SEQUENTIAL PROCEDURES FOR SELECTING THE BEST ONE OF SEVERAL BINOMIAL POPULATIONS[1]

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1. Introduction and summary. A problem that seems to be of some practical importance is how to select the best one of k experimental categories or populations when there is a fixed probability for each population that any measurement will be classified as a 'success' and the best population is defined as the one with the greatest probability of a success. For example, we might be interested in determining which of k new drugs offers the greatest probability of survival against a specified disease, or which of k new production techniques has the greatest probability of producing a 'good' item.

A treatment of this problem using a fixed sample size approach was given by Sobel and Huyett [6]. To describe their formulation of the problem, denote the populations by $\Pi_1, \Pi_2, \cdots, \Pi_k$, the corresponding probabilities by $p_1, p_2, \cdots, p_k$, and the ordered probabilities by $p_{[1]} \geqq p_{[2]} \geqq \cdots \geqq p_{[k]}$, and let $\Pi_{[j]}$ be the population associated with $p_{[j]}$. Then [6] described a statistical procedure and gave tables for determining the common sample size required with each population so that population $\Pi_{[1]}$ will be selected with probability $\geqq P^*$ whenever $p_{[1]} \geqq p_{[2]} + d$, where d and P^* are constants selected in advance of the experiment. This formulation of the problem, which we will call the main formulation, seems satisfactory when nothing is known about the magnitude of $(p_1, p_2, \cdots, p_k)$ or if there is some *a priori* information available which indicates that $p_{[1]}$ and $p_{[2]}$ do not differ too much from .5, say $.25 \leqq p_{[2]} \leqq p_{[1]} \leqq .75$. An alternative formulation of the problem when the *a priori* information indicates that $p_{[2]}$ and $p_{[1]}$ differ substantially from .5 was given in [6] as follows: the sample size is determined so that population $\Pi_{[1]}$ is selected with probability $\geqq P^*$ whenever $p_{[2]} \leqq p^*_{[2]}$ and $p_{[1]} \geqq p^*_{[2]} + d$, where $p^*_{[2]}$ is an additional constant determined in advance of the experiment on the basis of the *a priori* information about the probable value of $p_{[2]}$.

The present paper is based on a somewhat novel use of the Poisson distribution to obtain a random number of measurements from each population at every stage of experiment combined with the application of one-sided sequential confidence limits developed in [4]. Using these techniques we derive sequential procedures for selecting the best population both for the main formulation and for a generalization of the alternative formulation of the problem. Some Monte Carlo calculations which are summarized in Section 5 indicate that a substantial saving is possible with the sequential procedures.

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2. The sampling procedure. The basic approach of the present paper is to require that at each stage of the experiment the number of measurements taken from any population is a random variable having the Poisson distribution. In practise, this can be carried out using the published tabulations of the Poisson distribution such as [2] and a table of random numbers, such as [5].

Let $\{N_{ir}\}$ $(i = 1, 2, \cdots, k; r = 1, 2, \cdots)$ be a double sequence of independent random variables each having a Poisson distribution with mean $= J$, where J is a positive integer fixed in advance of the experiment. Let S_{ir} denote the number of successes and F_{ir} the number of failures when N_{ir} measurements are taken from population Π_i at the rth stage of the experiment. If $N_{ir} = 0$, we take $S_{ir} = F_{ir} = 0$. We assume that all measurements are independent and that there is a constant probability p_i that any measurement from Π_i is classified as a success. It is easy to prove (and is actually worked out as an example in [1], page 203) that for each i and r the (unconditional) joint probability distribution of S_{ir} and F_{ir} is that of two independent Poisson random variables with means Jp_i and $J(1 - p_i)$ respectively. The pair (S_{ir}, F_{ir}) is independent of (S_{jt}, F_{jt}) unless $i = j$ and $r = t$.

Now we let $q_i = 1 - p_i$ for $i = 1, 2, \cdots, k$, let $\Delta_{ij} = p_i - p_j$, $\theta_{ij} = q_i/q_j$, $\theta'_{ij} = p_i/p_j$, and let $V_{ijr} = S_{ir} + F_{jr}$, $W_{ijr} = S_{jr} + F_{ir}$, $Y_{ijr} = F_{ir} + F_{jr}$, and $Z_{ijr} = S_{ir} + S_{jr}$. Since the sum of independent random variables each having a Poisson distribution also has a Poisson distribution, it follows that V_{ijr} has a Poisson distribution with mean $= Jp_i + Jq_j = J(1 + \Delta_{ij})$, and W_{ijr} has a Poisson distribution with mean $= J(1 - \Delta_{ij})$.

We now consider $Y_{ijr} = F_{ir} + F_{jr}$. Since Y_{ijr} has a Poisson distribution with mean $= J(q_i + q_j)$, it is known (and easy to prove) that the conditional distribution of F_{ir} when Y_{ijr} is fixed is that a binomial distribution corresponding to Y_{ijr} trials with constant probability $p = q_i/(q_i + q_j) = \theta_{ij}/(1 + \theta_{ij})$ on each trial. By the same reasoning, the conditional distribution of S_{ir} when Z_{ijr} is fixed is that of a binomial distribution corresponding to Z_{ijr} trials with constant probability $p = p_i/(p_i + p_j) = \theta'_{ij}/(1 + \theta'_{ij})$.

3. Sequential confidence limits. In this section we will make use of the general procedure of [4] to obtain one-sided sequential confidence limits for Δ_{ij}, θ_{ij}, and θ'_{ij}. Let $f(v, w) = P[V_{ijr} = v \text{ and } W_{ijr} = w]$. From the discussion of Section 2, we have

$$f(v, w) = \{\exp[-J(1 + \Delta_{ij})]\}[J(1 + \Delta_{ij})]^v/v! \cdot \{\exp[-J(1 - \Delta_{ij})]\}[J(1 - \Delta_{ij})]^w/w!$$

Let $g(v, w)$ denote the probability distribution

$$g(v, w) = \{\exp[-J(1 + \Delta_{ij})/\lambda]\}[J(1 + \Delta_{ij})/\lambda]^v/v! \cdot \{\exp[-\lambda J(1 - \Delta_{ij})]\}[\lambda J(1 - \Delta_{ij})]^w/w!$$

Here $\lambda > 1$ is a constant whose selection is discussed in Section 4. If we let

$\alpha^* = 1 - P^*$, then (see [7], page 146)

$$P[\prod_{r=1}^{n} f(V_{ijr}, W_{ijr})/\prod_{r=1}^{n} g(V_{ijr}, W_{ijr}) > \alpha^* \quad \text{for all} \quad n, n = 1, 2, \cdots] \geqq P^*. \tag{3.1}$$

Taking logarithms to the base e and simplifying, (3.1) reduces to

$$\begin{aligned} P[\Delta_{ij} < (\lambda - 1)^2/(\lambda^2 - 1) \\ + [-\lambda \log \alpha^* + \textstyle\sum_{r=1}^{n} (V_{ijr} - W_{ijr})\lambda \log \lambda](nJ(\lambda^2 - 1))^{-1} \\ \text{for all} \quad n, n = 1, 2, 3, \cdots] \geqq P^*. \end{aligned} \tag{3.2}$$

Noting that $V_{ijr} - W_{ijr} = (S_{ir} - F_{ir}) - (S_{jr} - F_{jr})$, it follows from (3.2) that when $\Delta_{ij} = p_i - p_j > d$,

$$\begin{aligned} P[\textstyle\sum_{r=1}^{n} (S_{ir} - F_{ir}) \leqq \textstyle\sum_{r=1}^{n} (S_{jr} - F_{jr}) \\ + \{\lambda \log \alpha^* + nJ[d(\lambda^2 - 1) - (\lambda - 1)^2]\}(\lambda \log \lambda)^{-1} \\ \text{for at least one} \quad n, n = 1, 2, \cdots \mid p_i - p_j > d] \leqq \alpha^*. \end{aligned} \tag{3.3}$$

With a view to applications when *a priori* information indicates that $p_{[1]}$ and $p_{[2]}$ are large, we shall now derive a one-sided sequential confidence limits for $\theta_{ij} = q_i/q_j$. Let $f(f_{ir} \mid t_{ijr}) = P[F_{ir} = f_{ir} \mid F_{ir} + F_{jr} = t_{ijr}]$. From the discussion of Section 2,

$$f(f_{ir} \mid t_{ijr}) = \binom{t_{ijr}}{f_{ir}} p^{f_{ir}}(1 - p)^{t_{ijr} - f_{ir}}$$

where $p = q_i/(q_i + q_j) = \theta_{ij}/(1 + \theta_{ij})$.

Let $g(f_{ir} \mid t_{ijr}) = \binom{t_{ijr}}{f_{ir}}[1 - a_1(1 - p)]^{f_{ir}}[a_1(1 - p)]^{t_{ijr} - f_{ir}}$ where a_1 is a constant with $0 < a_1 < 1$ whose selection is discussed in Section 4. If $t_{ijr} = 0$, we take $f(0 \mid 0) = g(0 \mid 0) = 1$. Now let t_{ij} denote the sequence $(t_{ij1}, t_{ij2}, \cdots)$. Then we have

$$P[\prod_{r=1}^{n} f(F_{ir} \mid t_{ijr})/\prod_{r=1}^{n} g(F_{ir} \mid t_{ijr}) > \alpha^* \quad \text{for all} \quad n, n = 1, 2, \cdots \mid t_{ij}] \geqq P^*. \tag{3.4}$$

Since the lower bound on the conditional probability in (3.4) holds for each sequence $t_{ij} = (t_{ij1}, t_{ij2}, \cdots)$ the same relationship must hold for the unconditional probability, so that

$$P[\prod_{r=1}^{n} f(F_{ir} \mid F_{ir} + F_{jr})/\prod_{r=1}^{n} g(F_{ir} \mid F_{ir} + F_{jr}) > \alpha^* \quad \text{for all} \quad n, n = 1, 2, \cdots] \geqq P^*. \tag{3.5}$$

Upon simplifying, we obtain

$$\begin{aligned} P[\theta_{ij} > \{(\alpha^*)^{(\Sigma F_{ir})^{-1}} (a_1)^{(\Sigma F_{jr})(\Sigma F_{ir})^{-1}} (1 - a_1)\} \\ \cdot \{1 - (\alpha^*)^{(\Sigma F_{ir})^{-1}} (a_1)^{(\Sigma F_{jr})(\Sigma F_{ir})^{-1}}\}^{-1} \\ \text{for every } n \text{ such that } \textstyle\sum_{r=1}^{n} F_{ir} > 0] \geqq P^*, \end{aligned} \tag{3.6}$$

where each sum in (3.6) ranges from $r = 1$ to $r = n$. When $\theta_{ij} = q_i/q_j \leqq c_1 < 1$, it follows from (3.6) that

$$(3.7) \quad P[\textstyle\sum_{r=1}^{n} F_{ir} \geqq \{\log a_1 \sum_{r=1}^{n} F_{jr} + \log \alpha^*\} (\log \{c_1/(1 + c_1 - a_1)\})^{-1} \text{ for at least one } n, n = 1, 2, \cdots \mid q_i/q_j \leqq c_1] \leqq \alpha^*.$$

To obtain an equivalent inequality when $\theta'_{ij} = p_i/p_j \geqq c_2$, we note that $p_i/p_j \geqq c_2 > 1$ is equivalent to $p_j/p_i \leqq 1/c_2 < 1$ so from the analogue of (3.7) we obtain

$$(3.8) \quad P[\textstyle\sum_{r=1}^{n} S_{ir} \leqq \{[\log (1 + c_2 - a_2 c_2)] \sum_{r=1}^{n} S_{jr} + \log \alpha\}^*(-\log a_2)^{-1} \text{ for at least one } n, n = 1, 2, \cdots \mid p_i/p_j \geqq c_2] \leqq \alpha^*$$

where a_2 is a constant with $0 < a_2 < 1$. The choice of a_2 is discussed in Section 4.

4. The specification of the sequential procedures.

4.1. *Sequential procedures for the main formulation.* In this section we will first derive a class of sequential procedures when λ is restricted to the interval $1 < \lambda < (1 + d)/(1 - d)$ so that for any choice of λ in this interval the corresponding sequential procedure satisfies the basic requirement that population $\Pi_{[1]}$ is selected with probability $\geqq P^*$ whenever $p_{[1]} - p_{[2]} \geqq d$. In addition to specifying values of d, P^* and λ, it is also necessary to select a value for J in advance of the experiment. The choice of J will be a compromise based on the consideration that increasing J decreases the number of stages required by the procedure but also increases the average sample size. The choice of J is similar to the problem of selecting the group size in the standard theory of sequential analysis.

To describe the procedures, first set $\alpha = (1 - P^*)/(k - 1)$, let $A(\lambda) = J[d(\lambda^2 - 1) - (\lambda - 1)^2]/(\lambda \log \lambda)$, and let $N(\lambda)$ denote the largest integer which is less than $(-\log \alpha)/[A(\lambda) \log \lambda]$. Since $1 < \lambda < (1 + d)/(1 - d)$ we have $A(\lambda) > 0$, and without any real loss in the applicability of the results, we can suppose the parameters were chosen so that $N(\lambda) > 0$. We now start the sequential procedure by observing the values $N_{11}, N_{21}, \cdots, N_{k1}$ of k independent random variables each having a Poisson distribution with mean $= J$, using published tables such as [2] and [5]. Then we take N_{11} measurements from Π_1, N_{21} measurements from $\Pi_2, \cdots$, and N_{k1} measurements from Π_k. After the first stage of the experiment we reject any population Π_i for which

$$S_{i1} - F_{i1} \leqq \max_{1 \leqq j \leqq k}[S_{j1} - F_{j1}] + \log \alpha/\log \lambda + A(\lambda).$$

Proceeding by induction, suppose at the start of the rth stage of the experiment there are $T(r)$ populations $\Pi_{r_1}, \Pi_{r_2}, \cdots, \Pi_{r_{T(r)}}$ left after the $(r - 1)$st stage is completed. We then observe the values of $T(r)$ additional Poisson random variables $N_{r_1 r}, N_{r_2 r}, \cdots, N_{r_{T(r)} r}$, and then take $N_{r_1 r}$ measurements from $\Pi_{r_1}, \cdots, N_{r_{T(r)} r}$ measurements from $\Pi_{r_{T(r)}}$. We then reject any remaining population Π_i for which

$$(4.1) \quad \textstyle\sum_{\beta=1}^{r} (S_{i\beta} - F_{i\beta}) \leqq \max [\sum_{\beta=1}^{r} (S_{j\beta} - F_{j\beta})] + \log \alpha/\log \lambda + rA(\lambda),$$

where the max is taken over all $T(r)$ populations left after the $(r - 1)$st stage. The experiment is terminated as soon as only one population is left, in which case the remaining population is selected as best. Since $N(\lambda)$ was defined so that $\log \alpha/\log \lambda + rA(\lambda) < 0$ for $r \leqq N(\lambda)$, the populations can never be all eliminated before the $(N(\lambda) + 1)$st stage. If more than one population is left after the $N(\lambda)$th stage, the experiment is terminated at the $(N(\lambda) + 1)$st stage by selecting the remaining population Π_i for which $\sum_{\beta=1}^{N(\lambda)+1} (S_{i\beta} - F_{i\beta})$ is a maximum. If a tie for first place should occur at the $(N(\lambda) + 1)$st stage, the population to be selected as best can be obtained by selecting one among the tied populations by any random mechanism.

We now give the proof that whenever $p_{[1]} - p_{[2]} \geqq d$ population $\Pi_{[1]}$ will be selected with probability $\geqq P^*$ for any sequential procedure in this class, that is for any λ satisfying $1 < \lambda < (1 + d)/(1 - d)$. We first note that if $r = N(\lambda) + 1$, the inequality (4.1) is satisfied for every population left after the $N(\lambda)$th stage. The event that $\Pi_{[1]}$ is eliminated can occur if and only if for at least one value of $s(s = 2, 3, \cdots, K)$ there is an integer $n(s)$ such that

$$\sum_{\beta=1}^{n(s)} (S_{[1]\beta} - F_{[1]\beta}) \leqq \sum_{\beta=1}^{n(s)} (S_{[s]\beta} - F_{[s]\beta}) + \log \alpha/\log \lambda + n(s)A(\lambda).$$

It follows from (3.3) that when $p_{[1]} - p_{[2]} \geqq d$ the probability of this event is $\leqq (k - 1)\alpha = \alpha^* = 1 - P^*$, so the probability that $\Pi_{[1]}$ is selected is $\geqq P^*$.

An optimum choice of λ is unknown at present. We recommend using

$$\lambda = (1 + .75d)/(1 - .75d).$$

The calculations summarized in Section 5 seem to indicate that this choice of λ yields good results, since it results in a substantial reduction in the average sample size.

4.2. *Sequential procedures for an alternative formulation.* In this section we consider a different formulation of the problem which may be more useful than the preceding formulation if there is *a priori* knowledge either that $p_{[2]}$ is large (say $p_{[2]} \geqq .8$) or that $p_{[2]}$ is small (say $p_{[2]} \leqq .2$). This new formulation will include the alternative formulation of Sobel and Huyett (which was described in the introduction) as a special case. We will first consider the situation where $p_{[2]}$ is known to be large, and then more briefly the situation where $p_{[2]}$ is known to be small.

When $p_{[2]}$ is large the quantity $q_{[1]}/q_{[2]}$ seems to be a useful measure of the superiority of $p_{[1]}$ over $p_{[2]}$. We therefore consider the following formulation of the problem: to find a sequential procedure for selecting the best population so that $\Pi_{[1]}$ is selected with probability $\geqq P^*$ whenever $q_{[1]}/q_{[2]} \leqq c_1$, where c_1 is a constant <1 which is specified in advance of the experiment. Since the inequalities $p_{[2]} \leqq p_{[2]}^*$ and $p_{[1]} \geqq p_{[2]}^* + d$ imply that $q_{[1]}/q_{[2]} \leqq (1 - p_{[2]}^* - d)/(1 - p_{[2]}^*)$, this means that the alternative formulation given by Sobel and Huyett is included as a special case of this new formulation if we take $c_1 = (1 - p_{[2]}^* - d)/(1 - p_{[2]}^*)$.

We now restrict a_1 to the interval $c_1 < a_1 < 1$ and obtain a family of sequential procedures, one for each choice of a_1 in this interval, such that for each procedure in this family the basic requirement that $\Pi_{[1]}$ should be selected with probability $\geqq P^*$ whenever $q_{[1]}/q_{[2]} < c_1$ is satisfied.

We now give the class of sequential procedures satisfying the specified requirement. The selection of J and the general sampling procedure is the same as in Section 4.1. However instead of (4.1), we now eliminate at the rth stage ($r = 1, 2, \cdots$) any category Π_i remaining after the $(r - 1)$st stage for which

$$(4.2)\sum_{\beta=1}^{r} F_{i\beta} \geqq \{[\min_{(j)} (\sum_{\beta=1}^{r} F_{j\beta})] \log a_1 + \log \alpha\}\{\log [c_1/(1 + c_1 - a_1)]\}^{-1}$$

where the minimum is taken over all populations not eliminated in the first $(r - 1)$ stages. The experiment is terminated as soon as only one population is left, in which case the remaining population is selected as best. However, if at any stage (4.2) is satisfied for all the remaining populations, we terminate the experiment and select one of these populations with the minimum cumulative sum as the best.

The event that $\Pi_{[1]}$ is eliminated by the above procedure can occur if and only if for at least one integer s ($s = 2, 3, \cdots, k$) we have

$$\sum_{\beta=1}^{n} F_{[1]\beta} \geqq \{(\sum_{\beta=1}^{n} F_{[s]\beta}) \log a_1 + \log \alpha\}\{\log [c_1/(1 + c_1 - a_1)]\}^{-1}$$

for at least one integer n. When $q_{[1]}/q_{[2]} \leqq c_1$, it follows from (3.7) that the probability of this event is $\leqq (k - 1)\alpha = \alpha^* = 1 - P^*$, so the probability requirement is satisfied.

For each choice of a_1 in the interval $c_1 < a_1 < 1$ the resulting sequential procedure is partially closed in the sense that the number of stages in which at least one failure occurs with at least one of the remaining populations is bounded.

The optimum choice of a_1 is unknown. At present, we recommend using $a_1 = c_1 + .25(1 - c_1)$. The calculations in Section 5 indicate that this choice yields good results, since there is a substantial reduction in the average sample size compared to the corresponding fixed sample size procedure.

TABLE I

A comparison of the average total sample sizes for $k = 4$ and $P^* = .95$

Parameter Configuration (p_1, p_2, p_3, p_4)	Main Formulation		Alternative Formulation	
	Fixed Sample Size Procedure ($d = .1$)	Sequential Procedure ($d = .1, \lambda = 1.162$)	Fixed Sample Size Procedure ($d = .1, p^*_{[2]} = .8$)	Sequential Procedure ($c_1 = .5$, $a_1 = .625$)
(.8, .8, .8, .8)	848	699	428	350
		(20)		(11)
(.8, .8, .8, .9)	848	485	428	291
		(19)		(10)
(.6, .7, .8, .9)	848	347	428	195
		(13)		(7)

(1) The value $J = 1$ was used for each sequential procedure.

(2) Each number in parenthesis is the estimated standard deviation of the number directly above it.

We note that this sequential procedure only involves the random variables $\{F_{ir}\}$, and does not depend on the random variables $\{S_{ir}\}$. This feature is due to the fact that we were unable to discover any function of $S_{i\beta}$ and $S_{j\beta}$ whose probability distribution would depend only on the parameter q_i/q_j. We conjecture that when $p_{[2]}$ is large there is only a small loss of information in not using the $\{S_{ir}\}$, but further research on this and other matters would clearly be desirable. Similar considerations apply to the case when $p_{[2]}$ is small, which will now be discussed.

Returning briefly to the case when $p_{[2]}$ is small, we now work directly in terms of the parameter $p_{[1]}/p_{[2]}$, and look for a sequential procedure meeting the requirement that whenever $p_{[1]}/p_{[2]} \geqq c_2 > 1$, then $\Pi_{[1]}$ is selected with probability $\geqq P^*$. This satisfies the alternative formulation of [6] if we take $c_2 = (p^*_{[2]} + d)/p^*_{[2]}$. To obtain a sequential procedure meeting this requirement we use the previous general sampling procedure, but at the rth stage of the experiment now reject any remaining population Π_i for which

$$\sum_{\beta=1}^{r} S_{i\beta} \leqq \{\max_j \sum_{\beta=1}^{r} S_{j\beta}[\log(1 + c_2 - a_2c_2)] + \log \alpha\}(-\log a_2)^{-1},$$

where the maximum is taken over all populations left after the first $(r - 1)$ stages, and a_2 is restricted to the interval $1/c_2 < a_2 < 1$. We can show just as before on the basis of (3.8) that the probability requirement is met. For each a_2 with $1/c_2 < a_2 < 1$ we again get a partially closed procedure. At present we recommend using $a_2 = 1/c_2 + .25(1 - 1/c_2)$.

5. Some numerical results. In order to obtain some idea of the efficiency of the recommended sequential procedures using $\lambda = (1 + .75d)/(1 - .75d)$ for the main formulation and $a_1 = c_1 + .25(1 - c_1)$ or $a_2 = 1/c_2 + .25(1 - 1/c_2)$ for the alternative formulation, a number of sampling experiments were carried out. The results for the case $k = 4$, $P^* = .95$ and three different parameter configurations are summarized in Table I. The data of Table I shows that a substantial reduction in the average sample size is possible with the sequential procedures.

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