ON A STRONG LAW OF LARGE NUMBERS FOR MARTINGALES

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Let $x_1, x_2, \cdots$ be independent random variables with $Ex_n = 0$ for each $n \geqq 1$. Chung [3] proved the following theorem. If $\sum_{n=1}^{\infty} E|x_n|^{2r}/n^{1+r} < \infty$ for some $r \geqq 1$, then $\lim (x_1 + \cdots + x_n)/n = 0$ a.e. In [2], the author attempted but failed to extend Chung's result to the case in which the x_n's are martingale summands. However, the following result has been proved in [2].

LEMMA. *Let $Y_n = x_1 + \cdots + x_n$ be a martingale and C_k be a nonincreasing sequence of positive numbers. For $\alpha \geqq 1$ and $2\alpha \geqq \beta > 0$, if there exists i_0 such that for $i \geqq i_0$*

(1) $$E|Y_i|^{2\alpha} \leqq AE(\textstyle\sum_1^i x_k^2)^\alpha,$$

(2) $$i^{\alpha-1}C_i^{2\alpha-\beta} \leqq A, \qquad \textstyle\sum_i^\infty C_k^{2\alpha}k^{\alpha-2} \leqq AC_i^\beta,$$

where A is a constant, independent of i, and if

(3) $$\textstyle\sum_1^\infty C_k^\beta E|x_k|^{2\alpha} < \infty,$$

then

(4) $$\lim C_n Y_n = 0 \quad \text{a.e.}$$

Recently, Burkholder [1] proved that (1) is always satisfied. Therefore, by Lemma 1, immediately we have the following result.

THEOREM. *Let $Y_n = x_1 + \cdots + x_n$ be a martingale. If C_k is a nonincreasing sequence of positive numbers, then the conditions (2) and (3) imply (4). In particular, if $Y_n = x_1 + \cdots + x_n$ is a martingale such that*

$$\textstyle\sum_1^\infty E|x_k|^{2\alpha}/k^{1+\alpha} < \infty,$$

then $\lim Y_n/n = 0$ a.e.

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