A REMARK ON THE LAW OF THE ITERATED LOGARITHM[1]

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1. Introduction. Let $X_1, X_2, \cdots$ be random variables on a probability triple $(\Omega, \mathfrak{F}, P)$. Use E for expectation with respect to P, and $\exp x$ for e^x. Let $X(n) = X_1 + \cdots + X_n$.

Theorem 1. *Let* $S = \sup_{n \geq 3} (n \log \log n)^{-\frac{1}{2}} X(n)$. *If* $X_1, X_2, \cdots$ *are independent, identically distributed, uniformly bounded, and* $E(X_i) = 0$, *then* $E\{\exp (hS^2)\} < \infty$ *for any positive, finite* h.

This theorem will be proved in Section 2.

Let $\mathfrak{F}_n$ be an increasing sequence of sub-σ-fields of $\mathfrak{F}$, such that

$$X_n \text{ is } \mathfrak{F}_n\text{-measurable} \tag{1}$$

and

$$E\{X_{n+1} \mid \mathfrak{F}_n\} = 0. \tag{2}$$

Let $V_n = E\{X_{n+1}^2 \mid \mathfrak{F}_n\}$, $V(n) = 3 + V_1 + \cdots + V_n$, and

$$T = \sup_{n \geq 1} [V(n) \log \log V(n)]^{-\frac{1}{2}} X(n). \tag{3}$$

Theorem 2. *Under conditions* (1) *and* (2), *if* $X_1, X_2, \cdots$ *are uniformly bounded, then* $E\{\exp (hT)\} < \infty$ *for all finite* h.

The proof of this theorem is omitted. It is similar to that of Theorem 1, using (30) and (31) of [1] in place of (4) and (8).

Suppose now $X_1, X_2, \cdots$ are independent, $\mathfrak{F}_n$ is the σ-field spanned by $X_1, \cdots, X_n$, $E(X_i) = 0$, and $V_i = E(X_i^2) < \infty$. In particular, (1) and (2) hold.

Example 1. It can happen that $\sup_{n \geq 1} [V(n) \log V(n)]^{-\frac{1}{2}} X(n) = \infty$ a.e.

Example 2. Even if $|X_i| \leq 1$, it can happen that $E\{\exp (h|T|^{1+\epsilon})\} = \infty$ for any positive h and ϵ.

These examples will be constructed in Section 3.

2. Proof of Theorem 1. Let $a = \inf_{n \geq 1} P\{X(n) \geq 0\}$. The central limit theorem implies $a > 0$. As is well known,

$$P\{X(n) > y \text{ for some } n \leq m\} \leq a^{-1} P\{X(m) > y\}. \tag{4}$$

Plainly, for $1 < L < \infty$, $E\{\exp (hS^2)\}$ is no more than $L + I$, where

$$I = \int_L^\infty P\{\exp (hS^2) > w\}\, dw.$$

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Now $I = I_+ + I_-$, where $I_\pm = \int_L^\infty P\{\pm S > x\}\, dw$ and $x = h^{-\frac{1}{2}}(\log w)^{\frac{1}{2}}$. It is enough to show that for L large, $I_+ < \infty$, the argument for I_- being symmetric. Fix any $r > 1$. Let J be so large that $r^J > 3$. Then $\{S > x\} = \bigcup_{j=J}^\infty A_j$, where

$$A_J = \{(n \log \log n)^{-\frac{1}{2}} X(n) > x \text{ for some } n \text{ with } 3 \leqq n \leqq r^J\}$$

and for $j > J$,

$$A_j = \{(n \log \log n)^{-\frac{1}{2}} X(n) > x \text{ for some } n \text{ with } r^{j-1} < n \leqq r^j\}.$$

Plainly, $\int_L^\infty P(A_J)\, dw < \infty$, and it is enough to choose J and L so large that $\sum_{j=J+1}^\infty \int_L^\infty P(A_j)\, dw < \infty$. But A_j is a subset of $\{X(n) > y \text{ for some } n \leqq r^j\}$, where $y = [r^{j-1} \log (j-1)]^{\frac{1}{2}} x$. In view of (4), $P(A_j) \leqq a^{-1} p_j$, where $p_j = P\{X(r^j) > y\}$. Consequently, it is enough to choose J and L so large that

$$\sum_{j=J+1}^\infty \int_L^\infty p_j\, dw < \infty. \tag{5}$$

By Chebychev's inequality, for any $t > 0$,

$$P\{X(m) > y\} \leqq \exp(-ty)[E\{\exp(tX_1)\}]^m. \tag{6}$$

Plainly, there is a positive real number σ such that for all $t > 0$,

$$E\{\exp(tX_1)\} \leqq \exp(\sigma t^2). \tag{7}$$

Combine (6) and (7) to get

$$P\{X(m) > y\} \leqq \exp\{-ty + m\sigma t^2\}. \tag{8}$$

Put $t = y/(2m\sigma)$ in (8) to get

$$P\{X(m) > y\} \leqq \exp\{-y^2/(4m\sigma)\}. \tag{9}$$

Consequently, with $z = r^{-j/2} y$, the jth term in (5) is at most

$$\int_L^\infty \exp[-z^2/(4\sigma)]\, dw = b_j \int_{L_j}^\infty z \exp\{z^2[-1/(4\sigma) + rh/\log (j-1)]\}\, dz, \tag{10}$$

where

$$b_j = 2rh/\log (j-1)$$

and

$$L_j = [r^{-1} h^{-1} \log L \log (j-1)]^{\frac{1}{2}}.$$

For sufficiently large J, $j > J$ implies

$$-1/(4\sigma) + rh/\log (j-1) \leqq -1/(5\sigma).$$

Then the right side of (10) is bounded above by

$$d_j \exp[-L_j^2/(5\sigma)] = d_j (j-1)^{-f_j}, \tag{11}$$

where $d_j = 5\sigma rh/\log (j-1)$ and $f_j = (5\sigma rh)^{-1} \log L$. For sufficiently large L, the right side of (11) sums, completing the proof.

The boundedness of X_1 was used only for (7), the critical values of t being near 0 and ∞.

3. Examples.

EXAMPLE 1. The random variables $X_1, X_2, \cdots$ are independent, have mean 0, finite variances $V_1, V_2, \cdots$, such that, if $X(n) = X_1 + \cdots + X_n$, and $V(n) = 3 + V_1 + \cdots + V_n$, then

$$\limsup_{n\to\infty} [V(n) \log V(n)]^{-\frac{1}{2}} X(n) = \infty \quad \text{a.e.}$$

CONSTRUCTION. For $n = 1$, $P\{X_n = \pm 1\} = \frac{1}{2}$. For $n \geqq 2$,

$$P\{X_n = \pm[n(\log n)2^n]^{\frac{1}{2}}\} = (2n \log n)^{-1},$$

and $P\{X_n = 0\} = 1 - (n \log n)^{-1}$.

PROOF. Plainly, $E(X_n) = 0$, and for $n \geqq 2$, $V_n = E(X_n^2) = 2^n$, $V(n) = 2^{n+1}$. Suppose by way of contradiction that $P\{\limsup_{n\to\infty} [V(n) \log V(n)]^{-\frac{1}{2}} X(n) < \infty\}$ is positive. By the 0–1 law, this probability is 1. By symmetry, and because $V(n-1) \sim V(n)$,

$$P\{\limsup_{n\to\infty} [V(n) \log V(n)]^{-\frac{1}{2}} |X(n-1)| < \infty\} = 1,$$

so

$$P\{\limsup [V(n) \log V(n)]^{-\frac{1}{2}} X_n < \infty\} = 1.$$

But $X_n = [n(\log n)2^n]^{\frac{1}{2}}$ for infinitely many n, a.e., by the Borel-Cantelli lemma, completing the proof.

EXAMPLE 2. Let f be a positive function on $[3, \infty)$. The random variables $X_1, X_2, \cdots$, are independent, have mean 0, and are bounded by 1 in absolute value. Let $V_n = E(X_n^2)$, $X(n) = X_1 + \cdots + X_n$, $V(n) = 3 + V_1 + \cdots + V_n$, and $U(n) = f[V(n)]X(n)$. For any $\epsilon > 0$ and any $h > 0$,

$$\sup_{n \geqq 1} E\{\exp(h|U(n)|^{1+\epsilon})\} = \infty.$$

CONSTRUCTION. Let $0 = n_0 < n_1 < n_2 < \cdots$, where n_k grows sufficiently quickly. Let $\delta_k = 1/(n_k - n_{k-1})$. For $n_{k-1} < m \leqq n_k$, let X_m be $-\delta_k$ or $+1$, and have $E(X_m) = 0$. In particular, $E(X_m^2) = V_m = \delta_k$, and $V(n_k) = k + 3$.

PROOF. Let Z be a Poisson random variable with parameter 1. For any real number z, and any positive real numbers H and ϵ, $E\{\exp(H|Z + z|^{1+\epsilon})\} = \infty$. Now $\sum \{X_i : n_{k-1} < i \leqq n_k\}$ is essentially distributed like $Z - 1$. So, given X_i for $1 \leqq i \leqq n_{k-1}$, $U(n_k)$ is conditionally distributed almost like $f(k+3) \cdot [Z - 1 + X(n_{k-1})]$, and $E\{\exp h|U(n_k)|^{1+\epsilon}\}$ is very large, completing the proof. Naturally, the faster f decreases, the more quickly n_k must grow.

REFERENCE

[1] DUBINS, L. E. and FREEDMAN, D. A. (1965). A sharper form of the Borel-Cantelli Lemma and the Strong Law. *Ann. Math. Statist.* **36** 800–807.