BOUNDED EXPECTED UTILITY

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1. Introduction. The Blackwell-Girshick utility axioms [1], pp. 104–110, apply a preference-indifference relation $\precsim$ ("is not preferred to") to the set $\mathcal{P}_d$ of all discrete probability distributions defined on a set of consequences X. More precisely, with reference to a σ-algebra on X that contains $\{x\}$ for each $x \varepsilon X$, $\mathcal{P}_d$ is the set of all countably additive measures on the σ-algebra such that $P(A) = 1$ for some countable set A in the σ-algebra. The first purpose of this paper is to show that the Blackwell-Girshick utility theorem, which can be viewed as an extension of the standard von Neumann-Morgenstern result [3], can be obtained even on weakening their (B-G) denumerable "sure-thing" axiom. The second purpose is to show that versions of the new axiom, which is related to Savage's *P7* [2], p. 77, can be used in deriving the expected-utility property for other sets of probability measures on X, including general σ-additive measures and finitely-additive measures. Bounded utilities result in all cases considered except for the case where all distributions are simple.

2. The von Neumann-Morgenstern theory. The von Neumann-Morgenstern expected-utility theory serves as the base of our discussion.

Let $\bar{\alpha} = (1 - \alpha)$ when $\alpha \varepsilon [0, 1]$. An *abstract convex set* is a set $\mathcal{P} = \{P, Q, R, \cdots\}$ and an operation $\alpha P + \bar{\alpha} Q$ associating an element of $\mathcal{P}$ with each fraction in [0, 1] and each ordered pair of elements of $\mathcal{P}$, such that if $P, Q, R \varepsilon \mathcal{P}$ and $\alpha, \beta \varepsilon [0, 1]$ then

1. $1P + 0Q = P$,
2. $\alpha P + \bar{\alpha} Q = \bar{\alpha} Q + \alpha P$,
3. $\alpha(\beta P + \bar{\beta} Q) + \bar{\alpha} Q = \alpha\beta P + (1 - \alpha\beta) Q$.

With $\precsim$ a binary relation on $\mathcal{P}$, let $P \prec Q \Leftrightarrow [P \precsim Q$ and not $Q \precsim P]$, and $P \sim Q \Leftrightarrow [P \precsim Q$ and $Q \precsim P]$. $\precsim$ on $\mathcal{P}$ is a *weak order* if it is transitive and strongly connected ($P, Q \varepsilon \mathcal{P} \Rightarrow P \precsim Q$ or $Q \precsim P$).

The following axioms and theorem (proofs in [3], Appendix, and [2], Chapter 5) form the core of the theory. In all cases $P, Q, R \varepsilon \mathcal{P}$.

Axiom 0. *$\mathcal{P}$ is an abstract convex set.*

Axiom 1. *$\precsim$ on $\mathcal{P}$ is a weak order.*

Axiom 2. $[P \sim (\prec) Q, \alpha \varepsilon (0, 1)] \Rightarrow \alpha P + \bar{\alpha} R \sim (\prec) \alpha Q + \bar{\alpha} R$.

Axiom 3. $[P \prec Q, Q \prec R] \Rightarrow \alpha P + \bar{\alpha} R \prec Q$ *and* $Q \prec \beta P + \bar{\beta} R$ *for some* $\alpha, \beta \varepsilon (0, 1)$.

Theorem 1. [*Axioms* 0, 1, 2, 3] $\Rightarrow$ *there is a real function u on $\mathcal{P}$ such that if $P, Q \varepsilon \mathcal{P}$ and $\alpha \varepsilon [0, 1]$ then*

$$u(P) \leqq u(Q) \Leftrightarrow P \precsim Q, \tag{1}$$

$$u(\alpha P + \bar{\alpha} Q) = \alpha u(P) + \bar{\alpha} u(Q). \tag{2}$$

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3. The Blackwell-Girshick theory. Henceforth we interpret $\mathcal{P}$ as a set of probability distributions or measures on a set X, take $\alpha P + \bar{\alpha} Q$ as the ordinary convex combination of measures $P, Q \varepsilon \mathcal{P}$, and assume that $\mathcal{P}$ is closed under convex combinations so that Axiom 0 holds. In addition, all one-point distributions are assumed to be in $\mathcal{P}$ and no notational distinction will be made between $x \varepsilon X$ and the one-point distribution that assigns probability 1 to $\{x\}$. Because the one-point distributions are elements of $\mathcal{P}$, convex closure requires that $\mathcal{P}_s$, the set of all simple distributions on X (those that assign probability 1 to some finite subset), be a subset of $\mathcal{P}$, but does not imply that the whole of $\mathcal{P}_d$ is included in $\mathcal{P}$. However, we shall for the present take $\mathcal{P} = \mathcal{P}_d$ (for any σ-algebra that contains each $\{x\}$) since this is the Blackwell-Girshick setting. Not only is $\mathcal{P}_d$ an abstract convex set, but also if $\alpha_i \geqq 0$ and $P_i \varepsilon \mathcal{P}_d$ for $i = 1, 2, \cdots$ and $\sum_{i=1}^{\infty} \alpha_i = 1$, then $\sum_{i=1}^{\infty} \alpha_i P_i \varepsilon \mathcal{P}_d$.

The Blackwell-Girshick theory uses Axioms 1 and 3 and the following denumerable extension of Axiom 2 to strengthen Theorem 1 to obtain Theorem 2 (essentially their Theorem 4.4.2).

AXIOM 2^*. *$[P_i, Q_i \varepsilon \mathcal{P}_d, P_i \preccurlyeq Q_i, \alpha_i \geqq 0$ for $i = 1, 2, \cdots$ and $\sum_{i=1}^{\infty} \alpha_i = 1] \Rightarrow \sum_{i=1}^{\infty} \alpha_i P_i \preccurlyeq \sum_{i=1}^{\infty} \alpha_i Q_i$. If in addition $P_i \prec Q_i$ for some i for which $\alpha_i > 0$, then $\sum \alpha_i P_i \prec \sum \alpha_i Q_i$.*

THEOREM 2. *$[\mathcal{P} = \mathcal{P}_d$, Axioms 1, 2^*, 3$] \Rightarrow$ there is a bounded real function u on $\mathcal{P}_d$ such that* (1) *holds and*

$$u\left(\sum_{i=1}^{\infty} \alpha_i P_i\right) = \sum_{i=1}^{\infty} \alpha_i u(P_i) \tag{3}$$

whenever $\sum_{i=1}^{\infty} \alpha_i = 1$ and $\alpha_i \geqq 0$, $P_i \varepsilon \mathcal{P}_d$ for all i.

Expression (2) is a special case of (3), resulting when $\alpha_1 = \alpha$, $\alpha_2 = \bar{\alpha}$, $P_1 = P$, $P_2 = Q$.

4. Bounded utility. With $\sum_{i=1}^{\infty} \alpha_i P_i \varepsilon \mathcal{P}_d$ and $\sum_{i=n+1}^{\infty} \alpha_i > 0$ for all n, Axioms 1, 2, and 3 alone imply by the easy extension of (2) that

$$u\left(\sum_{i=1}^{\infty} \alpha_i P_i\right) = \sum_{i=1}^{n} \alpha_i u(P_i) + \left(\sum_{i=n+1}^{\infty} \alpha_i\right) u\left(\sum_{i=n+1}^{\infty} \alpha_i \left(\sum_{i=n+1}^{\infty} \alpha_i\right)^{-1} P_i\right), \tag{4}$$

which appears as the middle equation on p. 110 of [1]. From (4) we see that if u on $\mathcal{P}_d$ is bounded, then (3) follows. In addition, if (3) holds, then u on $\mathcal{P}_d$ is easily seen to be bounded. Thus, it seems that what Axiom 2^* adds to Axiom 2 when $\mathcal{P} = \mathcal{P}_d$ is that u on $\mathcal{P}_d$, satisfying (1) and (2), be bounded.

To verify that Axiom 2^* is indeed stronger than Axiom 2 when $\mathcal{P} = \mathcal{P}_d$, we first embed $\mathcal{P}_d$ in a vector space. Let $X = \{1, 2, \cdots\}$ and let $\mathfrak{J}$ be the set of all real-valued, countably-additive set functions on the class of all subsets of X so that $\mathcal{P}_d \subset \mathfrak{J}$. $\mathfrak{J}$ is easily seen to be a vector space over the real number field. The set of one-point measures in $\mathcal{P}_d$ is a linearly independent subset of $\mathcal{P}_d$ and of $\mathfrak{J}$, and the subspace of $\mathfrak{J}$ generated by the one-point measures includes $\mathcal{P}_s$ but does not contain any P in $\mathcal{P}_d$ that is not simple. Let $\mathfrak{S}$ be the class of all subsets of $\mathcal{P}_d$ that (i) contain each one-point measure and (ii) are linearly independent. Using Zorn's lemma on $\mathfrak{S}$ with $\subseteq$ as the order relation we readily find that $\mathfrak{S}$ has a maximal element, say $\mathcal{S} \subseteq \mathcal{P}_d$, such that (i) and (ii) hold for $\mathcal{S}$, and each $P \in \mathcal{P}_d$ is a unique linear combination of elements in $\mathcal{S}$.

Let $u(x) = x$ for each $x \in X$, $u(P) = 0$ for each nonsimple $P \in \mathcal{S}$, and extend u linearly to the subspace of $\mathfrak{J}$ generated by $\mathcal{S}$. This subspace includes $\mathcal{P}_d$. If, for $P, Q \in \mathcal{P}_d$, $P = \sum \alpha_i S_i$ and $Q = \sum \beta_i S_i$ where $S_i \in \mathcal{S}$ for each i, then $u(\alpha P + \bar{\alpha} Q) = u(\sum \alpha\alpha_i S_i + \sum \bar{\alpha}\beta_i S_i) = \sum \alpha\alpha_i u(S_i) + \sum \bar{\alpha}\beta_i u(S_i) = \alpha u(P) + \bar{\alpha} u(Q)$ so that (2) holds. Defining $P \preccurlyeq Q$ if and only if $u(P) \leqq u(Q)$, for all $P, Q \in \mathcal{P}_d$, (1) also holds and, as is easily seen, Axioms 1, 2, and 3 hold. However, u is unbounded, (3) is false, and Axiom 2* cannot hold.

5. Axiom 4 and variants. We now introduce an axiom that, along with Axioms 1, 2, and 3 implies that u on $\mathcal{P} = \mathcal{P}_d$ is bounded and hence yields the conclusion of Theorem 2. Axiom 2 and the new axiom, Axiom 4, can therefore replace Axiom 2* in the Blackwell-Girshick system. When $\mathcal{P} = \mathcal{P}_d$, both Axioms 2 and 4 are implied by Axiom 2*, but together they do not imply Axiom 2*. However, Axiom 2* does follow from Axioms 1, 2, 3, and 4 when $\mathcal{P} = \mathcal{P}_d$, as will be implied by Theorem 4. (See Theorems 7 and 8 and their proofs for contrasting conclusions.)

Along with Axiom 4 we state a weaker variant ($4W$) and a stronger variant ($4S$).

AXIOM 4. $[A \subseteq X, P(A) = 1, x_* \in X, x_* \preccurlyeq x$ *for all* $x \in A] \Rightarrow x_* \preccurlyeq P$. $[A \subseteq X, P(A) = 1, x^* \in X, x \preccurlyeq x^*$ *for all* $x \in A] \Rightarrow P \preccurlyeq x^*$.

AXIOM 4W. $[A \subseteq X, P(A) = 1, x_* \in X, x_* \prec x$ *for all* $x \in A] \Rightarrow x_* \preccurlyeq P$. $[A \subseteq X, P(A) = 1, x^* \in X, x \prec x^*$ *for all* $x \in A] \Rightarrow P \preccurlyeq x^*$.

AXIOM 4S. $[A \subseteq X, P(A) = 1, P_* \in \mathcal{P}, P_* \preccurlyeq x$ *for all* $x \in A] \Rightarrow P_* \preccurlyeq P$. $[A \subseteq X, P(A) = 1, P^* \in \mathcal{P}, x \preccurlyeq P^*$ *for all* $x \in A] \Rightarrow P \preccurlyeq P^*$.

Axiom 4S is Savage's final postulate, $P7$ [2], p. 77, translated into the context of this paper. Axiom 4 is obtained from this by replacing P_* and P^* with simple one-point distributions x_* and x^*. Axiom 4W is obtained from Axiom 4 by replacing $x_* \preccurlyeq x$ and $x \preccurlyeq x^*$ with $x_* \prec x$ and $x \prec x^*$. Clearly, Axiom 4S $\Rightarrow$ Axiom 4 $\Rightarrow$ Axiom 4W.

The following version of Savage's $P5$ [2], p. 31, will also be used in several of the theorems to follow.

AXIOM 5. $x \prec y$ *for some* $x, y \in X$.

6. Theorems. For convenience of comparison we now present a series of theorems, the first of which was proved in Section 4. The next two sections give proofs of the other theorems.

THEOREM 3. $[\mathcal{P} = \mathcal{P}_d$, *Axioms* 1, 2, 3, 5] *do not imply the conclusion of Theorem* 2.

THEOREM 4. $[\mathcal{P} = \mathcal{P}_d$, *Axioms* 1, 2, 3, 4] *imply the conclusion of Theorem* 2.

THEOREM 5. $[\mathcal{P} = \mathcal{P}_d$, *Axioms* 1, 2, 3, 4W] *do not imply the conclusion of Theorem* 2.

THEOREM 6. $[\mathcal{P} = \mathcal{P}_d$, *Axioms* 1, 2, 3, 4W, 5] *imply the conclusion of Theorem* 2.

In the next two theorems $\mathcal{P}_\sigma$ is the set of all σ-additive probability measures on any σ-algebra $\mathcal{R}$ on X that contains each $\{x\}$ for $x \in X$ and each $\preccurlyeq$-interval of X.

THEOREM 7. $[\mathcal{P} = \mathcal{P}_\sigma$, *Axioms* 1, 2, 3, 4] *imply that there is a bounded real func-*

tion u on $\mathcal{P}_\sigma$ such that (1) *holds and*

$$u(P) = \int u(x)\, dP(x) \tag{5}$$

for every $P \varepsilon \mathcal{P}_\sigma$.

THEOREM 8. [$\mathcal{P} = \mathcal{P}_\sigma$, *Axioms* 1, 2, 2^*, 3, 4W, 5] *do not imply the conclusion of Theorem* 7.

Savage [2], in his finitely-additive theory, says (p. 78) that "... perhaps, if the theory were worked out in a countably additive spirit from the start, little or no counterpart of *P*7 would be necessary." In terms of this paper Theorems 7 and 8 and Theorems 3 through 6 show that a little counterpart of *P*7 (Axiom 4 or 4W) is essential in obtaining a general expected-utility result in countably-additive situations.

In the two final theorems $\mathcal{P}_f$ is the set of all finitely-additive probability measures on the class of all subsets of X. We use the largest Boolean-algebra here because it is essentially the Boolean-algebra used by Savage [2], but any Boolean-algebra that contains each $\{x\}$ and each $\preccurlyeq$-interval can be seen to serve as well. Recall that Axiom 4S is essentially Savage's *P*7.

THEOREM 9. [$\mathcal{P} = \mathcal{P}_f$, *Axioms* 1, 2, 3, 4S] *imply that there is a bounded real function u on $\mathcal{P}_f$ such that* (1) *holds and* (5) *holds for every $P \varepsilon \mathcal{P}_f$.*

THEOREM 10. [$\mathcal{P} = \mathcal{P}_f$, *Axioms* 1, 2, 2^*, 3, 4, 5] *do not imply the conclusion of Theorem* 9.

7. Discussion and examples. This section proves the "do not imply" theorems, Theorems 5, 8, and 10, with specific examples. The next section then proves the other theorems.

PROOF OF THEOREM 5. Let u on $\mathcal{P}_d$ satisfy (1) and (2) with $u(x) = 0$ for all $x \varepsilon X$. Then Axioms 1, 2, and 3 are easily seen to hold and Axiom 4W is trivially satisfied. Following the analysis in Section 4, we have $u(x) = 0$ for all $x \varepsilon X$ and can let u be arbitrary (say, identically 1) on the distributions in $\mathcal{S}$ that are not simple. This reverses the approach taken in Section 4 and results in Axioms 1, 2, 3 and 4W holding with u on $\mathcal{P}_d$ unbounded.

PROOF OF THEOREM 8. Let $X = [0, 1]$, let $\mathcal{B}$ be the Borel sets on X, and let $u(x) = -1$ if $x < \frac{1}{2}$, $u(x) = 1$ if $x \geqq \frac{1}{2}$, $u(P) = \sum u(x)P(x)$ for all $P \varepsilon \mathcal{P}_\sigma$. Defining $P \preccurlyeq Q \Leftrightarrow u(P) \leqq u(Q)$, Axioms 1, 2, 2^*, 3, 4W, and 5 are easily verified. But Axiom 4 is violated by the uniform measure Q on $[\frac{1}{2}, 1]$ for $1 \preccurlyeq x$ for all $x \varepsilon [\frac{1}{2}, 1]$ yet $Q \prec 1$ since $u(Q) = 0$.

Thus we note that, although Axiom 2^* implies Axiom 4 when $\mathcal{P} = \mathcal{P}_d$, Axiom 2^* does not necessarily imply Axiom 4 even in the presence of the other axioms when $\mathcal{P}$ properly includes $\mathcal{P}_d$, which is the case here.

PROOF OF THEOREM 10. With $\mathcal{P} = \mathcal{P}_f$ we take our cue from Savage (p. 78) and let $X = \{1, 2, 3, \cdots\}$, $u(x) = x/(1 + x)$,

$$u(P) = \int u(x)\, dP(x) + \lim_{\epsilon \to 0} P(u(x) \geqq 1 - \epsilon),$$

and define $P \preccurlyeq Q \Leftrightarrow u(P) \leqq u(Q)$. Since (1) and (2) hold, Axioms 1, 2, and 3 are easily seen to hold, and Axiom 5 holds. Axiom 4 also holds: if $P(A) = 1$ and $x_* \preccurlyeq x$ for all $x \varepsilon A$, then $u(x_*) \leqq u(P)$; if $x \preccurlyeq x^*$ for all $x \varepsilon A$, then $u(P) \leqq u(x^*)$ since for any x^*, $u(x^*) < 1 - \epsilon$ for ϵ small. Moreover, Theorem 4 leads to the conclusion that Axiom 2* holds. Let P be any probability measure on the set of all subsets of X such that $P(x) = 0$ for all x. For this measure, $u(P) = 1 + 1 = 2$, which violates (5). It follows from Theorem 9 (or by easy modification of the example) that Axiom 4S does not hold.

8. Proofs of Theorems 4, 6, 7, and 9. Because the proofs of these theorems have common parts it will be efficient and instructive to prove them as a unit. In all four theorems Axioms 1, 2, and 3 are taken to hold, and $\mathcal{P}$ satisfies Axiom 0. Moreover $\mathcal{P}_d \subseteq \mathcal{P}$ in all cases since $\mathcal{P}_d \subseteq \mathcal{P}_\sigma$ and $\mathcal{P}_d \subseteq \mathcal{P}_f$. The u referred to is that provided by Theorem 1 and therefore satisfies (1) and (2).

With $x_i \varepsilon X$ and $P_i \varepsilon \mathcal{P}$ for all i, Theorem 1 implies by the easy extension of (2) that

$$(6) \qquad u\left(\sum_{i=1}^{\infty} (\tfrac{1}{2})^i P_i\right) = \sum_{i=1}^{n} (\tfrac{1}{2})^i u(P_i) + (\tfrac{1}{2})^n u\left(\sum_{i=1}^{\infty} (\tfrac{1}{2})^i P_{n+i}\right)$$

and, in particular,

$$(7) \qquad u\left(\sum_{i=1}^{\infty} (\tfrac{1}{2})^i x_i\right) = \sum_{i=1}^{n} (\tfrac{1}{2})^i u(x_i) + (\tfrac{1}{2})^n u\left(\sum_{i=1}^{\infty} (\tfrac{1}{2})^i x_{n+i}\right).$$

For each of Theorems 4, 6, 7, and 9 the special case of (6) represented by (7) leads to u on X being bounded. We prove this using the weakest of the Axiom 4 series, Axiom 4W. Suppose u on X is unbounded above. Take $u(x_i) \geqq 2^i$ for all i. Then, by (7),

$$u\left(\sum (\tfrac{1}{2})^i x_i\right) \geqq n + (\tfrac{1}{2})^n u\left(\sum (\tfrac{1}{2})^i x_{n+i}\right),$$

and for large enough n there is an $x_* \prec x_i$ for all $i > n$ so that $u(\sum (\frac{1}{2})^i x_i) \geqq n + (\frac{1}{2})^n u(x_*)$ by (1) and Axiom 4W. This implies that $u(\sum (\frac{1}{2})^i x_i) = +\infty$, a contradiction. Unboundedness below is dealt with symmetrically.

Hereafter let $a = \inf u(x)$, $b = \sup u(x)$; whence $a \leqq b$. We next show that, for each case, u on $\mathcal{P}$ on bounded. For Theorems 4 and 6 with $\mathcal{P} = \mathcal{P}_d$ this is all we shall require since (3) then follows from (4). However, for Theorems 7 ($\mathcal{P} = \mathcal{P}_\sigma$) and 9 ($\mathcal{P} = \mathcal{P}_f$) we shall need to show also that $a \leqq u(P) \leqq b$ for all $P \varepsilon \mathcal{P}$, and that, in general, $\inf_A u(x) \leqq u(P) \leqq \sup_A u(x)$ when $P(A) = 1$.

Suppose first that $u(X) = \{a, b\}$. Then $a \leqq u(P) \leqq b$ for all $P \varepsilon \mathcal{P}$ readily follows from Axiom 4 and (1) for Theorems 4, 7, and 9. For Theorem 6, Axiom 5 requires $a < b$. With $a < b$ we now show for Theorem 6 that Axiom 4W implies that u on $\mathcal{P}_d$ is bounded. For this let $\mathcal{P}_a = \{P \mid P \varepsilon \mathcal{P}_d, P(u(x) = a) = 1\}$, $\mathcal{P}_b = \{P \mid P \varepsilon \mathcal{P}_d, P(u(x) = b) = 1\}$, so that every $P \varepsilon \mathcal{P}_d$ can be written as a convex combination of a $P_a \varepsilon \mathcal{P}_a$ and a $P_b \varepsilon \mathcal{P}_b$. Hence if u is bounded on $\mathcal{P}_a \cup \mathcal{P}_b$ it is bounded on all of $\mathcal{P}_d$ by (2). For $\mathcal{P}_b$, $a \leqq u(P)$ for all $P \varepsilon \mathcal{P}_b$ by $a < b$, Axiom 4W, and (1). Suppose u on $\mathcal{P}_b$ is unbounded. Then with $u(P_i) \geqq 2^i$ and $P_i \varepsilon \mathcal{P}_b$ for all i, (6) implies that

$$(8) \qquad u\left(\sum_{i=1}^{\infty} (\tfrac{1}{2})^i P_i\right) \geqq n + (\tfrac{1}{2})^n u\left(\sum_{i=1}^{\infty} (\tfrac{1}{2})^i P_{n+i}\right)$$

which requires $u(\sum (\frac{1}{2})^i P_{n+i})$ to approach $-\infty$ as $n \to \infty$, for otherwise $u(\sum (\frac{1}{2})^i P_i)$ must be infinite. But clearly $\sum (\frac{1}{2})^i P_{n+i} \varepsilon \mathcal{P}_b$ so that $a \leqq u(\sum (\frac{1}{2})^i P_{n+i})$ for all n, and thus a contradiction is obtained. Hence u on $\mathcal{P}_b$ is bounded. A symmetric proof shows that u on $\mathcal{P}_a$ is bounded.

Next, suppose that $u(X) \neq \{a, b\}$ so that there is a $z \varepsilon X$ such that $a < u(z) < b$. Fix z, let $Z* = \{x \mid x \varepsilon X, x \sim z\}$, $Z_* = \{x \mid x \varepsilon X, x \prec z\}$, $Z^* = \{x \mid x \varepsilon X, z \prec x\}$ so that neither Z_* nor Z^* is null and $Z_* \cup Z* \cup Z^* = X$, and let

$$\mathcal{P}_* = \{P \mid P \varepsilon \mathcal{P}, P(Z_*) = 1\},$$

$$\mathcal{P}* = \{P \mid P \varepsilon \mathcal{P}, P(Z*) = 1\},$$

$$\mathcal{P}^* = \{P \mid P \varepsilon \mathcal{P}, P(Z^*) = 1\}.$$

Then, for any of the cases considered, each $P \varepsilon \mathcal{P}$ can be written as a convex combination of a measure in $\mathcal{P}_*$, a measure in $\mathcal{P}*$, and a measure in $\mathcal{P}^*$. Consequently, if u is bounded on $\mathcal{P}_* \cup \mathcal{P}* \cup \mathcal{P}^*$ it is bounded on all of $\mathcal{P}$; and if $a \leqq u(P) \leqq b$ for all $P \varepsilon \mathcal{P}_* \cup \mathcal{P}* \cup \mathcal{P}^*$, then $a \leqq u(P) \leqq b$ for all $P \varepsilon \mathcal{P}$. For Theorems 4, 7, and 9, $u(P) = u(z)$ for all $P \varepsilon \mathcal{P}*$ follows from (1) and Axiom 4. For Theorem 6, u on $\mathcal{P}*$ is bounded since there must be $x, y \varepsilon X$ such that $x \prec z$ and $z \prec y$ when $a < u(z) < b$: use (1) and Axiom 4W. We next observe that $u(z) \leqq u(P)$ for all $P \varepsilon \mathcal{P}^*$ by (1) and Axiom 4W. An analysis using (8) then shows that u on $\mathcal{P}^*$ is bounded. A symmetric proof shows that u on $\mathcal{P}_*$ is bounded. Hence, for all cases, u on $\mathcal{P}$ is bounded. This completes the proofs of Theorems 4 and 6. The other proofs are completed as follows.

Proof of Theorem 7. Let M be such that $u(z) \leqq u(P) \leqq M$ for all $P \varepsilon \mathcal{P}^*$. If $u(x) = b$ for some $x \varepsilon Z^*$, then $u(P) \leqq b$, by Axiom 4 and (1). If $u(x) < b$ for all $x \varepsilon Z^*$, proceed as follows. Let $A(\epsilon) = \{x \mid u(z) < u(x) < b - \epsilon\}$, $B(\epsilon) = \{x \mid b - \epsilon \leqq u(x) < b\}$, $0 < \epsilon < b - u(z)$. Given $P \varepsilon \mathcal{P}^*$ let $P_{A(\epsilon)}$ be the conditional distribution of P on $A(\epsilon)$ when $P(A(\epsilon)) > 0$, let $P_{B(\epsilon)}$ be the conditional distribution of P on $B(\epsilon)$ when $P(B(\epsilon)) > 0$; if $P(A(\epsilon)) = 0$ set $u(P_{A(\epsilon)}) = a$, and if $P(B(\epsilon)) = 0$ set $u(P_{B(\epsilon)}) = a$. By this and (2), $u(P) = P(A(\epsilon))u(P_{A(\epsilon)}) + P(B(\epsilon))u(P_{B(\epsilon)})$. By definition, Axiom 4 and (1), $u(P_{A(\epsilon)}) \leqq b$ and since $u(P_{B(\epsilon)}) \leqq M$ for all $\epsilon \varepsilon (0, b - u(z))$, $u(P) \leqq P(A(\epsilon))b + [1 - P(A(\epsilon))]M$. Because the measures in $\mathcal{R}$ are σ-additive, $P(A(\epsilon)) \to 1$ as $\epsilon \to 0$, so that $u(P) \leqq b$. Hence $u(z) \leqq u(P) \leqq b$ for all $P \varepsilon \mathcal{P}^*$. A symmetric proof shows that $a \leqq u(P) \leqq u(z)$ for all $P \varepsilon \mathcal{P}_*$. Thus $a \leqq u(P) \leqq b$ for all $P \varepsilon \mathcal{P}_\sigma$.

Using a standard partition proof, (5) follows readily from the fact that $\int u(x)\, dP(x)$ is well defined for all $P \varepsilon \mathcal{P}_\sigma$ [u, being monotone in X is $\mathcal{R}$-measurable, and it is bounded] and the fact that Axiom 4 and $a \leqq u(P) \leqq b$ for all $P \varepsilon \mathcal{P}_\sigma$ imply, by the foregoing proof, that for any $A \varepsilon \mathcal{R}$ of the form $A = \{x \mid c \leqq (<) u(x) \leqq (<) d\}$, $c, d \varepsilon [a, b]$, if $P(A) > 0$, then $c \leqq u(P_A) \leqq d$ for the conditional measure P_A of P on A.

Proof of Theorem 9. With $u(z) \leqq u(P)$ for all $P \varepsilon \mathcal{P}^*$ suppose that $b < u(P)$

for some $P \varepsilon \mathcal{P}^*$. Then a suitable convex combination Q of P and some $x \varepsilon Z^*$ yields, by (2), $b < u(Q) < u(P)$. Therefore, $x \prec Q$ for all $x \varepsilon Z^*$, and $P \preccurlyeq Q$ by Axiom 4S, so a contradiction is reached. Hence $u(z) \leqq u(P) \leqq b$ for all $P \varepsilon \mathcal{P}^*$, and, symmetrically, $a \leqq u(P) \leqq u(z)$ for all $P \varepsilon \mathcal{P}_*$ so that $a \leqq u(P) \leqq b$ for all $P \varepsilon \mathcal{P}_f$. From this and Axiom 4S it then follows that $\inf_A u(x) \leqq u(P) \leqq \sup_A u(x)$ when $P(A) = 1$. A standard partition proof then gives (5).

9. Summary. In summary we include a few generalizations whose validity is easily established with reference to the modes of proof for the preceding theorems.

We consider a set $\mathcal{P}$ of probability measures, on a Boolean algebra on X, for which Axioms 0, 1, 2, and 3 hold. The fixed algebra is assumed to contain each $\{x\}$ for $x \varepsilon X$ and each $\preccurlyeq$-interval of X. Moreover, not only is $\mathcal{P}$ an abstract convex set (Axiom 0) but also it

(a) is closed under countably infinite linear combinations,
(b) is closed under the formation of conditional probabilities,
(c) contains every one-point probability measure.

$\mathcal{P}_d$, $\mathcal{P}_\sigma$, and $\mathcal{P}_f$ are particular instances of $\mathcal{P}$ that satisfy these conditions.

Let u on $\mathcal{P}$ satisfy (1) and (2) of Theorem 1. Then, under the preceding specifications, u on $\mathcal{P}$ satisfies (5) for all $P \varepsilon \mathcal{P}$ if and only if Axiom 4S holds, and, when (5) holds, u on $\mathcal{P}$ is bounded. In fact, (5) holds if and only if the weaker version of Axiom 4S obtained on replacing $P_* \preccurlyeq x$ and $x \preccurlyeq P^*$ by $P_* < x$ and $x < P^*$ holds. This weakening of Axiom 4S, not mentioned above, corresponds to the weakening Axiom 4 that results in Axiom 4W.

If $\mathcal{P} = \mathcal{P}_d$ and u on $\mathcal{P}$ is bounded, then (5) holds. However, if $\mathcal{P}_d \subseteq \mathcal{P}$ and u on $\mathcal{P}$ is bounded, (5) can be false. (5) can even be false (see the Proof of Theorem 8) when u is bounded and $\inf u(x) \leqq u(P) \leqq \sup u(x)$ for all $P \varepsilon \mathcal{P}$. However, (5) holds when $P(A) = 1$ implies that $\inf_A u(x) \leqq u(P) \leqq \sup_A u(x)$.

In certain cases Axiom 4S can be weakened and (5) will still hold. If $\mathcal{P} = \mathcal{P}_d$, Axiom 4 implies (5), and Axiom 4W implies (5) provided that the very mild Axiom 5 is also adopted. If $\mathcal{P} = \mathcal{P}_\sigma$, Axiom 4 implies (5), but Axiom 4W does not imply (5) even when it is supplemented by Axioms 2^* and 5. When $\mathcal{P} = \mathcal{P}_f$, no weakening of Axiom 4S is sufficient for (5) except for that noted above ($P_* < x$ and $x < P^*$ in place of $P_* \preccurlyeq x$ and $x \preccurlyeq P^*$).

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REFERENCES

[1] BLACKWELL, D. and GIRSHICK, M. A. (1954). *Theory of Games and Statistical Decisions.* Wiley, New York.
[2] SAVAGE, L. J. (1954). *The Foundations of Statistics.* Wiley, New York.
[3] VON NEUMANN, J. and MORGENSTERN, O. (1947). *Theory of Games and Economic Behavior.* Princeton Univ. Press.