

OPTIMUM INVARIANT TESTS IN UNBALANCED VARIANCE COMPONENTS MODELS

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1. Summary. In this paper the problem of testing the hypothesis $\Delta \leq \Delta_0$ against $\Delta > \Delta_0$, where Δ is the ratio of variances in the one-way classification of the analysis of variance with variance components, is treated. The model is not restricted to equal class frequencies. It is found that the most powerful invariant test against an alternative Δ_1 depends upon Δ_1 , but has the property of maximizing the minimum power over the set of alternatives with $\Delta \geq \Delta_1$. The test statistic is distributed like a ratio of linear combinations of independent chi-square distributed random variables.

It is shown that a statistic used by Wald [6] to derive a confidence interval for Δ gives a test that is almost equal to the most powerful invariant tests against large alternatives Δ_1 . For the case $\Delta_0 = 0$ it is equal to the usual test in the fixed-effects model. In the balanced case the test reduces to the usual F -test which Herbach [2] has proved to be both uniformly most powerful invariant and uniformly most powerful unbiased.

2. Transformation to a canonical form. We define the model for the observations X_{ij} as follows

$$(2.1) \quad X_{ij} = \mu + U_i + V_{ij}, \quad j = 1, 2, \dots, n_i; \quad i = 1, 2, \dots, r,$$

where μ is an unknown constant, and where the U_i and V_{ij} are all independently normally distributed with expectation zero and variances τ^2 and σ^2 respectively.

Let $\Delta = \tau^2/\sigma^2$. The hypothesis to be tested is

$$H: \Delta \leq \Delta_0 \quad \text{against} \quad \Delta > \Delta_0.$$

To simplify the model we transform to a canonical form. Define

$$X_i' = [X_{i1}, X_{i2}, \dots, X_{in_i}], \quad V_i' = [V_{i1}, V_{i2}, \dots, V_{in_i}],$$

$$Y_i' = [Y_{i1}, Y_{i2}, \dots, Y_{in_i}],$$

and let P_i be a $n_i \times n_i$ orthogonal matrix with first row equal to $[n_i^{-1/2}, n_i^{-1/2}, \dots, n_i^{-1/2}]$. For each i we make the transformation $Y_i = P_i X_i$. Then

$$(2.2) \quad \begin{bmatrix} Y_{i1} \\ Y_{i2} \\ \vdots \\ Y_{in_i} \end{bmatrix} = n_i^{1/2}(\mu + U_i) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + P_i V_i.$$

Since P_i is orthogonal it follows that the elements of Y_i are independent.

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Clearly Y_i and Y_j are independent when $i \neq j$. Hence all Y_{ij} are independent. They are normally distributed, and the following expectations and variances are obtained from (2.2):

$$(2.3) \quad \begin{aligned} EY_{i1} &= n_i^{\frac{1}{2}}\mu; & \text{Var } Y_{i1} &= (n_i\Delta + 1)\sigma^2, & i &= 1, 2, \dots, r; \\ EY_{ij} &= 0; & \text{Var } Y_{ij} &= \sigma^2, & j &= 2, \dots, n_i; & i &= 1, 2, \dots, r. \end{aligned}$$

3. Most powerful invariant and similar test. The problem of testing H is invariant under a group of translations defined by

$$Y'_{i1} = Y_{i1} + n_i^{\frac{1}{2}}a, \quad -\infty < a < \infty, \quad i = 1, 2, \dots, r.$$

Maximal invariant under this group is

$$(3.1) \quad Z_i = Y_{i1} - (n_i/n_r)^{\frac{1}{2}}Y_{r1}, \quad i = 1, 2, \dots, r-1,$$

and Y_{ij} , $j = 2, 3, \dots, n_i$; $i = 1, 2, \dots, r$. $Z' = [Z_1, Z_2, \dots, Z_{r-1}]$ have a multinormal distribution with

$$(3.2) \quad \begin{aligned} EZ_i &= 0, & i &= 1, 2, \dots, r-1; \\ \text{Var } Z_i &= (2n_i\Delta + 1 + n_i/n_r)\sigma^2; \\ \text{Cov } (Z_i, Z_j) &= (n_in_j)^{\frac{1}{2}}(\Delta + n_r^{-1})\sigma^2, & i &\neq j. \end{aligned}$$

Let the covariance matrix of Z given by (3.2) be denoted $A(\Delta)\sigma^2$.

The density function of the invariant statistics is given by

$$(3.3) \quad C \exp \left(-(2\sigma^2)^{-1}(z'A(\Delta)^{-1}z + \sum_{i=1}^r \sum_{j=2}^{n_i} y_{ij}^2) \right)$$

where C is a constant which depends upon the parameters. The set of possible values of the parameters is $\Omega = \{(\sigma, \Delta) \mid 0 \leq \Delta < \infty, 0 < \sigma < \infty\}$, the set of values consistent with the hypothesis is $\omega = \{(\sigma, \Delta) \mid 0 \leq \Delta \leq \Delta_0, 0 < \sigma < \infty\}$, and the set of common accumulation points of ω and $\Omega - \omega$ is $\omega_0 = \{(\sigma, \Delta) \mid \Delta = \Delta_0, 0 < \sigma < \infty\}$. The distribution of Z and the Y_{ij} 's for $j \geq 2$ for $(\sigma, \Delta) = (1, \Delta_0)$ is given by

$$dF_{(1, \Delta_0)}(z, y) = C' \exp \left(-\frac{1}{2}(z'A(\Delta_0)^{-1}z + \sum_{i=1}^r \sum_{j=2}^{n_i} y_{ij}^2) \right) d\mu(z, y)$$

where μ denotes the Lebesgue measure. The distribution for any (σ, Δ) may now be written

$$(3.4) \quad \begin{aligned} dF_{(\sigma, \Delta)}(z, y) &= C'' \exp \left(-(2\sigma^2)^{-1}(z'A(\Delta)^{-1}z - z'A(\Delta_0)^{-1}z) \right. \\ &\quad \left. - ((2\sigma^2)^{-1} - \frac{1}{2})(z'A(\Delta_0)^{-1}z + \sum_{i=1}^r \sum_{j=2}^{n_i} y_{ij}^2) \right) dF_{(1, \Delta_0)}(z, y). \end{aligned}$$

In particular for $(\sigma, \Delta) \in \omega_0$,

$$\begin{aligned} dF_{(\sigma, \Delta_0)}(z, y) \\ = C''' \exp \left(-((2\sigma^2)^{-1} - \frac{1}{2})(z'A(\Delta_0)^{-1}z + \sum_{i=1}^r \sum_{j=2}^{n_i} y_{ij}^2) \right) dF_{(1, \Delta_0)}(z, y) \end{aligned}$$

which constitutes an exponential family of distributions.

Let $U = Z'A(\Delta_0)^{-1}Z + \sum_{i=1}^r \sum_{j=2}^{n_i} Y_{ij}^2$. According to Sverdrup [5], Theorem 3, the most powerful similar α -test φ on ω_0 against an alternative $(\sigma_1, \Delta_1) \in \Omega - \omega$ is found by setting $\varphi(z, y) = 1$ when

$$(3.5) \quad C'' \exp \left(-(2\sigma_1^2)^{-1}(z'A(\Delta_1)^{-1}z - z'A(\Delta_0)^{-1}z) - ((2\sigma_1^2)^{-1} - \frac{1}{2})u \right) > c(u)$$

and $\varphi(z, y) = 0$ when the inequality sign is reversed and where c is a function determined such that

$$(3.6) \quad E(\varphi(Z, Y) | U) = \alpha \quad \text{a.e. } F_{(1, \Delta_0)}.$$

Combining (3.5) and (3.6) we get the condition

$$P_{(1, \Delta_0)}(Z'A(\Delta_0)^{-1}Z - Z'A(\Delta_1)^{-1}Z > c'(U) | U) = \alpha$$

or

$$P_{(1, \Delta_0)}([Z'A(\Delta_0)^{-1}Z - Z'A(\Delta_1)^{-1}Z]U^{-1} > c''(U) | U) = \alpha.$$

Introduce

$$(3.7) \quad W = (Z'A(\Delta_0)^{-1}Z - Z'A(\Delta_1)^{-1}Z)(Z'A(\Delta_0)^{-1}Z + \sum_{i=1}^r \sum_{j=2}^{n_i} Y_{ij}^2)^{-1}.$$

The distribution of W does not depend upon σ , and $U = Z'A(\Delta_0)^{-1}Z + \sum_{i=1}^r \sum_{j=2}^{n_i} Y_{ij}^2$ is complete and sufficient when $(\sigma, \Delta) \in \omega_0$. Then by a theorem of Basu [1] W and U are independent when $(\sigma, \Delta) \in \omega_0$. It follows that $c''(u)$ must be a constant independent of u .

4. The distribution of the test statistic. Since the distribution of W does not depend upon σ , we put $\sigma = 1$. Then $Q = \sum_{i=1}^r \sum_{j=2}^{n_i} Y_{ij}^2$ have a chi-square distribution with $n - r$ degrees of freedom, and Q and Z are independent, ($n = \sum_{i=1}^r n_i$).

From (3.2) it is seen that the covariance matrix $A(\Delta)$ of Z may be written in the form

$$(4.1) \quad A(\Delta) = B\Delta + C$$

where the matrices B and C do not depend upon Δ . It is well known from matrix theory that there exists a nonsingular matrix P such that

$$(4.2) \quad PCP' = I \quad \text{and} \quad PBP' = \Lambda$$

where I is the identity matrix and Λ is a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \dots, \lambda_{r-1}$ which are the solutions of $|B - \lambda C| = 0$. The λ 's are positive since both B and C are positive definite matrices.

We make the transformation

$$(4.3) \quad R = PZ.$$

From (4.1) and (4.2) it is seen that the covariance matrix of R is $\Delta\Lambda + I$. Hence the elements R_1, R_2, \dots, R_{r-1} of R are independent with variances $\Delta\lambda_i + 1, i = 1, 2, \dots, r - 1$. It follows from (4.1)–(4.3) that

$$(4.4) \quad \begin{aligned} Z'A(\Delta_0)^{-1}Z &= R'(\Delta_0\Lambda + I)^{-1}R = \sum_{i=1}^{r-1} R_i^2 / (\Delta_0\lambda_i + 1); \\ Z'A(\Delta_1)^{-1}Z &= R'(\Delta_1\Lambda + I)^{-1}R = \sum_{i=1}^{r-1} R_i^2 / (\Delta_1\lambda_i + 1). \end{aligned}$$

Let $S_i = R_i/(\Delta\lambda_i + 1)^{\frac{1}{2}}$, $i = 1, 2, \dots, r-1$. Then S_1, S_2, \dots, S_{r-1} are independently and identically distributed as $N(0, 1)$. From (4.4)

$$(4.5) \quad \begin{aligned} Z'A(\Delta_0)^{-1}Z &= \sum_{i=1}^{r-1} (\Delta\lambda_i + 1)(\Delta_0\lambda_i + 1)^{-1}S_i^2; \\ Z'A(\Delta_1)^{-1}Z &= \sum_{i=1}^{r-1} (\Delta\lambda_i + 1)(\Delta_1\lambda_i + 1)^{-1}S_i^2. \end{aligned}$$

Substitution in (3.7) gives that W is distributed as

$$(4.6) \quad W(\Delta) = \frac{\sum_{i=1}^{r-1} \left(\frac{\Delta\lambda_i + 1}{\Delta_0\lambda_i + 1} - \frac{\Delta\lambda_i + 1}{\Delta_1\lambda_i + 1} \right) S_i^2}{\sum_{i=1}^{r-1} \frac{\Delta\lambda_i + 1}{\Delta_0\lambda_i + 1} S_i^2 + Q}$$

and in particular for $\Delta = \Delta_0$

$$W(\Delta_0) = \sum_{i=1}^{r-1} [1 - (\Delta_0\lambda_i + 1)/(\Delta_1\lambda_i + 1)] S_i^2 [\sum_{i=1}^{r-1} S_i^2 + Q]^{-1}.$$

To get a size α test a constant c must be determined such that $P(W(\Delta_0) > c) = \alpha$. The distribution of $W(\Delta_0)$ is then needed. For power calculations the distribution of $W(\Delta)$ is also needed.

5. Monotonicity of the power function. Let the power function of the most powerful invariant and similar test against Δ_1 be denoted $\beta(\Delta | \Delta_1)$. Then $\beta(\Delta | \Delta_1) = P(W(\Delta) > c)$ which by (4.6) can be written

$$\beta(\Delta | \Delta_1) = P(\sum_{i=1}^{r-1} (\Delta\lambda_i + 1)(\Delta_0\lambda_i + 1)^{-1}((\Delta_1 - \Delta_0)\lambda_i(\Delta_1\lambda_i + 1)^{-1} - c)S_i^2Q^{-1} > c).$$

Suppose that $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{r-1}$ (which can always be attained by relabelling) and let λ_m be the largest λ such that $(\Delta_1 - \Delta_0)\lambda_i/(\Delta_1\lambda_i + 1) < c$. Then

$$\begin{aligned} \beta(\Delta | \Delta_1) &= P(\sum_{i=m+1}^{r-1} (\Delta\lambda_i + 1)(\Delta_0\lambda_i + 1)^{-1}((\Delta_1 - \Delta_0)\lambda_i(\Delta_1\lambda_i + 1)^{-1} - c)S_i^2Q^{-1} \\ &\quad - \sum_{i=1}^m (\Delta\lambda_i + 1)(\Delta_0\lambda_i + 1)^{-1} \\ &\quad \cdot [c - (\Delta_1 - \Delta_0)\lambda_i(\Delta_1\lambda_i + 1)^{-1}]S_i^2Q^{-1} > c). \end{aligned}$$

We now multiply with $(\Delta'\lambda_m + 1)/(\Delta\lambda_m + 1)$ on both sides of the inequality sign within the parenthesis, and observe that

$$(\Delta'\lambda_m + 1)(\Delta\lambda_i + 1)((\Delta\lambda_m + 1)(\Delta_0\lambda_i + 1))^{-1} \geq (\Delta'\lambda_i + 1)(\Delta_0\lambda_i + 1)^{-1}$$

as $(\Delta' - \Delta)(\lambda_i - \lambda_m) \leq 0$. Hence for $\Delta < \Delta'$,

$$\begin{aligned} \beta(\Delta | \Delta_1) &< P(\sum_{i=m+1}^{r-1} (\Delta'\lambda_i + 1)(\Delta_0\lambda_i + 1)^{-1}[(\Delta_1 - \Delta_0)\lambda_i(\Delta_1\lambda_i + 1)^{-1} - c]S_i^2Q^{-1} \\ &\quad - \sum_{i=1}^m (\Delta'\lambda_i + 1)(\Delta_0\lambda_i + 1)^{-1} \\ &\quad \cdot [c - (\Delta_1 - \Delta_0)\lambda_i(\Delta_1\lambda_i + 1)^{-1}]S_i^2Q^{-1} > c) \\ &= \beta(\Delta' | \Delta_1). \end{aligned}$$

Thus $\beta(\Delta | \Delta_1)$ is an increasing function of Δ . In particular it is seen that the test is unbiased.

6. An alternative test. The test discussed in the preceding sections depends upon the alternative Δ_1 . If we cannot or will not specify any alternative, we can use as a general principle that it is desirable to have great power for large deviations from the hypothesis, i.e., for large values of Δ . Using this principle, let $\Delta_1 \rightarrow \infty$. Then

$$Z'A(\Delta_1)^{-1}Z = \sum_{i=1}^{r-1} R_i^2(\Delta_1\lambda_i + 1)^{-1} \rightarrow 0$$

and the limiting form of the test statistic W becomes

$$(6.1) \quad W' = Z'A(\Delta_0)^{-1}Z(Z'A(\Delta_0)^{-1}Z + Q)^{-1}.$$

To reject the hypothesis when $W' > \text{constant}$ is the same as to reject when

$$(6.2) \quad T = Z'A(\Delta_0)^{-1}ZQ^{-1} > \text{constant}.$$

From (4.5) it is seen that T is distributed as

$$(6.3) \quad T(\Delta) = \sum_{i=1}^{r-1} (\Delta\lambda_i + 1)(\Delta_0\lambda_i + 1)^{-1}S_i^2Q^{-1}$$

where in particular $T(\Delta_0) = \sum_{i=1}^{r-1} S_i^2Q^{-1}$. Hence $(n - r)(r - 1)^{-1}T$ have an F -distribution with $(r - 1)$ and $(n - r)$ degrees of freedom when $\Delta = \Delta_0$. Let $f_{1-\alpha}$ denote the upper α -point of this distribution, then we shall reject the hypothesis when $T > (n - r)(r - 1)^{-1}f_{1-\alpha}$. It is easily seen from (6.3) that the power function of this test is an increasing function of Δ .

The statistic T may also be used to derive a confidence interval for Δ . Let $f_{\alpha/2}$ and $f_{1-\alpha/2}$ be the lower and upper $\alpha/2$ -point of the above mentioned F -distribution. Then

$$(6.4) \quad P(f_{\alpha/2} < (n - r)(r - 1)^{-1}Z'A(\Delta)^{-1}ZQ^{-1} < f_{1-\alpha/2}) = 1 - \alpha.$$

As in (4.4)

$$(6.5) \quad Z'A(\Delta)^{-1}Z = \sum_{i=1}^{r-1} R_i^2(\Delta\lambda_i + 1)^{-1}$$

which shows that $Z'A(\Delta)^{-1}Z$ is a decreasing function of Δ . Hence (6.4) will give an interval for Δ .

7. The test statistics expressed by means of the original observations. Let $\bar{X}_i = n_i^{-1} \sum_{j=1}^{n_i} X_{ij}$, then $Y_{i1} = n_i^{\frac{1}{2}}\bar{X}_i$. Hence by Section 2

$$(7.1) \quad \begin{aligned} Q &= \sum_{i=1}^r \sum_{j=2}^{n_i} Y_{ij}^2 = \sum_{i=1}^r (\sum_{j=1}^{n_i} Y_{ij}^2 - Y_{i1}^2) \\ &= \sum_{i=1}^r \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2. \end{aligned}$$

By definition

$$Z_i = Y_{i1} - (n_i/n_r)^{\frac{1}{2}}Y_{r1} = n_i^{\frac{1}{2}}(\bar{X}_i - \bar{X}_r), \quad i = 1, 2, \dots, r - 1.$$

Define for any Δ ,

$$Z_r = \sum_{i=1}^r n_i(n_i\Delta + 1)^{-1}\bar{X}_i.$$

Then Z_i and Z_r are independent for $i = 1, 2, \dots, r - 1$, and $\text{Var } Z_r = a = \sum_{i=1}^r n_i(\Delta n_i + 1)^{-1}$. The covariance matrix of $Z^{*'} = [Z', Z_r]$ is:

$$\begin{bmatrix} A(\Delta) & 0 \\ 0 & a \end{bmatrix}.$$

Let D be the covariance matrix of $X = [\bar{X}_1, \bar{X}_2, \dots, \bar{X}_r]$. Then since Z^* is a linear transformation of X

$$Z^{*'} \begin{bmatrix} A(\Delta) & 0 \\ 0 & a \end{bmatrix}^{-1} Z^* = Z' A(\Delta)^{-1} Z + \frac{Z_r^2}{a}$$

is equal to

$$X' D^{-1} X = \sum_{i=1}^r n_i \bar{X}_i^2 (\Delta n_i + 1)^{-1}.$$

Combining this with the definition of Z_r we get

$$(7.2) \quad Z' A(\Delta)^{-1} Z = \sum_{i=1}^r n_i (\Delta n_i + 1)^{-1} (\bar{X}_i - \bar{X})^2$$

where

$$\bar{X} = \left(\sum_{i=1}^r n_i (\Delta n_i + 1)^{-1} \right)^{-1} \sum_{i=1}^r n_i (\Delta n_i + 1)^{-1} \bar{X}_i.$$

From (7.1) and (7.2) the test statistics W and T may now be computed. In particular it is seen that

$$(7.3) \quad T = \sum_{i=1}^r n_i (\Delta n_i + 1)^{-1} (\bar{X}_i - \bar{X})^2 \left(\sum_{i=1}^r \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 \right)^{-1}$$

where \bar{X} is computed with $\Delta = \Delta_0$. From this it is seen that the confidence interval based on T proposed in Section 7 is exactly the same as the one proposed by Wald [6].

By inserting (7.1) and (7.2) in the expression (3.7) for W , it is easily seen that, when the model is balanced the test based upon W is equal to the usual F -test which rejects when

$$m(\Delta_0 m + 1)^{-1} \sum_{i=1}^r (\bar{X}_{ij} - \bar{X})^2 \left(\sum_{i=1}^r \sum_{j=1}^m (X_{ij} - \bar{X}_i)^2 \right)^{-1} > (n - r)(r - 1)^{-1} f_{1-\alpha}$$

where $n_1 = n_2 = \dots = n_r = m$.

8. Optimum properties. In Section 3 it was proved that a maximal invariant under the group of translations is Z_1, \dots, Z_{r-1} and the Y_{ij} for $j > 1$. The problem of testing the hypothesis H is also invariant under the group of all orthogonal transformations of the variables Y_{ij} for $j > 1$, and under the group of change of scale of all variables. It is easily seen that a maximal invariant under the group G of all these transformations is

$$(8.1) \quad [Z_1 Q^{-\frac{1}{2}}, Z_2 Q^{-\frac{1}{2}}, \dots, Z_{r-1} Q^{-\frac{1}{2}}].$$

The distribution of (8.1) depends only upon Δ . Hence any test which depends upon (8.1) must be similar on ω_0 defined in Section 3. The class of tests invariant under the group G is thus contained in the class of tests considered in Section 3,

i.e., class of tests invariant under translations and similar on ω_0 . From (3.7) it is seen that the optimum test in the latter class depends only upon (8.1). Hence it is the optimum test in the former class too.

The discussion in the preceding paragraph shows that we can consider the test based upon W either as the most powerful test which is invariant under the group G , or as the most powerful test which is similar on ω_0 and invariant under a group of translations.

It may at this point be of interest to compare with Herbach's [2] results for the balanced case. Herbach proves that the usual F -test may either be considered as the most powerful test invariant under a group of transformations including translation, change of scale and orthogonal transformation, or as the most powerful similar test. (Only the case $\Delta_0 = 0$ is considered by Herbach, but it is easily proved that the results are true for any Δ_0). The difference is that for the unbalanced model it has not been possible to use similarity alone to derive a test.

The group of transformations satisfies the conditions of the Hunt-Stein theorem (see Lehmann [3] chapter 8). The most powerful invariant test against Δ_1 then maximizes the minimum power over the set of all alternatives with $\Delta = \Delta_1$, and since the power function of the test increases with Δ the test also maximizes the minimum power over the set of alternatives with $\Delta \geq \Delta_1$. Hence it is a maximin test over $\Delta \geq \Delta_1$, where the term maximin is used in the sense of Lehmann [3].

The statistic T given by (7.3) is the one used by Scheffé ([4] Chapter 7) to test the hypothesis $\Delta_0 = 0$. In this case T is equal to the usual test statistic in the fixed effects model. The test based on T may be interpreted as being almost equal to the most powerful invariant tests against large alternatives Δ_1 .

T was introduced to avoid to specify any alternative Δ_1 . Another way out is given by the following argument: It should be possible to determine a small number β such that if the power is greater than $1 - \beta$ it would be regarded as good enough. Then Δ_1 should be determined as the smallest Δ_1 such that the probability of not rejecting the hypothesis when $\Delta > \Delta_1$ is less than β . Since the power functions are monotone in Δ it is seen that Δ_1 should be taken as the solution of $\beta(\Delta_1 | \Delta_1) = 1 - \beta$. In this way the shortest possible interval (Δ_0, Δ_1) of not satisfactory power is obtained, and because of the maximin property of the test this is true even without the restriction to invariant tests.

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