ALTERNATIVE PROOFS FOR CERTAIN UPCROSSING INEQUALITIES

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1. Introduction. The purpose of this paper is to give different methods of proof for certain upcrossing and downcrossing inequalities that appear in martingale theory and to obtain sometimes improved inequalities. They include the fundamental one, due to Doob for martingales, and others due to Bishop, Dubins, Hunt and Snell. Except for the last two sections, the paper deals with an inductive approach which is exhibited first to show the main idea in the case of Snell's extension to submartingales of Doob's inequality. For the other cases fewer details will be given. For the sake of completeness we shall repeat some definitions.

Given a finite sequence of points $\mathcal{C} = \{c_1, \cdots, c_n\}$ in the two-point compactification of the real line and two real numbers $a < b$, we say that $\mathcal{C}$ *upcrosses* (*downcrosses*) $[a, b]$ *at least m times* if there exist m pairs of integers: $j_1 < k_1 < \cdots < j_m < k_m$ such that $c_{j_i} \leqq a$, $c_{k_i} \geqq b$, $(c_{j_i} \geqq b, c_{k_i} \leqq a)$. We say that $\mathcal{C}$ *upcrosses m times the interval* $[a, b]$ if $\mathcal{C}$ upcrosses it at least m times but not $m + 1$. A finite sequence of measurable functions in a probability space (Ω, Σ, P), $\{f_1, \cdots, f_n\}$, is said to be a *submartingale* if they are integrable and $\int_{F_j} f_j \, dP \leqq \int_{F_j} f_{j+1} \, dP$, for any j and any $F_j \in \mathcal{B}(f_1, \cdots, f_j)$, the least σ-algebra making measurable $f_1, \cdots, f_j$. It is said to be a *supermartingale* if their negatives constitute a submartingale, and is called a *martingale* if it is simultaneously a sub and a supermartingale.

We want to prove the following inequality (cf. [1], [6], [2]):

$$E(U) \leqq (b - a)^{-1} E[(f_n - a)^+], \tag{1}$$

where $U = U_{a,b,R}(\omega)$ denotes the number of times the submartingale $R = \{f_1, \cdots, f_n\}$ upcrosses $[a, b]$ at ω. The finite sequence $S = \{(f_1 - a)^+/(b - a), \cdots, (f_n - a)^+/(b - a)\}$ is also a submartingale, and everywhere, we have:

$$U_{0,1,S}(\omega) = U_{a,b,R}(\omega).$$

Therefore, it is sufficient to prove (1) in case of a nonnegative submartingale and $a = 0$, $b = 1$:

$$E(U_{0,1}) \leqq E(f_n). \tag{2}$$

(In general, we shall drop superfluous indices without comment.) When $n = 2$, (2) follows immediately:

$$P(f_1 = 0, f_2 \geqq 1) = E(U) \leqq \int_{\{f_1 = 0\}} f_2 \, dP. \tag{3}$$

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Assume it is true for $n \geqq 2$. Given the submartingale $S = \{f_1, \cdots, f_n, f_{n+1}\}$, let us consider the sequence $T = \{f_1, \cdots, f_{n-1}, g\}$ where $g = f_n$ on the sets $E_1 = \{f_n \geqq 1\}$, $E_2 = \{f_n = 0\}$. On $E_3 = \{0 < f_n < 1\}$ define $g = f_{n+1}$. T is a submartingale. In fact:

$$\int_{F_{n-1}} f_{n-1}\, dP \leqq \int_{F_{n-1}} f_n\, dP \leqq \int_{F_{n-1}(E_1 \cup E_2)} f_n\, dP + \int_{F_{n-1}E_3} f_{n+1}\, dP = \int_{F_{n-1}} g\, dP.$$

Since by inductive hypothesis we suppose the theorem true for T, it only remains to estimate $|U_{0,1,S} - U_{0,1,T}|$. On E_1, $f_n \geqq 1$, and therefore f_{n+1} does not contribute to any upcrossing for S. Then, $U_S(\omega) = U_T(\omega)$ if $\omega \varepsilon E_1$. On E_3, $0 < f_n < 1$, therefore, f_n does not contribute to any upcrossing for S and since $g(\omega) = f_{n+1}(\omega)$ for $\omega \varepsilon E_3$, we have in this case: $U_S(\omega) = U_T(\omega)$. On E_2, S has one upcrossing more than T exactly on the set $E_4 = E_2 \cap \{f_{n+1} \geqq 1\}$. Then, $U_S = U_T + I_{E_4}$, and,

$$EU_S = EU_T + PE_4 \leqq Eg + PE_4. \tag{4}$$

On the other hand we have:

$$Eg = \int_{E_1} f_n\, dP + \int_{E_3} f_{n+1}\, dP \leqq \int_{E_1 \cup E_3} f_{n+1}, \tag{5}$$

$$PE_4 = P(f_n \leqq 0, f_{n+1} \geqq 1) \leqq \int_{E_2} f_{n+1}\, dP. \tag{6}$$

From (4), (5) and (6), (2) follows. Q.E.D.

We have considered first the case $n = 2$ and not the trivial one $n = 1$, because it permits a guess of the form of the final inequality.

Observe that (2) can be strengthened as follows:

$$EU + Ef_1^+ = P(f_2 \geqq 1, f_1 = 0) + \int_{\{f_1 > 0\}} f_1\, dP \leqq Ef_2. \tag{7}$$

Then, following the same pattern of proof, we would get instead of (1):

$$(b - a)EU + E(f_1 - a)^+ \leqq E(f_n - a)^+. \tag{8}$$

Moreover, assume that a and b are random variables $\mathcal{B}(f_1)$-measurable and $a(\omega) < b(\omega)$ everywhere. Let U denote the number of upcrossings of the *random interval* $[a, b]$, where, by definition, a and b are integrable. The same proof that gives (8) applied directly to the process $\{(f_n - a)^+\}$ proves the following inequality:

$$E((b - a)U) + E(f_1 - a)^+ \leqq E((f_n - a)^+ I_G), \tag{9}$$

where $G = \{f_{n-1} > a\} \cup \{f_{n-1} \leqq a, f_n \geqq b\}$, (cf. [2] and Section 3). The integrability condition of the random interval could be relaxed but we shall not enter into this sort of detail.

Professor J. L. Doob pointed out to the author that the induction could be made on the first variables instead of the last ones, which, in a sense might be a more natural procedure when dealing with a submartingale.

2. Downcrossing inequalities. Analogous formulae to (1) and (7) of Section 1 could be obtained for downcrossings of an interval. Moreover, a slightly more delicate argument provides more precise estimations as shown in the next theorem which is a generalization of certain results due to L. E. Dubins, [4]. To avoid subscripts we write $I[A]$ instead of I_A.

THEOREM 1. *Let* $\{f_1, \cdots, f_n\}$ *be a submartingale and* $D(\omega)$ *the number of downcrossings at* ω *of the random interval* $[a(\omega), b(\omega)]$. *Then,*

(1) $E((b-a)I[D > k]) \leqq E((f_n - b)^+I[D = k, f_{n-1} > a]), \quad k = 0, 1, \cdots.$

Proof. Reasoning as in the introduction, we see it is sufficient to prove (1) for a nonnegative submartingale and a random interval $[0, b(\omega)]$. When $n = 2$, $\{D > 1\} = \phi$ and:

$$E(bI[D > 0]) = E(bI[f_1 \geqq b, f_2 = 0]) \leqq E(f_1 I[f_1 \geqq b, f_2 = 0])$$
$$\leqq E((f_1 - f_2)I[f_1 \geqq b, f_2 = 0]).$$

By submartingale property the last integral is not greater than:

$$E((f_2 - f_1)I[f_1 \geqq b, f_2 > 0]) \leqq E((f_2 - b)^+I[D = 0, f_1 > 0]),$$

from where next inequalities follow ($n = 2$):

(2) $E(bI[D > k]) \leqq E((f_n - b)^+I[D = k, f_{n-1} > 0]), \quad k = 0, 1, 2, \cdots.$

Given the nonnegative submartingale $S = \{f_1, \cdots, f_{n+1}\}$, $n > 2$, consider the nonnegative submartingale $T = \{f_1, \cdots, f_{n-1}, g\}$ where $g = f_n$ on $E_1 = \{f_n \geqq b\}$ and on $E_2 = \{f_n = 0\}$, $g = f_{n+1}$ on $E_3 = \{0 < f_n < b\}$. Call $E_4 = \{f_{n+1} = 0\}$. If D^S and D^T are the number of downcrossings for S and T respectively, it is easy to see—considering the different cases $\omega \in E_i$, $i = 1, 2, 3$, and then summing up on j—that:

(3) $\{D^S = j\} = (\{D^T = j\} - E_1E_4) + \{D^T = j - 1\}E_1E_4 \quad \text{for} \quad j > 0,$

(4) $\{D^S > k\} \subseteq \{D^T > k\} + \{D^T = k\}\cdot E_1E_4 \quad \text{for} \quad k \geqq 0.$

By inductive hypothesis, it is legitimate to use (2) for the submartingale T and therefore from (4) we get:

(5) $$E(bI[D^S > k]) \leqq E((g - b)^+I[D^T = k]) + E(bI[A]),$$

where A denotes the auxiliary set $\{D^T = k\}E_1E_4$. Since in E_2, $(f_n - b)^+ = 0$ and in E_3, $D^T = D^S$, we obtain:

(6) $$E(g - b)^+I[D^T = k] \leqq E(f_{n+1} - b)^+I[\{D^S = k\}E_3]$$
$$+ E(f_n - b)^+I[\{D^T = k\}E_1].$$

The sum of the last expectations on (5) and (6) is equal to:

$$Ef_nI[\{D^T = k\}E_1\} - EbI[\{D^T = k\}(E_1 - E_4)]$$

(7) $$\leqq Ef_{n+1}I[\{D^T = k\}E_1] - EbI[D^T = k, f_n \geqq b, f_{n+1} > 0]$$
$$= E(f_{n+1} - b)I[D^T = k, f_n \geqq b, f_{n+1} > 0]$$
$$\leqq E(f_{n+1} - b)^+I[\{D^S = k\}E_1).$$

From (5), (6) and (7), (2) follows for the submartingale S. Q.E.D.

When a and b are constants, Theorem 1 takes the form:

(8) $P(D > k) \leqq (b - a)^{-1} \int_{G_k} (f_n - b)^+ dP, G_k = \{D = k, f_{n-1} > a\}, \quad k \geqq 0,$

and summing up on k (cf. [1], [4]):

(9) $ED \leqq (b - a)^{-1} \int_{\{f_{n-1}>a\}} (f_n - b)^+ dP \leqq (b - a)^{-1} \| (f_n - b) I[J] \|_1,$

where J is the set of points where $f_{n-1} > a$ and $f_n > b$.

As another application (cf. [3]) assume that $b - a = y$ and $(f_n - b)^+ \leqq d < \infty$ almost surely; d and y are constants. Call $A = \{D > 0\}$ and $\complement A$, the complement of A. Then, from (1) it follows: $PA \leqq (d/y)P(\complement A)$. Therefore $P(\complement A) > 0$ and $PA/P(\complement A) \leqq d/y$. Observe now that $\exp[(z - 1)/z] < z$ for $0 < z < 1$. When $z = P(\complement A)$ and $PA > 0$, we have:

(10) $PA \leqq 1 - P(\complement A) < 1 - \exp[-PA/P(\complement A)] \leqq 1 - \exp[-d/y].$

If $0 \leqq f_n \leqq x < \infty$, x a constant, $a = 0, b = y$, (10) becomes:

(11) $P(D > 0) < 1 - \exp[-PA/P(\complement A)] \leqq 1 - e \exp[-x/y].$

3. Upcrossing inequalities. The analog of (1), Section 2, for upcrossings instead of downcrossings is discussed next because of its different behaviour.

THEOREM 1. *If $U(\omega)$ denotes the number of upcrossings at ω, under the hypothesis of Theorem* 1, *Section* 2,

(1) $E((b - a)I[U \geqq k]) \leqq E((f_n - a)^+ I[U = k]), \quad k = 1, 2, \cdots.$

PROOF. As in Section 2, it is sufficient to prove (1) for nonnegative submartingales, $a \equiv 0$. The case $n = 2$ is analogous to the inequality (2), Section 1, and is trivial. Given S, T, E_1, E_2, E_3 as in the proof of Theorem 1, Section 2, define $E_4 = \{f_{n+1} \geqq b\}$. If the random upcrossing variables for S and T are represented by U^S and U^T respectively, it is easy to obtain the analogues to (3) and (4) of Theorem 1, Section 2:

(2) $\{U^S = j\} = (\{U^T = j\} - E_2E_4) + \{U^T = j - 1\}E_2E_4, \quad j \geqq 1,$

(3) $\{U^S \geqq k\} \subseteq \{U^T \geqq k\} + \{U^T = k - 1\} \cdot E_2E_4, \quad k \geqq 1.$

Defining $A = \{U^T = k - 1\}E_2E_4$, and making use of the inductive hypothesis we finally obtain for $k \geqq 1$:

(4) $EbI[U^S \geqq k] \leqq E(gI[U^T = k] + bI[A]) = E(gI[\{U^T = k\}\complement E_2]) + EbI[A]$

(5) $EbI[A] = E(bI[\{U^T = k - 1\}\{f_n = 0\}\{f_{n+1} \geqq b\}) \leqq Ef_{n+1}I[\{U^S = k\}E_2].$

On the other hand, $\omega \in \complement E_2 = E_1 + E_3$ implies $\{U^T = k\} = \{U^S = k\}$ and besides $E_1 \cdot \{U^T = k\} \in \mathfrak{B}(f_1, \cdots, f_n)$, then the first expectation in the last member of (4) is not greater than:

(6) $$E(f_{n+1}I[\{U^S = k\}\complement E_2]).$$

From (4), (5) and (6), (1) follows. Q.E.D.

Summation on k gives:

$$\int (b - a)U\,dP \leqq \int_H (f_n - a)^+\,dP, \tag{7}$$

where $H = \{U > 0\}$ is the set where some upcrossing occurs.

4. Application. In this section we want to draw some consequences of the two preceding theorems (cf. [2], [3], [4)). For a submartingale $\{f_1, \cdots, f_n\}$ and the interval $[a, b]$, a, b constants, we have:

$$(b - a)P(D > k) \leqq \int_{\{D=k\}} (f_n - b)^+\,dP, \qquad k \geqq 0, \tag{1}$$

$$(b - a)P(U \geqq k) \leqq \int_{\{U=k\}} (f_n - a)^+\,dP, \qquad k \geqq 1, \tag{2}$$

where the first inequality is due to Dubins [4].

Suppose now that a and b are negative numbers and the submartingale is nonpositive. From (1) we get $P(D > k) \leqq (-b/(b - a))P(D = k)$ and therefore:

$$P(D > k) \leqq (b/a)P(D \geqq k), \qquad k \geqq 0. \tag{3}$$

Analogously, from (2), $P(U \geqq k) \leqq -a\cdot P(U = k)/(b - a)$, and then:

$$P(U > k) \leqq (b/a)P(U \geqq k), \qquad k \geqq 1. \tag{4}$$

(However, *in* (1) $>$ *cannot be replaced by* $\geqq$, because this would imply in case of a nonpositive submartingale that $(b - a)P(D > k) \leqq (a - 2b)P(D = k)$ and therefore, a contradiction, since the second member could be strictly negative). From (3) and (4), (5) and (6) follow for $k \geqq 0$:

$$P(D \geqq k + 1) = P(D > k) \leqq (b/a)P(D \geqq k) \leqq \cdots \leqq (b/a)^k P(D > 0). \tag{5}$$

$$P(U \geqq k + 1) \leqq (b/a)P(U \geqq k) \leqq \cdots \leqq (b/a)^k P(U > 0). \tag{6}$$

(The submartingale $\{-I_{(0,1)}, -2I_{(0,\frac{1}{2})}, -I_{(0,\frac{1}{2})}, -2I_{(0,\frac{1}{2})}\}$, defined in the unit interval, shows that for $a = -2$, $b = -1$: $P(D \geqq 2) = (b/a)P(D > 0)$ and therefore that *the exponent k in* (5) *cannot be replaced by* $k + 1$.) From (5) and (6), we finally obtain (cf. [5], [2], [3]):

$$E(D \mid \{D > 0\}) \leqq |a/(b - a)| \geqq E(U \mid \{U > 0\}). \tag{7}$$

From (7), Section 3, we obtain, *in general*:

$$E(U \mid \{U > 0\}) \leqq \|(f_n - a)^+\|_\infty/(b - a). \tag{8}$$

5. A separation inequality. A set of points of R^2, $\Gamma = \{u^1, v^1; \cdots; u^n, v^n\}$ will be called a *finite system of pairs* (fsp) if the second coordinates satisfy: $v_2^i > u_2^i$, and the first ones: $v_1^i = u_1^i < u_1^{i+1} = v_1^{i+1}$. Consider two straight lines, A, B, with slopes a and b passing through $x \in R^2$. Call $w(x, a, b, \Gamma)$ the number of pairs of Γ with first coordinate greater than x_1 and separated by A and B, i.e., if u^k, v^k, is such a pair, the straight line joining x with $u^k(v^k)$ has slope less (greater) than or equal to $a(b)$.

THEOREM 1. *Let* Γ *be a* fsp *and* a *and* b *as above. If* $w(t) = w(t, \Gamma) = \sup_{x_1=t} w(x, a, b, \Gamma)$, *then,*

$$\int w(t)\, dt \leqq \sum_{i=1}^{n} (v_2^i - u_2^i)/(b - a).$$

This theorem is analogous to a lemma due to E. Bishop, (cf. [0], p. 2). We have no restrictive hypothesis on Γ but our definition of w is different.

PROOF OF THEOREM 1. The function $w(t)$ is a step function. Let $I_1, \cdots, I_n$ be nonoverlapping, left closed, right open intervals that cover the support of w and such that:

(i) w is constant on each $I_i = [c_i, d_i)$,

(ii) given j, $I_j \subseteq [u_1^k, u_1^{k+1}]$ for some k.

Consider all the triangles Δ^i with vertices u^i, v^i, and the intersection σ^i of the straight line with slope b passing through v^i with the straight line with slope a passing through u^i. We shall consider such triangles "right open," that is, without its vertical side. Let V_t be the vertical line of points with abscissa t. Since separation from x of exactly k pairs of Γ is equivalent to that x belongs exactly to k triangles Δ^i, and precisely those determined by the separated points, it follows that on V_{c_i} there is a set of points belonging exactly to $w(I_i)$ triangles where $w(I_i)$ is the common value of w on the points of I_i. Pick one of these points: s^i. If m^i and M^i are the intersections of V_{d_i} with the straight lines through s^i with slopes a and b, then by (ii) the (right open) triangle $\delta^i = (m^i; M^i; s^i)$ consists of points x such that $w(x, a, b, \Gamma) = w(I_i)$. That is, δ^i lies in exactly $w(I_i)$ triangles Δ^j. Then, if B^k represents the interval (m^k, M^k), we have:

$$(1) \quad \int w(t)\, dt = \sum_k w(I_k)|I_k| = \sum w(I_k)|B^k|/(b - a) = \sum_{j=1}^{n} \sum_{\{k;\, \delta^k \subseteq \Delta^j\}} |B^k|/(b - a).$$

Since the δ^k's contained in a Δ^j have nonoverlapping projections on the x-axis, the sum of the lengths of their vertical bases is not greater than the length of the basis of Δ^j. Therefore, the last sum of (1) is not greater than:

$$\sum_{j=1}^{n} (v_2^j - u_2^j)/(b - a). \qquad \text{Q.E.D.}$$

Assume now that the points of Γ are located on straight lines making an angle γ with the x-axis and such that $\alpha \neq \gamma \neq \beta$, where α and β are the angles with $tg\alpha = a$, $tg\beta = b$, $-\pi/2 < \alpha, \beta, \gamma < \pi/2$. If d_i denotes the distance between u^i and v^i, we obtain with the same proof.

$$(2) \quad \int w(t)\, dt \leqq (\sin(\gamma - \beta) \sin(\alpha + \gamma)/\sin(\beta - \alpha) \sin\gamma) \sum_{i=1}^{n} d_i .$$

6. Remarks. Let Γ be a fs of n pairs and $w(t)$ the function defined in the preceding section. Assume that $a = 0$, $b = 1$ (the general case could be treated with formula (2), Section 5). *There exists a finite, nonnegative submartingale* S *defined on the support of* w, $I = \{t; w(t) \neq 0\}$, *and with respect to Lebesgue measure,* *such that:*

(1) $$\int U_{0,1,S}\,dt = \int w(t)\,dt = \int \text{ last element of } S.$$

In fact, put $E_i = \{t; w(t) = i\}, i = 1, 2, \cdots, k = \max w(t)$. From the proof of Theorem 1, Section 5, one can see that $w(t)$ actually takes all the values 1 to k on sets of positive measure. Then, for $t \varepsilon I$, it is easy to check that the following submartingale satisfies our claim:

$$S: X_1 = 0; \quad X_2 = 1; \quad X_{2j+1} = X_{2j} \text{ on } E_1 \cup E_2 \cup \cdots \cup E_{j-1}$$
$$0 \text{ on } E_{j+1} \cup \cdots \cup E_k, = 1 + \mu(E_{j+1} \cup \cdots \cup E_k)/\mu(E_j) \text{ on } E_j;$$
$$X_{2j+2} = X_{2j+1} \vee 1, \qquad 1 \leqq j \leqq k.$$

As a matter of fact, S *upcrosses* $[0, 1]$ *exactly* j *times on* E_j and

$$\int X_{2k}\,dt = \mu(E_1) + 2\mu(E_2) + \cdots + k\mu(E_k) = \int U_{0,1}\,dt = \int w\,dt.$$

In the preceding section and in this one, is implicit the formula that relates the sum of measures of nonoverlapping sets with the sum of measures of the subsets that cover a point the same number of times. That this is quite natural is shown by the following consideration. If $\{f_1, \cdots, f_n\}$ is a nonnegative submartingale and $f_1 = 0$, $U = U_{0,1}$, define:

$$B_k = \{\omega; (X_{k+1} \wedge 1)^m - (X_k \wedge 1)^m \rightarrow_{m\to\infty} 1\}.$$

Then, it is easy to see that $U(\omega)$ coincides with the number of sets B_k containing ω.

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