ON THE EXPECTED VALUE OF A STOPPED SUBMARTINGALE

By Y. S. Chow[1]

Purdue University

Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a submartingale in a probability space $(\Omega, \mathfrak{F}, P)$. A stopping time t is a positive integer-valued $(+\infty$ included) random variable such that for each $n = 1, 2, \cdots$, the set $[t = n] \varepsilon \mathfrak{F}_n$. If a stopping time t is finite, i.e., $P[t = \infty] = 0$, then t is said to be a stopping rule. For a stopping time t, $E|x_t|$ is defined to be $\int_{[t<\infty]} |x_t| \, dP = \int_{[t<\infty]} |x_t|$. In a recent paper of Dubins and Freedman [2], the following theorem has been proved. Let $(x_n, \mathfrak{F}_n, n > 1)$ be a martingale. (a) If $\sup E|x_n| = \infty$, then there exists a stopping time t such that $E|x_t| = \infty$. (b) If $\sup E|x_n| < \infty$, then $Ex_t = Ex_1$ for every stopping rule t if and only if x_n's are uniformly integrable. However, the proof is somewhat complicated. In an oral communication, D. Siegmund has given a simpler proof of (b), by using the standard martingale arguments. In this note, Dubins and Freedman's results (a) and (b) are extended to submartingales as follows:

Theorem. *Let* $(x_n, \mathfrak{F}_n, n \geqq 1)$ *be a submartingale* (a). *If* $\sup E(x_n^-) = \infty$, *then there exists a stopping time t such that* $E|x_t| = \infty$. (b) If $\sup E|x_n| < \infty$,

$$Ex_m \leqq Ex_t \leqq \sup Ex_n \tag{1}$$

for every stopping rule t satisfying $P[t \geqq m] = 1$, *if and only if* x_n*'s are uniformly integrable.*

Proof. (a) We can assume that $E|x_n| < \infty$ for each n and $Ex_1 = 0$. Put $C_0 = \Omega$. Then there exists an integer $n_1 > 1$ such that $\int_{C_0} x_{n_1}^- \geqq 1$. Put $A_1 = [x_{n_1} \leqq 0]C_0$ and $B_1 = C_0 - A_1$. Then either $\sup \int_{A_1} x_n^- = \infty$ or $\sup \int_{B_1} x_n^- = \infty$. Hence we can choose $C_1 = A_1$ or B_1 so that $\sup \int_{C_1} x_n^- = \infty$. Put $D_1 = A_1$ if $C_1 = B_1$ and $D_1 = B_1$ if $C_1 = A_1$. When $D_1 = A_1$, by the definition of n_1,

$$\int_{D_1} |x_{n_1}| = \int_{A_1} |x_{n_1}| = \int_{C_0} x_{n_1}^- \geqq 1,$$

and when $D_1 = B_1$, by submartingale property and $Ex_1 = 0$,

$$\int_{D_1} |x_{n_1}| = \int_{B_1} |x_{n_1}| = \int_{C_0} x_{n_1}^+ \geqq 1.$$

Assume that we have defined n_k, A_k, B_k, C_k and D_k for a positive integer k. Choose $n_{k+1}, A_{k+1}, B_{k+1}, C_{k+1}$ and D_{k+1} as follows. Since $\sup \int_{C_k} x_n^- = \infty$, there exists an integer $n_{k+1} > n_k$ such that $\int_{C_k} x_{n_{k+1}}^- \geqq 1$. Put $A_{k+1} = [x_{n_{k+1}} \leqq 0]C_k$ and $B_{k+1} = C_k - A_{k+1}$. Then either $\sup \int_{A_{k+1}} x_n^- = \infty$ or $\sup \int_{B_{k+1}} x_n^- = \infty$. Hence we can choose $C_{k+1} = A_{k+1}$ or B_{k+1} so that $\sup \int_{C_{k+1}} x_n^- = \infty$. Put $D_{k+1} = A_{k+1}$ if $C_{k+1} = B_{k+1}$ and $D_{k+1} = B_{k+1}$ if $C_{k+1} = A_{k+1}$. When $D_{k+1} = A_{k+1}$, by the definition of n_{k+1},

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$$(2) \qquad \int_{D_{k+1}} |x_{n_{k+1}}| = \int_{A_{k+1}} |x_{n_{k+1}}| = \int_{C_k} x_{n_{k+1}}^- \geqq 1,$$

and when $D_{k+1} = A_{k+1}$, by submartingale property and $Ex_1 = 0$,

$$(2) \qquad \int_{D_{k+1}} |x_{n_{k+1}}| = \int_{B_{k+1}} |x_{n_{k+1}}| = \int_{C_k} x_{n_{k+1}}^+ \geqq 1.$$

Therefore, n_k and D_k are well-defined for $k = 1, 2, \cdots$. Let

$$(3) \qquad t = \inf \{n_k \mid x_{n_k} \varepsilon D_k\}.$$

Since $D_k \varepsilon \mathfrak{F}_{n_k}$ for each k, t is a stopping time, and since D_k's are disjoint,

$$(4) \qquad E|x_t| = \sum_{k=1}^{\infty} \int_{[t=n_k]} |x_{n_k}| = \sum_{k=1}^{\infty} \int_{D_k} |x_{n_k}| = \infty.$$

The proof of (a) is completed.

(b) The "if" part is well known. For the "only if" part, note that the condition $\sup E|x_n| < \infty$ implies that $\lim x_n = x_\infty$ a.e., $E|x_\infty| \leqq \sup E|x_n| < \infty$ and $E|x_t| < \infty$ for every stopping rule t. Put $y_n = E(x_\infty \mid \mathfrak{F}_n)$. Then $(y_n, \mathfrak{F}_n, \infty \geqq n \geqq 1)$ is a martingale, where $y_\infty = x_\infty$. For $\epsilon > 0$ and $m = 1, 2, \cdots$, let

$$(5) \qquad t = \inf \{n \mid x_n \leqq y_n + \epsilon, n \geqq m\}.$$

Obviously, t is a stopping time and $P[t \geqq m] = 1$. Since x_∞ is finite a.e. and $\lim y_n = \lim x_\infty$ a.e., $P[t < \infty] = 1$. Hence (1) holds and since $(y_n, \mathfrak{F}_n, n \geqq 1)$ is a closed martingale with the last element x_∞,

$$Ex_m \leqq Ex_t \leqq Ey_t + \epsilon = Ex_\infty + \epsilon.$$

Therefore $Ex_\infty \geqq \sup Ex_n$. Similarly, $Ex_\infty \leqq \sup Ex_n$. Hence

$$(6) \qquad Ex_\infty = \sup Ex_n.$$

Now we prove that $(x_n, \mathfrak{F}_n, \infty \geqq n > 1)$ is a submartingale. Put $A_n = [y_n < x_n]$. If $PA_n > 0$, then

$$\int_{A_n} x_\infty = \int_{A_n} y_n < \int_{A_n} x_n - \epsilon$$

for some $\epsilon > 0$. Let $t = n$ on A_n, and off A_n, define

$$(7) \qquad t = \inf \{m \mid y_m < x_m + \epsilon, m > n\}.$$

As before, we can prove that t is a stopping rule and $P[t \geqq n] = 1$. From (1), we have

$$Ex_\infty = Ey_t = \sum_{k=n}^{\infty} \int_{[t=k]} y_k < \int_{A_n} x_n - \epsilon + \sum_{k=n+1}^{\infty} \int_{[t=k]} x_k + \epsilon$$
$$= Ex_t \leqq \sup Ex_n,$$

which is contradictory to (6). Therefore $PA_n = 0$ and

$$(8) \qquad x_n \leqq y_n = E(x_\infty \mid \mathfrak{F}_n) \quad \text{a.e.}$$

By a theorem of Doob ([1], p. 325), (6) and (8) imply that x_n's are uniformly integrable. Hence the proof is completed.

The proof of (a) is simpler than that of Dubins and Freedman, and the proof (6) is an adoption of D. Siegmund's approach for matingales.

REFERENCES

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[2] DUBINS, L. E. and FREEDMAN, D. A. (1966). On the Expected Value of a Stopped Martingale. *ann. Math. Statist.*. **37** 1505–1509.