{jj}^*(t) = M_{jj}(t)$ implies that $G_{jj}^*(t) = G_{jj}(t)$ so that the reversed process is recurrent and $G_{ij}^*(+\infty) = 1$. Setting $j = 0$, one obtains $1 = c_0 c_i^{-1} m_i$ or $c_i = c_0 m_i$, completing the proof.

COROLLARY 3.4. *For any two states $i, k \,\varepsilon\, I^+$,*

$${}_iM_{ij}(+\infty)/{}_kM_{kj}(+\infty) = C$$

for all $j \,\varepsilon\, I^+$. The constant $C = {}_iM_{ik}(+\infty)$ is independent of j.

PROOF. This can be proved directly by means of the Doeblin-ratio theorems proved in [11]. However, it also follows from Theorem 3.2 since the fact that $\pi(j, x) = m_j \tilde{H}_j(x)$ is the unique stationary measure does not depend on which state is fixed; in this paper state 0 was fixed and m_j defined as ${}_0M_{0j}(+\infty)$. Thus since ${}_iM_{ij}(+\infty)\tilde{H}_j(x)$ and ${}_kM_{kj}(+\infty)\tilde{H}_j(x)$ are both stationary measures, one must have that ${}_iM_{ij}(+\infty) = C\, {}_kM_{kj}(+\infty)$. That $C = {}_iM_{ik}(+\infty)$ follows by letting $j = k$ and noting that ${}_kM_{kk}(+\infty) = 1$ for any $k \,\varepsilon\, I^+$.

The proof that π is also the unique σ-finite stationary measure for the (Z, U)-process now follows easily.

THEOREM 3.3. *If α is a σ-finite measure for the (Z, U)-process, then $\alpha = c\pi$ for some positive constant c.*

PROOF. Any σ-finite stationary measure for this process satisfies the following equation:

$$\alpha(j, A) = \sum_{i,k} p_{ik} \int_{0-}^{+\infty} S_k(j, A, \cdot) * H_i(u, \cdot)(t)\alpha(i, du) + \int_{A-t} [1 - H_j(u, t)]\alpha(j, du). \tag{3.23}$$

By using an argument similar to that of Lemma 3.1,

$$\alpha(j, A) \geqq \int_{A-t}[1 - H_j(u, t)]\alpha(j, du)$$

implies that $\alpha(j, x) < +\infty$ for $x < x_0 = \inf \{0 < u \leqq +\infty : H_i(u) = 1\}$. By Fubini's theorem, this again implies that $E_\alpha[N_j(t)]$ is finite and measurable so that $E_\alpha[N_j(t)] = c_j t$. For $A = [0, x]$ and $t \geqq x$, (3.23) becomes

$$\alpha(j, x) = c_j \int_0^t K_j(u, x)\, du = c_j \int_0^x [1 - H_j(u)]\, du.$$

If P_α denotes the measure on the (Z, U)-process induced by the initial measure α, then

$$P_\alpha[Z_t = i, V_t \leqq x] = \int_0^{+\infty} H_i(u, x)\alpha(i, du) = c_i \int_0^x [1 - H_i(u)]\, du = \alpha(i, x),$$

and

$$P_\alpha[Z_t = i, V_t \leqq x] = \sum_j \int_0^{+\infty} P[Z_t = i, V_t \leqq x \mid Z_0 = j, U_0 = u]\alpha(j, du)$$
$$= \sum_{j,k} p_{jk} R_k(i, x, \cdot) * \alpha(j, du) + \alpha(j, [t, x + t]).$$

Therefore by (3.1) and (3.7), these last computations show that α is also a σ-finite stationary measure for the (Z, V)-process so that $c_i = cm_i$ by Theorem 3.2, completing the proof.

4. Remarks. 1. An important special case of Theorems 2.1, 3.2, and 3.3 is the result that $\pi(x) = \tilde{F}(x) = \int_0^x [1 - F(u)]\, du$, as given in (1.2), is the unique positive σ-finite stationary measure for the U- or V-process of a renewal process with common distribution function F. This extends the result of Doob [4]. (In connection with this it is important to observe that in order to view a renewal process as a special case of an MRP of the type considered in this paper, one must be careful how one represents the Z-process. For example, one may construct a 2-state MRP with $p_{10} = p_{01} = 1$ and $H_0 = H_1 = F$. Some such artifice is needed since we have assumed for convenience that U_t and V_t measure the time since the last and next discontinuity of the Z-process.)

2. In Sections 2 and 3 it was pointed out that

$$\sum x_i p_{ij} = x_j \tag{4.1}$$

need not have a unique solution since the embedded chain determined by (p_{ij}) may be transient. (It must be transient if the MRP is not strongly regular.) Derman, in [3], gives examples of transient chains for which the above equation has no solutions, one solution, or many solutions. One of his examples may be modified to give an example of an irreducible, recurrent non-lattice MRP of the type discussed in this paper for which the embedded MC is transient and for which (4.1) does not have a unique solution.

Let $\{Z_t^*; t \geqq 0\}$ be a continuous parameter MC with state space $\{-1, 0, 2, 4, \cdots\}$ obtained from an asymmetrical discrete random walk with transition probabilities $p_{i,i+2} = p > \frac{1}{2}$, $p_{i,i-2} = 1 - p = q$ $(i = 2, 4, 6, \cdots)$, $p_{02} = p$, $p_{0,-1} = q$, and $p_{-1,0} = 1$. By a suitable choice of the holding time parameters, say $q_{-1} = 1$, and $q_i = (p/q)^i$ $(i = 0, 2, 4, \cdots)$, it may be shown that Z_t^* explodes with probability one (see Miller [8]). Let $\{Z_t^*; t < \tau_1\}$ be the stopped continuous parameter MC where τ_1 is the random variable which records the time of the first explosion. To this stopped process, hitch a descending escalator (as in Chung [1], II.20) with state space $\{\cdots, 5, 3, 1\}$. The combined process, call it Z_t, is now defined for $t < \tau_1 + \tau_2$ where τ_2 is the random variable which measures the time required to come down the descending escalator. At the end of the holding time in state 1, the process moves to state -1 $(p_{1,-1} = 1)$ and the process described above is repeated a countable number of times so that Z_t is defined for all $t \geqq 0$. $\{Z_t; t \geqq 0\}$ is then a recurrent, irreducible, non-lattice MRP. The embedded MC with state space $\{-1, 0, 1, 2, \cdots\}$ has a transition matrix, $\mathbf{P}$, given by $p_{2i,2i+2} = p$, $p_{2i,2i-2} = q$ $(i = 2, 4, 6, \cdots)$, $p_{02} = p$, $p_{0,-1} = q$, $p_{-1,0} = 1$, $p_{2i+1,2i-1} = 1$ $(i = 0, 2, 4, 6, \cdots)$. The reader may verify that $x_{2i} = c_1(p/q)^i + c_2$ $(i = 0, 1, 2, \cdots)$, $x_{-1} = c_1 q + c_2 p$, and $x_{2i+1} = c_2(p - q)$ $(i = 0, 1, 2, \cdots)$ is a solution of (4.1) for any choice of $c_1, c_2 \geqq 0$ such that $c_1 + c_2 = 1$. Hence, (4.1) does not have a unique solution.

This example serves to illustrate several points. First, a stationary measure cannot be constructed for the (Z, V)- or (Z, U)-processes by multiplying $\tilde{H}_i$ by c_i where $\{c_i\}$ is *any* solution of (4.1). For then, by Theorem 3.2, $c_i = cm_i$, contradicting the fact that not all of the solutions in the example are proportional to m_i.

Secondly, although (4.1) is the equation of stationarity for a recurrent MC, it is not for a recurrent MRP. The basic equation for such processes (see Lemma 2.1 and Corollary 3.2) is

$$x_j = \sum_{i,k} x_i p_{ik}(1 - H_i) * M_{kj}(t) \tag{4.2}$$

for all $t \geqq 0$. This equation implies (4.1) by setting $t = 0$. What has been shown in Section 3 is that there is a unique solution to (4.2).

Thirdly, the example illustrates that $m_i = {}_0M_{0i}(+\infty)$, which is a solution to (4.1), need not be the same quantity as ${}_0M_{0i}^*(+\infty)$, the expected number of visits by the embedded chain to i before a visit to zero, starting at zero. This follows since m_{-1} is easily evaluated to be p so that $m_i = 1$ $(i = 0, 2, 4, \cdots)$, and $m_i = p - q$ $(i = 1, 3, 5, \cdots)$, whereas ${}_0M_{0i}^*(+\infty) = 0$ if $i = 1, 3, 5, \cdots$.

Lemma 2.1 shows that any transient MC which may be considered as the embedded MC of an irreducible, recurrent, non-lattice MRP of the type discussed in this paper, has at least one solution to (4.1). An example of a *recurrent*, irreducible, non-lattice MRP for which the embedded MC has no solution is constructed as follows. Construct an ascending escalator on I^+. Hitch another ascending escalator on I^+ onto this stopped process. Continue to define a process, Z_t, by this method for all $t \geqq 0$. The embedded MC has transition probabilities given by $p_{i,i+1} = 1$ for $i = 0, 1, 2, \cdots$. There is therefore no positive solution to (4.1) since this equation would imply that $x_0 = \sum_i x_i p_{i0} = 0$ because $p_{i0} = 0$ for all i. Observe that this MRP does not satisfy the conditions of this paper, since $Z_u \to \infty$ as $u \nearrow t$ does not imply $Z_u \to \infty$ as $u \searrow t$.

3. To obtain the stationary measure for the more general MRP for which the factorization $Q_{ij}(\cdot;\cdot) = p_{ij}H_i(\cdot;\cdot)$ does not hold, proceed as follows. Let $Z_t^* = (Z_t, Z_t^+)$. As discussed in the third paragraph following (1.3) of Section 1, $\{(Z_t^*, V_t); t \geqq 0\}$ and $\{(Z_t^*, U_t); t \geqq 0\}$ are MRP's to which Theorems 2.1, 3.2, and 3.3 may be applied. Choose $i, j, k, m \,\varepsilon\, I^+$ such that $p_{ij} > 0$ and $p_{km} > 0$. Then $H_{(k,m)} = p_{km}^{-1}Q_{km}$ and ${}_{(i,j)}M_{(i,j),(k,m)}(+\infty) = C\,{}_jM_{jk}(+\infty)p_{km}$ where C is a constant independent of k and m. The latter result may be obtained by direct evaluation or by showing that ${}_jM_{jk}p_{km}$ satisfies equation (4.2), so that by Remark 2, ${}_{(i,j)}M_{(i,j),(k,m)} = C\,{}_jM_{jk}(+\infty)p_{km}$. Setting $(k, m) = (i, j)$ and recalling that ${}_{(i,j)}M_{(i,j),(i,j)} = 1$, C is evaluated as $p_{ij}^{-1}\,{}_iM_{ij}(+\infty)$. Thus, ${}_{(i,j)}M_{(i,j),(k,m)}(+\infty) = p_{ij}^{-1}\,{}_iM_{ik}(+\infty)p_{km}$. Hence, if π is any σ-finite stationary measure, then

$$\begin{aligned}\pi(k, m, x) &= C_{(i,j)}M_{(i,j),(k,m)}(+\infty)\int_0^x [1 - H_{(k,m)}(u)]\,du \\ &= C'm_k \int_0^x [p_{km} - Q_{km}(u)]\,du = C'm_k\tilde{Q}_{km}(x).\end{aligned}$$

4. A delayed Markov renewal process (called a general Markov renewal process in [10]) is one for which $U_0 = 0$, the distribution of Z_0 is governed by a set $\{a_i\}$ of non-negative numbers (these numbers forming a probability distribution only if $\sum a_i = 1$), the distribution of (Z_0^+, V_0) is governed by one set of transition functions $\{Q'_{ij}(\cdot)\}$ whereas from then on the (Z, V)-process is governed by $\{Q_{ij}(\cdot)\}$. Theorems 2.1 and 3.2 show that an MRP will give rise to a stationary delayed MRP if and only if a_i and Q'_{ij} are chosen so that $a_iQ'_{ij}(x) = cm_i\tilde{Q}_{ij}(x)$.

In particular, if the process is positive recurrent (that is, $\mu_{jj} < \infty$, where μ_{jj} is the mean of G_{jj}), it follows that m_j is proportional to μ_{jj}^{-1} and that $\sum \eta_i\mu_{ii}^{-1} = 1$, where η_i is the mean of H_i. Setting $a_i = \eta_i\mu_{ii}^{-1}$ and $Q'_{ij}(x) = \eta_i^{-1}\tilde{Q}_{ij}(x)$, one obtains the unique stationary probability measure. In the case of a strongly-regular positive recurrent MRP, this gives the exact relationship between the "stationary" probabilities $\{\eta_j\mu_{jj}^{-1} = \lim_{t\to\infty} P_{jj}(t)\}$ of the Z-process and the stationary probabilities $\{\mu_{jj}^{*-1}\}$ of the embedded MC, where μ_{jj}^* denotes the mean recurrence time of state j in the embedded MC. (Cf. [10]). It is shown in [10] that $\mu_{jj} = C\mu_{jj}^*$. Therefore, the "stationary" probability of state j in the Z-process is proportional to η_j times the stationary probability of state j in the embedded MC. This is related to a result of Miller [8] for stable, continuous parameter MC's.

5. From Theorem 3.2, one can deduce an interesting fact about the recurrence-time distribution of the various states, namely

THEOREM 4.1. *For any states* $i, j \in I^+$ *of a recurrent, irreducible, non-lattice* MRP,

$$(4.3) \qquad (1 - G_{jj}) * M_{ii}(t) \to {}_jM_{ji}(+\infty)$$

as $t \to +\infty$.

PROOF. If the MRP is positive recurrent, this follows from the key renewal theorem since

$$(1 - G_{jj}) * M_{ii}(t) \to \mu_{ii}^{-1} \int_0^\infty [1 - G_{jj}(u)]\, du = \mu_{ii}^{-1} \mu_{jj} = {}_jM_{ji}(+\infty).$$

If the MRP is null recurrent this approach does not work since the limit reduces to $+\infty/+\infty$. However, in this case one obtains from (3.22), (since $M_{ij}^* = G_{ij}^* * M_{jj}^* + \delta_{ij}$ and $M_{jj}^* = M_{jj}$),

$$G_{ij}^*(t) = {}_iM_{ij}(+\infty) H_i * K_{ij} * M_{ii} * (1 - G_{jj})(t).$$

Since $G_{ij}^*(t) \to 1$, $H_i(t) \to 1$, and $K_{ij}(t) \to 1$ as $t \to +\infty$, one obtains $1 = {}_iM_{ij}(+\infty) \lim_{t \to \infty} M_{ii} * (1 - G_{jj})(t)$ which is the desired result.

5. Stationarity of a strongly regular MRP. Throughout this section it is assumed that the MRP is strongly regular, that is, $N(t) = \sum_i N_i(t) < +\infty$ a.s., and that $\eta_j = \int_0^\infty [1 - H_j(x)]\, dx < \infty$ for all $j \in I^+$. Set

$$(5.1) \qquad q_{ij} = (p_{ij} - \delta_{ij})/\eta_i .$$

The following theorem generalizes a result of Miller [8] for continuous parameter MC's to a result for MRP's by means of a shorter non-analytic proof.

THEOREM 5.1. *A strongly-regular, irreducible* MRP *is positive recurrent* (i.e. $\mu_{jj} < \infty$ *for all* $j \in I^+$) *if and only if* $\eta_j < +\infty$ *for* $j \in I^+$ *and there exists a convergent sequence* $\{y_i\}$ *of positive numbers such that* $\sum y_i q_{ij} = 0$. *The sequence is unique* (*up to a multiplicative constant*).

PROOF. Assume the MRP is positive recurrent. Then the finiteness of μ_{jj} implies that of each η_i, a consequence of the identity

$$(5.2) \qquad \mu_{jj} = \sum_i {}_jM_{ji}(\infty)\eta_i$$

(see equation (4.9) of [11]) since each ${}_jM_{ji}(\infty)$ is positive as may be seen from the relationship, ${}_jM_{ji}(\infty) = {}_0M_{0i}(\infty)/{}_0M_{0j}(\infty) = m_i/m_j$, which follows from Corollary 3.4. Also, by substituting this last relationship into (5.2) it follows that $m_j = \mu_{00}/\mu_{jj}$ for each j. Therefore, upon setting $y_i = \eta_i/\mu_{ii}$ one obtains that $\sum_i y_i = 1$ and $\sum_i y_i q_{ij} = 0$, the latter using the fact that $\{m_i\} = \{\mu_{00}/\mu_{ii}\}$ satisfies (4.1).

If $\eta_j < +\infty$, $\sum_i y_i < +\infty$, and $\sum_i y_i q_{ij} = 0$ for some positive sequence of y's, set $c_i = y_i \eta_i^{-1}$. Then $\sum_i \eta_i c_i < +\infty$ and $\sum_i c_i p_{ij} = c_i$. Set $Q_{ij}^*(x) = c_j c_i^{-1} p_{ji} H_i(x)$ where p_{ji} and H_i are the functions which determine the original strongly-regular MRP. $\mathbf{Q}^*$ is a matrix of mass functions and hence also determines a strongly-regular MRP. (This can be shown using Theorem 4.1 of [9].) The renewal function, $\mathbf{M}^*(t)$, associated with the latter MRP is uniquely determined by $\mathbf{Q}^*(t)$ and is given in terms of the original renewal function $\mathbf{M}(t)$ by

$$M^*_{ij}(t) = c_j c_j^{-1} H_i * \textstyle\sum_k p_{jk} M_{ki}(t) + \delta_{ij} .$$

From this and (2.7.2) of [11] one obtains

$$P^*_{ij}(t) = c_i^{-1} H_i * c_j \textstyle\sum_k p_{jk}(1 - H_j) * M_{ki}(t) + \delta_{ij}[1 - H_j(t)].$$

Summing over j, one obtains

$$1 = 1 - H_i(t) + c_i^{-1} H_i * \textstyle\sum_{j,k} c_j p_{jk}(1 - H_j) * M_{ki}(t)$$

or

$$c_i H_i(t) = H_i * \textstyle\sum_{j,k} c_j p_{jk}(1 - H_j) * M_{ki}(t). \tag{5.3}$$

Upon integrating both sides of (5.3) from 0 to s, one obtains

$$\begin{aligned} c_i \textstyle\int_0^s H_i(t)\, dt &= H_i * \textstyle\sum_{j,k} c_j p_{jk} \tilde{H}_j * M_{ki}(s) \\ &= H_i * \{ \textstyle\sum_{j,k \neq i} c_j p_{jk} \tilde{H}_j * G_{ki} + \textstyle\sum_j c_j p_{ji} \tilde{H}_j \} * M_{ii}(t). \end{aligned} \tag{5.4}$$

As $s \to +\infty$, the left side of (5.4) approaches $+\infty$. On the other hand, since the factor within parentheses on the right hand side is bounded by its value at $+\infty$, namely

$$\textstyle\sum_j c_j [\sum_{k \neq i} p_{jk} \eta_j G_{ki}(\infty) + p_{ji} \eta_j] \leqq \sum_j c_j \eta_j < +\infty,$$

it follows that $M_{ii}(+\infty) = +\infty$ so that the MRP is recurrent. Since the process is strongly-regular, the embedded MC is also recurrent so that the solution $\{c_i\}$ is unique. Therefore $c_i = C\ {}_jM_{ji}(\infty)$ for any choice of $j \in I^+$. Thus $\mu_{jj} = \sum_i {}_jM_{ji}(\infty)\eta_i = C \sum_i c_i \eta_i < +\infty$. This states that the MRP is positive recurrent as required. Also, since $y_i = C m_i \eta_i$, it follows that $\{y_i\}$ is unique, thereby completing the proof.

Theorem 4.1 shows that in the strongly-regular case with each $\eta_j < \infty$, positive recurrence depends only on the equation $\sum_i y_i q_{ij} = 0$ or equivalently on $\sum_i x_i p_{ij} = x_j$. Therefore, in this case, (4.1) must imply equation (4.2).

6. Stationarity of Markov renewal processes with auxiliary paths. In most applications involving a Renewal process, the process arises as the successive occurrence times of some regenerative event, such as the event that a server is free in a single-server queueing model with exponential inter-arrival times and arbitrary service times. (This queueing model is commonly denoted by $(M, G, 1)$.) Likewise, many Markov chains arise in practice as embedded processes, such as the sequence of queue lengths of a $(M, G, 1)$ queueing model observed at the successive times of completion of service. One of the shortcomings of this method of embedded Markov chains is that the continuous time parameter of the basic stochastic process has been replaced by the discrete time parameter of an MC. In particular, in many applications of this method to queueing theory, the stationary distribution of the embedded MC is obtained and applied even though it is not the stationary distribution of the original process. It is much more natural and informative to study the embedded MRP rather than the embedded MC since the time scale of the original process is retained. Thus, for the $(M, G, 1)$ queueing model, it is reasonable to study the embedded MRP which makes transitions at each completion of service and whose states indicate the queue lengths after these completions. (We remark that the relationship between the stationary distribution of the embedded MRP and the corresponding embedded MC is given in Section 5 above.)

Although the embedded MRP in the queueing model is more informative than the embedded MC, it does not contain all of the information yielded by the model. It does not, for example, contain the information about the arrival times of customers. However, it is possible to generalize the concept of an MRP to include the complete queueing process. This generalization allows for auxiliary stochastic processes to occur between successive transitions of the MRP. An example of an MRP with auxiliary processes is given by the $(M, G, 1)$ queueing model as follows. If with each state of the embedded MRP, which measures the queue length after each completion of service, one associates appropriately a Poisson process to describe the arrivals to the queue during the subsequent service time, the original queueing process may be obtained. Another example involving Brownian motion is described as follows: Consider the MRP (actually an S-MP) which measures at time t the last integer value taken on by a Brownian motion which is equal to zero at $t = 0$. It is then possible to attach to each state of this MRP an appropriate Brownian motion in such a way that the resulting process is equivalent to the original Brownian motion.

A formal definition of an MRP with auxiliary paths is given as follows: It is an extension of the regenerative processes with *tours* introduced by Smith [14]. Consider an MRP which satisfies the definition given in Section 1. Let there be a collection of stochastic processes, $\{Y_i(u): u \geqq 0\}$, one for each state $i \varepsilon I^+$, with values in some fixed abstract measurable space $(E, \mathcal{A})$.

For $n \geqq 1$, let $\{Y_i^{(n)}(u); u \geqq 0\}$ be independent versions of the Y_i-process. Let these processes and the MRP be defined on the same probability space. At the instant, T_{jr}, that the given MRP enters state j for the rth time, the auxiliary path on $[T_{jr}, T_{jr} + X_{jr})$ is defined as the process $\{Y_j^{(r)}(u); 0 \leqq u < X_{jr}\}$ where $T_{jr} + X_{jr}$ is the time until the next transition. It is assumed that each auxiliary path is completely independent of past states visited, holding times in these states, and the auxiliary paths described while in these states.

For $t \geqq 0$, define $W_t = Y_j^{(r)}(U_t)$ on the event, $[T_{jr} \leqq t < T_{jr} + X_{jr}]$. The W-process will be called the *auxiliary path process* associated with the given MRP.

In order to relate the W-process to the MRP it will be assumed that there are functions $Q_{ij}(\cdot, \cdot; \cdot)$ and $K_i(\cdot, \cdot; \cdot, \cdot, \cdot)$, the former defined on $[0, +\infty) \times (-\infty, +\infty) \times \mathcal{A}$ for each $i, j \varepsilon I^+$ and the latter on $[0, +\infty) \times E \times I^+ \times$

$[0, +\infty) \times \mathcal{A}$ for each $i \in I^+$, such that

$$(6.1)\quad P[W_{t+u} \in A,\ V_t > u,\ Z_t^+ = j \mid Z_t = i,\ U_t = x; (Z_s, U_s, W_s), 0 \leqq s < t - x] = Q_{ij}(x, u; A)$$

and

$$(6.2)\quad P[Z_t^+ = j,\ V_t > u,\ W_{t+u} \in A \mid Z_t = i,\ U_t = x,\ W_t = w; (Z_s, U_s, W_s),\ 0 \leqq s < t] = K_i(x, w; j, u, A)$$

for all $t \geqq 0$. Note that (6.2) says that at time t, the state next to be visited and the time until the next transition may be dependent upon W_t.

Set $Q_{ij}(u; A) = Q_{ij}(0, u; A)$ and $H_i(u; A) = \sum_j Q_{ij}(u; A)$. It is clear that $H_i(u; E) = 1 - H_i(u)$ and it may be shown as in Lemma 1.1 that $Q_{ij}(x, u; A) = Q_{ij}(u + x; A)[1 - H_i(x)]^{-1}$. Under assumptions (6.1) and (6.2), it follows that the (Z, U, W)-process is a Markov process with a stationary transition function given by

$$\begin{aligned} P[Z_t &= j,\ U_t \leqq y,\ W_t \in A \mid Z_0 = i,\ U_0 = x,\ W_0 = w] \\ &= \sum_k \int_{0-}^{t} \int_{(t-u-y)-}^{t-u} H_j(t - u - v; A)\, dM_{kj}(v)\, d_u L_i(x, w, k, u) \qquad \text{if } x + t > y \\ (6.3)\qquad &= \sum_k \int_{0-}^{t} \int_{0-}^{t-u} H_j(t - u - v; A)\, dM_{kj}(v)\, d_u L_i(x, w, k, u) \\ &\quad + \delta_{ij} T_i(x, w; t, A) \qquad \text{if } x + t \leqq y \end{aligned}$$

where $L_i(x, w, k, u) = K_i(x, w; j, 0, E) - K_i(x, w; j, u, E)$ and $T_i(x, w; t, A) = \sum_k K_i(x, w; k, t, A)$.

Let α be a measure defined on the Borel sets of $I^+ \times [0, +\infty) \times E$. Let $\alpha(i, x, A)$ denote the measure of the set $\{i\} \times [0, x] \times A$ for each $A \in \mathcal{A}$. The main result of this section shows that if the underlying MRP is recurrent, then there is a unique stationary measure for the (Z, U, W)-process amongst a natural class of measures.

THEOREM 6.1. *If the underlying* MRP *is recurrent then*

$$\pi(i, y, A) = m_i \int_0^y H_i(u; A)\, du$$

is the unique positive stationary measure for the (Z, U, W)-process amongst all measures α for which $\alpha(\cdot, \cdot, E)$ is a σ-finite measure for the (Z, U)-process.

PROOF. For any initial measure α, let P_α be the measure induced on the process and let

$$P_\alpha(j, y, A, t) = \sum_k \int_{0-}^{+\infty} P[Z_t = j, U_t \leqq y, W_t = A \mid Z_0 = k, U_0 = x, W_0 = w]\alpha(k, dx, dw).$$

Showing that π is a stationary measure is equivalent to showing that $\pi(j, y, A) = P_\pi(j, y, A, t)$ for each $j \in I^+$, $y \geqq 0$, $A \in \mathcal{A}$, and for all $t \geqq 0$. For $t > y$,

$$
\begin{aligned}
P_\pi(j, y, A, t) &= \textstyle\sum_{k,r} \int_{0-}^{+\infty} \int_{0-}^{t} \int_{(t-u-y)-}^{t-u} H_j(t-u-v; A)\, dM_{i,j}(v) \\
&\qquad \cdot d_u L_k(x, w, r, u)\pi(k, dx, dw) \\
&= \textstyle\sum_{k,r} \int_{0-}^{t} \int_{(t-u-y)-}^{t-u} H_j(t-u-v; A)\, dM_{ij}(v) \\
&\qquad \cdot d_u\{\int_{0-}^{+\infty} \int_E L_k(x, w, r, u)\pi(k, dx, dw)\}.
\end{aligned} \tag{6.4}
$$

The bracketed expression may be simplified as follows:

$$
\begin{aligned}
\textstyle\int_{0-}^{+\infty} \int_E L_k(x, w, r, u)\pi(k, dx, dw) &= P_\pi[Z_0 = k, Z_0^+ = r, V_0 \leqq u] \\
&= \textstyle\int_{0-}^{+\infty} P[Z_0^+ = r, V_0 \leqq u \mid Z_0 = k, U_0 = x]\alpha(k, dx, E) \\
&= \textstyle\int_{0-}^{+\infty} Q_{kr}(x; u)[1 - H_k(x)]\, dx = m_k p_{kr} \int_0^u [1 - H_k(x)]\, dx.
\end{aligned}
$$

Thus, (6.4) becomes

$$
\begin{aligned}
&P_\pi(j, y, A, t) \\
&\quad = \textstyle\sum_{k,r} m_k p_{kr} \int_0^t \int_{(t-u-y)-}^{t-u} H_j(t-u-v; A)[1 - H_k(u)]\, dM_{rj}(v)\, du \\
&\quad = m_j \textstyle\int_0^y H_j(s, A)\, ds = \pi(j, y, A)
\end{aligned}
$$

using (2.1) and the change of variable $t - u - v = s$.

If $t \leqq y$, one obtains

$$
\begin{aligned}
&P_\pi(j, y, A, t) \\
&= \textstyle\sum_{k,r} \int_{0-}^{+\infty} \int_E \int_{0-}^{t} \int_{0-}^{t-u} H_j(t-u-v; A)\, dM_{rj}(v)\, d_u L_k(x, w, r, u)\pi(k, dx, dw) \\
&\qquad + \textstyle\int_{0-}^{y-t} \int_E T_j(x, w; t, A)\pi(j, dx, dw).
\end{aligned}
$$

The first term on the right side of this equation reduces to $\pi(j, t, A)$ and the second to

$$
\begin{aligned}
&P_\pi[Z_0 = j, U_0 \leqq y - t, V_0 > t, W_t \,\varepsilon\, A] \\
&\qquad = \textstyle\int_0^{y-t} P[V_0 > t, W_t \,\varepsilon\, A \mid Z_0 = j, U_0 = x]\pi(j, dx, E) \\
&\qquad = m_j \textstyle\int_0^{y-t} H_j(x, t; A)[1 - H_j(x)]\, dx \\
&\qquad = m_j \textstyle\int_0^{y-t} H_j(x + t; A)\, dx = m_j \int_t^y H_j(s, A)\, ds \\
&\qquad = \pi(j, y, A) - \pi(j, t, A)
\end{aligned}
$$

so that $P_\pi(j, y, A, t) = \pi(j, y, A)$ and π is a stationary measure.

Now let α be any other stationary measure. Then $\beta(i, x) = \alpha(i, x, E)$ is a measure on the Borel sets of $I^+ \times [0, +\infty)$. The equations of stationarity become

$$
\begin{aligned}
\alpha(j, y, A) &= \textstyle\sum_{k,r} \int_{0-}^{t} \int_{(t-u-y)-}^{t-u} H_j(t-u-v; A)\, dM_{rj}(v) \\
&\qquad \cdot d_u\{\int_{0-}^{+\infty} \int_E L_k(x, w, r, u)\alpha(k, dx, dw)\} \quad \text{if} \quad t > y \\
&= \textstyle\sum_{k,r} \int_{0-}^{t} \int_{0-}^{t-u} H_j(t-u-v; A)\, dM_{rj}(v) \\
&\qquad \cdot d_u\{\int_{0-}^{+\infty} \int_E L_k(x, w, r, u)\alpha(k, dx, dw)\} \\
&\qquad + \textstyle\int_{0-}^{y-t} \int_E T_j(x, w; t, A)\alpha(k, dx, dw) \quad \text{if} \quad t \leqq y.
\end{aligned} \tag{6.5}
$$

The bracketed expression in each equation of (6.5) reduces to

$$P_\alpha[Z_0 = k,\, Z_0^+ = r,\, V_0 \leqq u] = \int_{0-}^{+\infty} Q_{kr}(x;\, u)\beta(k,\, dx)$$

and the second term in the second equation reduces to

$$P_\alpha[Z_0 = j,\, U_0 \leqq y - t,\, V_0 > t,\, W_t \,\varepsilon\, A] = \int_0^{y-t} H_j(x,\, t;\, A)\beta(i,\, dx).$$

Using these results and letting $A = E$ in (6.5), one obtains

$$\begin{aligned}\beta(j,\, y) &= \textstyle\sum_{k,r} p_{kr} \int_{0-}^{+\infty} S_r(j,\, y;\, \cdot\,) * H_k(x;\, \cdot\,)(t)\beta(k,\, dx) && \text{if } t > y \\ (6.6) \qquad &= \textstyle\sum_{k,r} p_{kr} \int_{0-}^{+\infty} P_{rj}(\,\cdot\,) * H_k(x;\, \cdot\,)(t)\beta(k,\, dx) \\ &\qquad + \int_0^{y-t} [1 - H_j(x;\, t)]\beta(j,\, dx) && \text{if } t \leqq y.\end{aligned}$$

That is, β is a σ-finite stationary measure for the $(Z,\, U)$-process. Hence, $\beta(j, x) = cm_j \int_0^x [1 - H_j(u)]\, du$ by Theorem 3.3 and from the first equation in (6.5), one obtains

$$\begin{aligned}\alpha(j, y, A) &= c \textstyle\sum_{k,r} m_k p_{kr} \int_{0-}^{t} \int_{(t-u-y)-}^{t-u} H_j(t - u - v, A)[1 - H_k(u)]\, dM_{rj}(r)\, du \\ &= cm_j \int_0^y H_j(s, A)\, ds = c\pi(j, y, A)\end{aligned}$$

as before and the proof is complete.

Two very simple but interesting examples of auxiliary paths are obtained by letting $Y_i^{(r)}(u) = \max(X_{ir} - u,\, 0)$ and $Y_i^{(r)}(u) \equiv Z_{(T_{ir}+X_{ir})}$. In these cases $W_t = V_t$ and $W_t = Z_t^+$. The reader can verify that in the first example $Q_{ij}(x,\, u;\, [0,\, y]) = p_{ij}[H_i(x + y + u) - H_i(x + u)][1 - H_i(x)]^{-1}$ and $K_i(x,\, w;\, j,\, u,\, [0,\, y]) = p_{ij}$ or 0 depending as $u < w \leqq u + y$ or not; while in the second example $Q_{ij}(x,\, u;\, k) = \delta_{jk}p_{ij}[1 - H_i(x;\, u)]$ and $K_i(x,\, m,\, j,\, u,\, k) = \delta_{jk}\delta_{km}[1 - p_{im}^{-1}Q_{im}(x;\, u)]$. From these one obtains $H_i(s,\, [0,\, y]) = H_i(s + y) - H_i(s)$ in the first example and $H_i(s,\, k) = p_{ik}[1 - H_i(s)]$ in the second example so that, not unexpectedly, the unique stationary measures for the $(Z,\, U,\, V)$-process and $(Z,\, Z^+,\, U)$-process are, respectively, $m_i \int_0^x [H_i(s + y) - H_i(s)]\, ds = \pi(i,\, x) + \pi(i, y) - \pi(i,\, x + y)$ and $m_i p_{ik} \int_0^x [1 - H_i(s)]\, ds = p_{ik}\pi(i,\, x)$.

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