A NOTE ON CONSERVATIVE CONFIDENCE REGIONS FOR THE MEAN OF A MULTIVARIATE NORMAL

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1. Introduction. Suppose $x_i = (x_{1i}, \cdots, x_{mi})'$ $(i = 1, \cdots, n)$ are independent observations from a m-variate normal population with mean vector μ and co-variance matrix Σ. Let $\bar{x}_{i\cdot} = \sum_j x_{ij}/n$ and $s_i^2 = \sum_j (x_{ij} - \bar{x}_{i\cdot})^2/(n-1)$. If Σ is a diagonal matrix, a confidence region for μ can be constructed from

$$(1) \qquad \Pr\{|z_i| \leqq c_i, i = 1, \cdots, m\} = \prod_1^m \Pr\{|z_i| \leqq c_i\}$$

with $z_i = n^{\frac{1}{2}}(\bar{x}_{i\cdot} - \mu_i)/\sigma_i$ if the diagonal elements, σ_i^2, of Σ are known, and $z_i = n^{\frac{1}{2}}(\bar{x}_{i\cdot} - \mu_i)/s_i$ otherwise. Dunn [1] conjectured that, for any Σ,

$$(2) \qquad \Pr\{|z_i| \leqq c_i, i = 1, \cdots, m\} \geqq \prod_1^m \Pr\{|z_i| \leqq c_i\}.$$

She proved the conjecture when Σ is of a special form, and in general for $m = 2$ and $m = 3$, and used the relation to construct conservative confidence limits for μ. The purpose of this note is to provide a general proof of the conjecture. When the variances are known, the conjecture has been proved with a different method by Sidak [2].

2. Diagonal elements of Σ kown. In this case $z = (z_1, \cdots, z_m)'$ has a normal distribution with $E(z_i) = 0$, $E(z_i^2) = 1$ and covariance matrix AA' say ($\sum_j a_{ij}^2 = 1$, $i = 1, \cdots, m$). It is always possible to choose A so that $a_{i1} = 0$ $(i = 2, \cdots, m)$. Let $y = A^{-1}z$, and let R_m, R_{m-1} be the regions $\{y: |\sum_{j=1}^m a_{ij}y_j| \leqq c_i, i = 1, \cdots, m\}$ and $\{y: |\sum_{j=2}^m a_{ij}y_j| \leqq c_i, i = 2, \cdots, m\}$ respectively. Since $a_{i1} = 0$ $(i = 2, \cdots, m)$, $R_m = R_{m-1} \cap \{|\sum_{j=1}^m a_{1j}y_j| \leqq c_1\}$.

Lemma 1. $\Pr\{y \varepsilon R_m\} \geqq [\Phi(c_1) - \Phi(-c_1)] \Pr\{y \varepsilon R_{m-1}\}$

Proof. The Lemma has been proved essentially by Dunn for $m = 2$. If $m > 2$, it is enough to show that

$$(3) \qquad \Pr\{R_m \mid y \varepsilon P\} \geqq [\Phi(c_1) - \Phi(-c_1)] \Pr\{R_{m-1} \mid y \varepsilon P\}$$

for every plane P containing the x_1 — axis.

Choose such a plane P. By an orthogonal transformation of $(y_2, \cdots, y_m)'$, P can be taken to be the co-ordinate plane $\{y: y_i = 0, i = 3, \cdots, m\}$. Then equation (3) becomes

$$(4) \quad \Pr\{|a_{11}y_1 + a_{12}y_2| \leqq c_1, |y_2| \leqq c_2'\} \geqq [\Phi(c_1) - \Phi(-c_1)] \Pr\{|y_2| \leqq c_2'\}.$$

But this follows immediately from the case for $m = 2$ [since $a_{11}^2 + a_{12}^2 \leqq 1$]. Now

$$\Pr\{|z_i| \leqq c_i, i = 1, \cdots, m\} = \Pr\{y \varepsilon R_m\}.$$

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and

$$\Pr\{|z_i| \leqq c_i\} = \Phi(c_i) - \Phi(-c_i).$$

Theorem 1 then follows from Lemma 1 by induction.

THEOREM 1. *If the diagonal elements of* Σ *are known*

$$\Pr\{|z_i| \leqq c_i, i = 1, \cdots, m\} \geqq \prod_1^m \Pr\{|z_i| \leqq c_i\}$$

3. Diagonal elements of Σ unknown. Let A and y be as in the preceding section.

LEMMA 2. $\Pr\{|\sum_j a_{ij}y_j| \geqq c_i, i = 1, \cdots, m\} \geqq \prod_1^m \Pr\{|y_i| \geqq c_i\}$

PROOF. Suppose $m = 2$,

$$\begin{aligned}\Pr\{|a_{11}y_1 + a_{12}y_2| \leqq c_1\} &= \Pr\{|a_{11}y_1 + a_{12}y_2| \leqq c_1, |y_2| \leqq c_2\} \\ &\qquad + \Pr\{|a_{11}y_1 + a_{12}y_2| \leqq c_1, |y_2| \geqq c_2\} \\ &\geqq \Pr\{|y_1| \leqq c_1, |y_2| \leqq c_2\} \\ &\qquad + \Pr\{|a_{11}y_1 + a_{12}y_2| \leqq c_1, |y_2| \geqq c_2\}\end{aligned}$$

by Theorem 1.

But

$$\begin{aligned}\Pr\{|a_{11}y_1 + a_{12}y_2| \leqq c_1\} &= \Pr\{|y_1| \leqq c_1\} \qquad (a_{11}^2 + a_{12}^2 = 1) \\ &= \Pr\{|y_1| \leqq c_1, |y_2| \leqq c_2\} + \Pr\{|y_1| \leqq c_1, |y_2| \geqq c_2\}\end{aligned}$$

Therefore

$$\Pr\{|a_{11}y_1 + a_{12}y_2| \leqq c_1, |y_2| \geqq c_2|\} \leqq \Pr\{|y_1| \leqq c_1, |y_2| \geqq c_2\}$$

so that

$$\Pr\{|a_{11}y_1 + a_2y_2| \geqq c_1, |y_2| \geqq c_2\} \geqq \Pr\{|y_1| \geqq c_1, |y_2| \geqq c_2\}$$

The proof for $m > 2$ proceeds just as the proof of Theorem 1.

Let V be the matrix with elements $v_{ij} = (x_{ij} - \mu_i)/\sigma_i$ let H be an $n \times n$ orthogonal matrix with nth column equal to $(1/n^{\frac{1}{2}}, 1/n^{\frac{1}{2}}, \cdots, 1/n^{\frac{1}{2}})'$ and let $U = VH$. Then the columns of U are independent and identically distributed with $E(u_{ij}) = 0$ and $E(u_{ij}^2) = 1$. Moreover $z_i = (n - 1)^{\frac{1}{2}}u_{in}/(\sum_{j=1}^{n-1} u_{ij}^2)^{\frac{1}{2}}$. Let the covariance matrix of each column vector be BB' with B chosen so that $b_{i1} = 0$ $(i = 2, \cdots, m)$, and let $Y = B^{-1}U$. Then

$$\begin{aligned}\Pr\{|z_i| \leqq c_i, i = 1, \cdots, m\} &= \Pr\{u_{ni}^2 \leqq [c_i^2/(n - 1)] \textstyle\sum_{j=1}^{n-1} u_{ji}^2, \\ &\qquad i = 1, \cdots, m\} \\ &= \Pr\{[\textstyle\sum_k b_{ik}y_{kn}]^2 \leqq [c_i^2/(n - 1)] \sum_{j=1}^{n-1} [\sum_k b_{ik}y_{kj}]^2, \\ &\qquad i = 1, \cdots, m\} \\ &\geqq \Pr\{y_{in}^2 \leqq [c_i^2/(n - 1)] \textstyle\sum_{j=1}^{n=1} [\sum_k b_{ik}y_{kj}]^2, \\ &\qquad i = 1, \cdots, m\}\end{aligned}$$

by Theorem 1

$$\geqq \Pr\{y_{in}^2 \leqq [c_i^2/(n-1)][\sum_{j=1}^{n-2} (\sum_k b_{ik} y_{kj})^2 + y_{in-1}^2], \quad i = 1, \cdots, m\}$$

$$\vdots$$

$$\geqq \Pr\{y_{in}^2 \leqq [c_i^2/(n-1)] \sum_{j=1}^{n-1} y_{ij}^2, i = 1, \cdots, m\}$$

by repeated application of Lemma 2

$$= \prod_1^n \Pr\{|y_{in}| \leqq [c_i/(n-1)^{\frac{1}{2}}][\sum_{j=1}^{n-1} y_{ij}^2]^{\frac{1}{2}}, \quad i = 1, \cdots, m\}$$

$$= \prod_1^n \Pr\{|z_i| \leqq c_i\}.$$

This proves:

THEOREM 2. *If* $z_i = n^{\frac{1}{2}}(m_i - \mu_i)/s_i$, *then*

$$\Pr\{|z_i| \leqq c_i, i = 1, \cdots, m\} \geqq \prod_1^m \Pr\{|z_i| \leqq c_i\}$$

4. Acknowledgment. I would like to thank the referee for pointing out reference [2].

REFERENCES

[1] DUNN, OLIVE JEAN. (1958). Estimation of the means of dependent variables. *Ann. Math. Statist.* **29** 1095–1111.

[2] SIDAK, ZBYNEK (1965). Rectangular confidence regions for means of multivariate normal distributions. 35th Session of the International Statistical Institute, Belgrade.

CORRECTION NOTE

CORRECTION TO

CALCULATION OF EXACT SAMPLING DISTRIBUTION OF RANGES FROM A DISCRETE POPULATION

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Correction to page 530, *Ann. Math. Statist.* **26,** 530–532, the lower limit on the summation in equation (2) should read $j = i$ not $j = 1$, as it was printed.