PERMUTATIONS WITHOUT RISING OR FALLING w-SEQUENCES

BY MORTON ABRAMSON AND W. O. J. MOSER

McGill University

0. Introduction. A permutation of degree $n > 1$ is said to contain the sequence $ijk \cdots st$ if, in the permutation, i immediately precedes j, j immediately precedes $k, \cdots$, and s immediately precedes t. The rising w-sequences ($w \geqq 2$) are those in the left column in Table I; the falling w-sequences are in the right column.

The enumeration of permutations without rising 2-sequences is given by Whitworth [8], p. 102. Permutations without rising or falling 2-sequences have been treated by Kaplansky [2] in the form of what he calls the n king problem: in how many ways can n kings be placed on an $n \times n$ board, no two in a row or column and no two attacking each other? (See also [1], [3], [4], [6], [9].) Riordan [5] enumerated permutations without rising 3-sequences. In Section 1 we obtain expressions for the number of permutations containing exactly $r \geqq 0$ rising and/or falling w-sequences. In Section 2 we obtain corresponding results in the "circular" case, where the integers 1 and n are considered adjacent.

1. Straight line case. Call a k-choice

$$x_1 < x_2 < \cdots < x_k \tag{1}$$

from $\{1, 2, \cdots, n\}$ a $(n: k \mid a, b, c, \cdots)$-choice if

$$a = \sum_{x_i - x_{i-1} > 1} 1, \qquad b = \sum_{x_i - x_{i-1} > 2} 1, \qquad c = \sum_{x_i - x_{i-1} > 3} 1, \cdots. \tag{2}$$

Clearly $a \geqq b \geqq c \geqq \cdots$, and any $(n: k \mid a, b, \cdots, p, q)$-choice is also a $(n: k \mid a, b, \cdots, p)$-choice (but not in general conversely). For example, for $n \geqq 21$,

$$2, 3, 5, 9, 10, 11, 13, 17, 18, 19, 20, 21 \tag{3}$$

is a $(n: 12 \mid 4, 2, 2, 0, \cdots)$-choice. Let $((n: k \mid a, b, c, \cdots, p, q))$ denote the number of $(n: k \mid a, b, c, \cdots, p, q)$-choices.

As usual we take

$$\binom{n}{r} = n!/r!(n-r)! \quad \text{when} \quad 0 \leqq r \leqq n,$$
$$= 0 \qquad \text{otherwise.}$$

THEOREM 1.

$$((n: k \mid a, b, c, \cdots, p, q)) = \binom{k-1}{a}\binom{a}{b}\binom{b}{c} \cdots \binom{p}{q}\binom{n-k-a-b-\cdots-p+1}{q+1}, \qquad 1 \leqq k \leqq n. \tag{4}$$

We require the well-known [7], p. 92,

LEMMA. *The number of ways of putting n like objects into m differents cells is $\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$. When no cell is empty the number of ways is $\binom{n-1}{m-1}$.*

Received 20 December 1966.

TABLE I

1	2	$\cdots$	w	w	$w-1$	$\cdots$	1
2	3	$\cdots$	$w+1$	$w+1$	w	$\cdots$	2
		$\vdots$				$\vdots$	
$n-w+1$	$n-w+2$	$\cdots$	n	n	$n-1$	$\cdots$	$n-w+1$
	Rising				Falling		

Proof of Theorem 1. Clearly, a k-choice from n (i.e., from $\{1, 2, \cdots, n\}$) is conveniently represented by $n-k$ symbols 0 (one for each integer not in the k-choice) and k symbols 1 (one for each integer in the k-choice) arranged along a straight line (rising order being left to right). For example, with $n = 23$, the 12-choice (3) is represented by

$$01101000111010001111100. \tag{5}$$

We find the arrangements representing the $(n\colon k \mid a, b, \cdots, p, q)$-choices as follows. Place $a+1$ boxes in a row, forming $a+2$ "cells": a cells between pairs of adjacent boxes and one cell at each end. Distribute the k symbols 1 into the $a+1$ boxes, none empty, in $\binom{k-1}{a}$ ways. Place a single symbol 0 in each of the a "in-between" cells. Choose b of the a cells, in $\binom{a}{b}$ ways, and place a second symbol 0 in each. Choose c of the b cells, in $\binom{b}{c}$ ways, and place a third symbol 0 in each. Continue until q cells have been chosen, in $\binom{p}{q}$ ways, from the p cells and an additional symbol 0 has been placed in each of the q cells. Now distribute, without restriction, the remaining $n-k-a-b-c-\cdots-p-q$ symbols 0 into the $q+2$ cells—the q cells involved in the previous step and the 2 cells at the ends—in $\binom{n-k-a-b-\cdots-q+q+2-1}{q+2-1} = \binom{n-k-[a+b+\cdots+p]+1}{q+1}$ ways. The result follows.

In a k-choice, a part is a sequence of consecutive integers not contained in a longer one; the length of a part is the number of integers in it. For example, the 12-choice (3) contains 5 parts:

$$(2, 3),\ (5),\ (9, 10, 11),\ (13),\ (17, 18, 19, 20, 21), \tag{6}$$

of lengths 2, 1, 3, 1, 5 respectively. Thus, a $(n\colon k \mid a, b, c, \cdots)$-choice contains $a+1$ parts. Hence, the number of k-choices from n containing exactly r parts is

$$((n\colon k \mid r-1)) = \binom{k-1}{r-1}\binom{n-k+1}{r}, \tag{7}$$

in agreement with [1].

Formula (4) may also be obtained from (7) and repeated use of the relation

$$((n\colon k \mid a, b, c, \cdots, p, q)) = ((n-k+1\colon a+1 \mid b, c, \cdots, p, q))\binom{k-1}{a}, \tag{8}$$

which is obtained as follows. Place $n-k$ symbols 0 along a straight line, forming $n-k+1$ cells. If we choose $a+1$ of the cells to be a $(n-k+1\colon a+1 \mid b, c, \cdots, p, q)$-choice, and then distribute the k symbols 1 into these $a+1$ cells with none empty, the resulting sequence of 0's and 1's corresponds to a $(n\colon k \mid a, b, c, \cdots, p, q)$-choice, and (4) follows immediately.

The relation

$$\begin{aligned}((n{:}\,k \mid a_1, \cdots, a_w)) &= ((n-1{:}\,k \mid a_1, \cdots, a_w)) \\ &+ ((n-1{:}\,k-1 \mid a_1, \cdots, a_w)) - ((n-2{:}\,k-1 \mid a_1, \cdots, a_w)) \\ &+ ((n-2{:}\,k-1 \mid a_1-1, a_2, \cdots, a_w)) \\ &- ((n-3{:}\,k-1 \mid a_1-1, a_2, \cdots, a_w)) \\ &+ ((n-3{:}\,k-1 \mid a_1-1, a_2-1, a_3, \cdots, a_w)) \\ &- ((n-4{:}\,k-1 \mid a_1-1, a_2-1, a_3, \cdots, a_w)) \\ &\vdots \\ &+ ((n-w{:}\,k-1 \mid a_1-1, \cdots, a_{w-1}-1, a_w)) \\ &- ((n-w+1{:}\,k-1 \mid a_1-1, \cdots, a_{w-1}-1, a_w)) \\ &+ ((n-w-1{:}\,k-1 \mid a_1-1, \cdots, a_w-1))\end{aligned}$$

is obtained by observing that the first term on the right counts those choices which do not contain the integer n; the next two terms count those choices which contain n and $n-1$; the next two terms count those choices which contain n and $n-2$ but not $n-1$; the next two terms count those choices containing n and $n-3$ but not $n-1$ and $n-2$; $\cdots$; the third and second to last terms count those choices containing n and $n-w$ but not $n-1, n-2, \cdots, n-w+1$; the last term counts those choices containing n but not $n-1, n-2, \cdots, n-w$.

This relation is valid for $a_i \geqq 0$ provided we agree that $((n{:}\,k \mid a, b, c, \cdots)) = 0$ whenever one (or more) of the $a, b, c, \cdots$ is negative.

The special case $w = 1$ is

$$\begin{aligned}((n{:}\,k \mid a)) = ((n-1{:}\,k \mid a)) &+ ((n-1{:}\,k-1 \mid a)) \\ &- ((n-2{:}\,k-1 \mid a)) + ((n-2{:}\,k-1 \mid a-1)),\end{aligned}$$

in agreement with [1], since a $(n{:}\,k \mid a)$- choice has $a+1$ parts, and also in agreement with the lemma in [5], since a $(n{:}\,k \mid a)$-choice has $k-a-1$ successions.

Let $A = (a_{ij})$, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$, be an $n \times m$ matrix. Let $g_{n,k}(m \mid v)$ denote the number of ways of choosing k of the nm entries such that if $a_{\alpha\beta}$ and $a_{\gamma\delta}$ are any two of the chosen entries then $\alpha \neq \gamma$, and $\beta = \delta$ when $|\alpha - \gamma| \leqq v$. It is easy to see that

$$(9) \quad g_{n,k}(m \mid v) = \sum_{a_1=0}^{k-1} \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \cdots \sum_{a_v=0}^{a_{v-1}} m^{a_v+1} \cdot ((n{:}\,k \mid a_1, a_2, \cdots, a_v)), \qquad v \geqq 1.$$

Indeed, if the rows are numbered $1, 2, \cdots, n$, then, for a particular k-choice of the n rows, forming a $(n{:}\,k \mid a_1, \cdots, a_v)$-choice, say, there are m^{a_v+1} ways of choosing k entries of the matrix—one from each of these k rows—so that the given conditions are satisfied.

In a permutation of $1, 2, \cdots, n$, the joint occurrence of k of the $2(n - w + 1)$ events listed in the two columns of Table I is possible if and only if

(i) no two of the events come from the same row and

(ii) two which come from rows s and t with $|s - t| \leqq w - 1$ must come from the same column.

Let us focus our attention on a partiuclar choice of k consistent events from Table I. We prove that if the rows, from which these k events come, form a $(n - w + 1: k \mid a_1, a_2, \cdots, a_{w-1})$-choice (of the rows) then the number of permutations containing these k particular events is

$$(10) \qquad (n - k - a_1 - a_2 - \cdots - a_{w-2} - w + 2)!, \qquad w \geqq 2,$$

with the understanding that when $w = 2$ the above expression is $(n - k)!$.

The following proof is valid for $w > 2$. If two events come from rows s and t say, $1 \leqq t - s \leqq w - 1$, then they are in the same column of Table I, and a permutation containing them must also contain the events, in that column, which are in the rows between i.e., in rows $s + 1, s + 2, \cdots, t - 1$. Thus the permutation contains

$$(11) \quad k + (a_1 - a_2) + 2(a_2 - a_3) + \cdots + (w - 2)(a_{w-2} - a_{w-1}) = k + a_1 + a_2 + \cdots + a_{w-2} - (w - 2)a_{u-1}$$

events (from Table I), and their rows form a

$$(n - w + 1: k + a_1 + a_2 + \cdots + a_{w-2} - (w - 2)a_{w-1} \mid b_1, b_2, \cdots, b_{w-1})\text{-choice},$$
$$b_1 = b_2 = \cdots = b_{w-1} = a_{w-1}.$$

This choice of the $n - w + 1$ rows has exactly $a_{w-1} + 1$ parts. Furthermore, two events in rows belonging to different parts of the above choice involve no common integers. Let $\alpha_1, \alpha_2, \cdots, \alpha_{a_{w-1}+1}$ be the lengths of the $a_{w-1} + 1$ parts. They involve, respectively,

$$(12) \qquad \alpha_1 + w - 1, \qquad \alpha_2 + w - 1, \cdots, \alpha_{a_{w-1}+1} + w - 1$$

different integers, so, by the previous remark, the k events involve

$$(13) \qquad \textstyle\sum_{i=1}^{a_{w-1}+1} (\alpha_i + w - 1) = \sum_{i=1}^{a_{w-1}+1} \alpha_i + (w - 1)(a_{w-1} + 1)$$

different integers altogether. Of course,

$$(14) \qquad \textstyle\sum_{i=1}^{a_{w-1}+1} \alpha_i = k + a_1 + a_2 + \cdots + a_{w-2} - (w - 2)a_{w-1}$$

and hence there are

$$(15) \qquad n - k - a_1 - a_2 - \cdots - a_{w-2} - a_{w-1} - w + 1$$

integers not involved in the events. Treating these integers and the $a_{w-1} + 1$ parts as

$$(16) \qquad n - k - a_1 - a_2 - \cdots - a_{w-2} - w + 2$$

distinct entities, we deduce that there are

$$(n - k - a_1 - a_2 - \cdots - a_{w-2} - w + 2)! \tag{17}$$

permutations containing any particular k chosen events whose rows form a $(n - w + 1 : k \mid a_1, \cdots, a_{w-1})$-choice.

In the case $w = 2$, the proof of (10) is similar.

Using the well-known principle of inclusion and exclusion [7], p. 53, we now have

THEOREM 2. (i) *The number of permutations of* 1, 2, $\cdots$, n *containing exactly* $r \geqq 1$ *rising and/or falling* w-*sequences is*

$$\begin{aligned} N_r(n, w) = \sum_{i=0}^{n-w+1-r} (-1)^i \tbinom{r+i}{r} \sum_{a_1=0}^{r+i-1} \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \cdots \sum_{a_{w-1}=0}^{a_{w-2}} 2^{a_{w-1}+1} \\ \cdot ((n - w + 1 : r + i \mid a_1, a_2, \cdots, a_{w-1})) \\ \cdot (n - r - i - a_1 - a_2 - \cdots - a_{w-2} - w + 2)!, \qquad r \geqq 1. \end{aligned}$$

(ii) *The number of permutations without rising or falling* w-*sequences is*

$$\begin{aligned} N_0(n, w) = n! + \sum_{i=1}^{n-w+1} (-1)^i \sum_{a_1=0}^{i-1} \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \cdots \sum_{a_{w-1}=0}^{a_{w-2}} 2^{a_{w-1}+1} \\ \cdot ((n - w + 1 : i \mid a_1, a_2, \cdots, a_{w-1}))(n - i - a_1 - a_2 - \cdots - a_{w-2} - w + 2)! \end{aligned}$$

(iii) *The number of permutations containing exactly* $r \geqq 1$ *rising (falling)* w-*sequences is*

$$\begin{aligned} R_r(n, w) = \sum_{i=1}^{n-w+1} (-1)^i \tbinom{r+i}{i} \sum_{a_1=0}^{r+i-1} \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \\ \cdots \sum_{a_{w-1}=0}^{a_{w-2}} ((n - w + 1 : r + i \mid a_1, a_2, \cdots, a_{w-1})) \\ \cdot (n - r - i - a_1 - a_2 - \cdots - a_{w-2} - w + 2)! \end{aligned}$$

(iv) *The number of permutations without rising (falling)* w-*sequences is*

$$\begin{aligned} R_0(n, w) = n! + \sum_{i=1}^{n-w+1} (-1)^i \sum_{a_1=0}^{i-1} \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \\ \cdots \sum_{a_{w-1}=0}^{a_{w-2}} ((n - w + 1 : i \mid a_1, a_2, \cdots, a_{w-1})) \\ \cdot (n - i - a_1 - a_2 - \cdots - a_{w-2} - w + 2)! \end{aligned}$$

Using Vandermonde's formula [7], p. 9,

$$\sum_{k=0} \tbinom{m}{k} \tbinom{n}{r-k} = \tbinom{m+n}{r}, \tag{18}$$

we have the following special cases.

$$\begin{aligned} \text{(I)} \quad R_0(n, 2) &= n! + \sum_{i=1}^{n-1} (-1)^i \sum_{a=0}^{i-1} ((n - 1 : i \mid a))(n - i)! \\ &= n! + \sum_{i=1}^{n-1} (-1)^i \sum_{a=0}^{i-1} \tbinom{i-1}{a} \tbinom{n-i}{a+1} (n - i)! && \text{by (7)} \\ &= n! + \sum_{i=1}^{n-1} (-1)^i (n - i)! \sum_{a=0}^{i-1} \tbinom{i-1}{a} \tbinom{n-i}{n-i-1-a} \\ &= \sum_{i=0}^{n-1} (-1)^i (n - i)! \tbinom{n-1}{i} && \text{by (18)} \end{aligned}$$

in agreement with Whitworth [8], p. 102.

(II) $R_0(n, 3) = n! + \sum_{i=1}^{n-2} (-1)^i \sum_{a=0}^{i-1} \sum_{b=0}^{a} ((n-2:i \mid a, b))$

$\cdot (n - i - a - 1)!$

$= n! + \sum_{i=1}^{n-2} (-1)^i \sum_{a=0}^{i-1} \sum_{b=0}^{a} \binom{i-1}{a}\binom{a}{b}\binom{n-a-i-1}{b+1}$

$\cdot (n - i - a - 1)!,$ by (4),

$= n! + \sum_{i=1}^{n-2} (-1)^i \sum_{a=0}^{i-1} \binom{i-1}{a}\binom{n-i-1}{a+1}$

$\cdot (n - i - a - 1)!$ by (18)

(and putting $a = i - j - 1$)

$$R_0(n, 3) = n! + \sum_{i=0}^{n-2} (-1)^i \sum_{j=0}^{i-1} \binom{i-1}{j}\binom{n-i-1}{i-j}(n - 2i + j)!$$

in agreement with Riordan [5], p. 747, expression (2).

(III) $N_0(n, 2) = n! + \sum_{i=1}^{n-1} (-1)^i \sum_{a=0}^{i-1} 2^{a+1}((n-1:i \mid a))$

$\cdot (n - i)!$

$= n! + \sum_{i=1}^{n-1} (-1)^i \sum_{a=0}^{i-1} 2^{a+1}\binom{i-1}{a}\binom{n-i}{a+1}(n - i)!$ by (4)

$= n! + \sum_{i=1}^{n-1} (-1)^i \sum_{r=1}^{i} 2^{r}\binom{i-1}{r-1}\binom{n-i}{r}(n - i)!$ by (18)

in agreement with Abramson and Moser [1], last expression combined with (9) on p. 271.

On an $n \times n$ chess-board, a positioning of n counters with no two in a row or column is described by a permutation $(i_1, i_2, \cdots, i_n)$ of degree n: the counters are in the squares, column j and row i_j, $j = 1, 2, \cdots, n$. Thus $N_0(n, w)$ is the number of ways of placing n counters on an $n \times n$ board, no two in a row or column and no w lined up consecutively along a diagonal. In particular, $N_0(n, 2)$ is the solution to the n-kings problem. The numbers $N_r(n, w)$ and $R_r(n, w)$ also have chess-board interpretations.

2. Circular case. Now we turn our attention to the circular case. The clockwise w-sequences are those in (A) in Table II; the counterclockwise w-sequences are in (B).

TABLE II

(A)					(B)				
1	2	$\cdots$	$w-1$	w	w	$w-1$	$\cdots$	2	1
2	3	$\cdots$	w	$w+1$	$w+1$	w	$\cdots$	3	2
3	4	$\cdots$	$w+1$	$w+2$	$w+2$	$w+1$	$\cdots$	4	3
		$\vdots$					$\vdots$		
$n-w+1$		$\cdots$	$n-2$	n	n	$n-1$	$\cdots$		$n-w+1$
$n-w+2$		$\cdots$	n	1	1	n	$\cdots$		$n-w+2$
		$\vdots$					$\vdots$		
n	1		$w-2$	$w-1$	$w-1$	$w-2$		1	n
Clockwise					Counterclockwise				

We will enumerate permutations which contain exactly r clockwise and/or counterclockwise w-sequences.

Define, for $1 \leqq i \neq j \leqq n$,

$$\overline{i, j} = j - i \qquad \text{if} \quad i < j$$
$$= j - i + n \qquad \text{if} \quad j < i. \tag{19}$$

Call a k-choice

$$x_1 < x_2 < \cdots < x_k \tag{20}$$

from $\{1, 2, \cdots, n\}$ a $\langle n\colon k \mid a, b, c, \cdots\rangle$-choice if (with $x_0 = x_k$)

$$a = \textstyle\sum_{\overline{x_i, x_{i+1}} > 1} 1, \qquad b = \sum_{\overline{x_i, x_{i+1}} > 2} 1, \qquad c = \sum_{\overline{x_i, x_{i+1}} > 3} 1, \cdots, \tag{21}$$

sums taken with i running through $0, 1, 2, \cdots, k-1$. Clearly $a \geqq b \geqq c \geqq \cdots$, and any $\langle n\colon k \mid a, b, \cdots, p, q\rangle$-choice is also a $\langle n\colon k \mid a, b, \cdots, p\rangle$-choice (but not in general conversely).

These "circular" k-choices are best seen when arranged in a circle. For example

(22)

2
21 3
20 5
19 9
18 10
17 11
13

is a $\langle 23\colon 12 \mid 5\rangle$-choice, a $\langle 23\colon 12 \mid 5, 3\rangle$-choice, a $\langle 23\colon 12 \mid 5, 3, 3, 0\rangle$-choice.

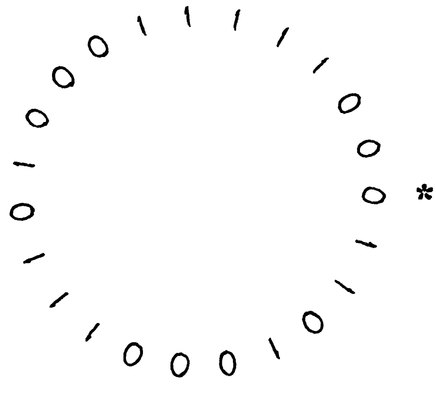
Let $\langle\langle n\colon k \mid a, b, c, \cdots, p, q\rangle\rangle$ denote the number of $\langle n\colon k \mid a, b, c, \cdots, p, q\rangle$-choices.

THEOREM 3. *For* $0 < k < n$ *and* $0 < q$,

$$\langle\langle n\colon k \mid a, b, c, \cdots, p, q\rangle\rangle = (n/a)\binom{k-1}{a-1}\binom{a}{b}\binom{b}{c}\cdots\binom{p}{q}\binom{n-k-a-b-\cdots-p-1}{q-1}. \tag{23}$$

PROOF. Clearly a "circular" k-choice from n is conveniently represented by $n - k$ symbols 0 and k symbols 1 arranged in a circle with one of n symbols marked, by a * say, to indicate that it corresponds to the integer 1. For example, for $n = 23$, the $\langle 23\colon 12 \mid 5, 3, 3, 0\rangle$-choice (22) is represented by

(24)



We find the arrangements representing the $\langle n\colon k \mid a, b, c, \cdots, p, q\rangle$-choices as follows. Place a distinguishable boxes in a circle, forming a cells. Distribute the k symbols 1 into the boxes in $\binom{k-1}{a-1}$ ways. Place a single 0 into each cell. Choose b of the a cells in $\binom{a}{b}$ ways, and place an additional single 0 in each. Choose c of the b cells, in $\binom{b}{c}$ ways, and place an additional single 0 in each. Continue until q cells have been chosen, in $\binom{p}{q}$ ways, from the p cells, and an additional 0 has been placed in each of these q cells. Distribute the remaining $n - k - a - b - \cdots - p - q$ symbols 0 into the last q cells, without restriction, in (by the lemma) $\binom{n-k-a-b-\cdots-p-q+q-1}{q-1}$ ways. Mark one of the n symbols with a *. The resulting configurations fall into sets of a each which are the same by rotation, and the result follows.

The special case

$$\langle\langle n\colon k \mid r\rangle\rangle = (n/r)\binom{k-1}{r-1}\binom{n-k-1}{r-1} = (n/(n-k))\binom{k-1}{r-1}\binom{n-k}{r} \tag{25}$$

is the number of "circular" k-choices from n with exactly r parts, in agreement with [1], p. 270.

Formula (23) may also be obtained from (25) and the relation

$$\langle\langle n\colon k \mid a, b, c, \cdots, p, q\rangle\rangle = \langle\langle n-k\colon a \mid b, c, \cdots, p, q\rangle\rangle\binom{k-1}{a-1}(n/(n-k)) \tag{26}$$

which is easily established.

In a permutation of 1, 2, $\cdots$, n the joint occurrence of k of the $2n$ events in Table II is possible if and only if

(i) no two come from the same row,

(ii) two which come from rows v and s with $\overline{v, s} \leqq w - 1$ or $\overline{s, v} \leqq w - 1$ must come from the same column, and

(iii) if the k events all come from the same column then their rows must be a

$$\langle n\colon k \mid a_1, \cdots, a_{w-1}\rangle\text{-choice (of the rows) with } a_{w-1} \geqq 1.$$

Conditions (i) and (ii) are obvious. Condition (iii) is necessary, for if $a_{w-1} = 0$ and the k events all come from the same column then the permutations containing them would contain all the events in that column—and this is impossible.

Now an argument similar to the one used to deduce (10) shows that the number of permutations containing any particular set of k consistent events, from Table II, whose rows form a $\langle n: k \mid a_1, a_2, \cdots, a_{w-1}\rangle$-choice, is

$$(n - k - a_1 - a_2 - \cdots - a_{w-2})!.$$

Using exclusion and inclusion we deduce

THEOREM 4. (i) *The number of permutations of* 1, 2, $\cdots$, n *containing exactly* $r \geqq 1$ *clockwise and/or counterclockwise w-sequences is*

$$M_r(n, w) = \sum_{i=0}^{n-w+1-r} (-1)^i \binom{r+i}{r} \sum_{a_1=1}^{r+i} \sum_{a_2=1}^{a_1} \sum_{a_3=1}^{a_2} \cdots \sum_{a_{w-1}=1}^{a_{w-2}} 2^{a_{w-1}} \cdot \langle\langle n: r + i \mid a_1, a_2, \cdots, a_{w-1}\rangle\rangle (n - r - i - a_1 - a_2 - \cdots - a_{w-2})!$$

(ii) *The number of permutations without clockwise or counterclockwise w-sequences is*

$$M_0(n, w) = n! + \sum_{i=1}^{n-w+1} (-1)^i \sum_{a_1=1}^{i} \sum_{a_2=1}^{a_1} \sum_{a_3=1}^{a_2} \cdots \sum_{a_{w-1}=1}^{a_{w-2}} 2^{a_{w-1}} \cdot \langle\langle n: i \mid a_1, a_2, \cdots, a_{w-1}\rangle\rangle (n - i - a_1 - a_2 - \cdots - a_{w-2})!$$

(iii) *The number of permutations containing exactly $r \geqq 1$ clockwise (counterclockwise) w-sequences is*

$$Q_r(n, w) = \sum_{i=0}^{n-w+1-r} (-1)^i \binom{r+i}{r} \sum_{a_1=1}^{r+i} \sum_{a_2=1}^{a_1} \sum_{a_3=1}^{a_2} \cdots \sum_{a_{w-1}=1}^{a_{w-2}} \langle\langle n: r + i \mid a_1, a_2, \cdots, a_{w-1}\rangle\rangle (n - r - i - a_1 - a_2 - \cdots - a_{w-2})!$$

(iv) *The number of permutations without clockwise (counterclockwise) w-sequences is*

$$Q_0(n, w) = n! + \sum_{i=1}^{n-w+1} (-1)^i \sum_{a_1=1}^{i} \sum_{a_2=1}^{a_1} \sum_{a_3=1}^{a_2} \cdots \sum_{a_{w-1}=1}^{a_{w-2}} \langle\langle n: i \mid a_1, a_2, \cdots, a_{w-1}\rangle\rangle (n - i - a_1 - a_2 - \cdots - a_{w-2})!$$

In the simple case when $w = 2$ we have

$$Q_0(n, 2) = n! + \sum_{i=1}^{n-1} (-1)^i \sum_{a=1}^{i} \langle\langle n: i \mid a\rangle\rangle (n - i)!$$
$$= n! + \sum_{i=1}^{n-1} (-1)^i \sum_{a=1}^{i} (n/(n - i)) \binom{i-1}{a-1} \binom{n-i}{a} (n - i)! \quad \text{by (25)}$$

TABLE III

n	$N_0(n, 2)$	$M_0(n, 2)$	$N_0(n, 2)/n!$	$M_0(n, 2)/n!$
2	0	0	0	0
3	0	0	0	0
4	2	0	.08333	0
5	14	10	.11667	.08333
6	90	60	.12500	.08333
7	646	462	.12817	.09167
8	5242	3920	.13001	.09722
9	47622	36954	.13123	.10183
10	479306	382740	.13208	.10547

$$= n! + \sum_{i=1}^{n-1} (-1)^i \binom{n-1}{i} (n/(n-i))(n-i)! \qquad \text{by (18)}$$
$$= n! \sum_{i=0}^{n-1} (-1)^i/i!$$

in agreement with Whitworth [8], p. 104.

If we mark the top row of an $n \times n$ chess-board, and then identify the top and bottom edges of the board, we obtain a cylindrical board. A positioning of n counters (on this board) with no two in a row or column is described by a permutation of degree n. Now $M_0(n, w)$ is the number of ways of placing n counters on this cylindrical board no two in a row or column, and no w lined up consecutively along a diagonal. Two positionings, described by permutations $(i_1, \cdots, i_n)$ and $(j_1, \cdots, j_n)$ may be considered equivalent if one can be obtained from the other by a rotation of the board i.e., if for some k, $i_r + k \equiv j_r \pmod n$, $r = 1, \cdots, n$. For example $M_0(n, 2)/n$ is the solution to the cylindrical n-kings problem: in how many ways can n kings be placed on a cylindrical board, no two in a row or column, no two attacking each other (and no two positionings equivalent by rotation).

As an illustration the 14 permutations

I	II	III
1 3 5 2 4	4 2 5 3 1	2 4 1 5 3
2 4 1 3 5	5 3 1 4 2	3 5 1 4 2
3 5 2 4 1	1 4 2 5 3	3 1 5 2 4
4 1 3 5 2	2 5 3 1 4	4 2 5 1 3
5 2 4 1 3	3 1 4 2 5	

are those counted in $N_0(5, 2) = 14$. The 10 permutations in columns I and II are those counted in $M_0(5, 2) = 10$. The permutations (13524) and (42531) represent the two equivalence classes which are counted in $M_0(5, 2)/5$.

A short table of the numbers $N_0(n, 2)$ and $M_0(n, 2)$ follows (Table III).

REFERENCES

[1] ABRAMSON, M. and MOSER, W. O. J. (1966). Combinations, successions and the n-kings problem. *Math. Mag.* **39** 269–273.

[2] KAPLANSKY, I. (1944). Symbolic solution of certain problems in permutations. *Bull. Amer. Math. Soc.* **50** 906–914.

[3] KAPLANSKY, I. (1945). The asymptotic distribution of runs of consecutive elements. *Ann. Math. Statist.* **16** 200–203.

[4] MENDELSOHN, N. S. (1956). The asymptotic series for a certain class of permutation problems. *Canad. J. Math.* **8** 234–244.

[5] RIORDAN, J. (1945). Permutations without 3-sequences. *Bull. Amer. Math. Soc.* **51** 745–748.

[6] RIORDAN, J. (1965). A recurrence for permutations without rising or falling successions. *Ann. Math. Statist.* **36** 708–710.

[7] RIORDAN, J. (1958). *Introduction to Combinatorial Analysis*. Wiley, New York.

[8] WHITWORTH, W. A. (1959). *Choice and Chance*. Hafner, New York.

[9] WOLFOWITZ, J. (1943). Additive partition functions and a class of statistical hypotheses. *Ann. Math. Statist.* **13** 247–279.