EQUIVALENT GAUSSIAN MEASURES WITH A PARTICULARLY SIMPLE RADON-NIKODYM DERIVATIVE[1]

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1. Introduction. We consider two Gaussian probability measures P_ρ and P_r determined by covariance functions $\rho(s, t)$ and $r(s, t)$ respectively (the mean functions will be assumed to vanish). The well known Feldman-Hájek theorem asserts that P_ρ and P_r are either equivalent or perpendicular. If they are equivalent, the Radon-Nikodym derivative $(dP_\rho/dP_r)(x)$ exists and is the exponential of a quadratic form in x. This quadratic form may be diagonal, i.e., expressible as $\int_a^b f(t)x^2(t)\,dt$. If P_r is Wiener measure, L. A. Shepp [4], p. 352, has shown precisely when this happens. His method allows him to calculate $E\{\exp[-\frac{1}{2}\int_0^T f(t)x^2(t)\,dt]\}$ and this in turn permits him to prove an interesting zero-one law for the Wiener process. The purpose of this paper is to extend these results to an arbitrary Gaussian process.

We will use $r(s, t)$ consistently to denote a *continuous* covariance function defined on $[a, b] \times [a, b]$. For $f(t) \geqq 0$, we let $K(s, t) = [f(s)f(t)]^{\frac{1}{2}}r(s, t)$ which is then a positive (semi-definite) kernel and hence has nonnegative characteristic values [3], p. 237. Let λ_1 be the largest of these values. Finally, let $D(\lambda)$ and $K_\lambda(s, t)$ be the Fredholm determinant [3], p. 173, and resolvent kernel [3], pp. 151–158, corresponding to K.

Theorem 1. *Let $f(t)$ be nonnegative, bounded and measurable on $[a, b]$, and let $r(s, t)$, $D(\lambda)$ and λ_1 be as above. If $\lambda < 1/\lambda_1$, then*

$$E^r\{\exp[\tfrac{1}{2}\lambda\int_a^b f(t)x^2(t)\,dt]\} = [D(\lambda)]^{-\frac{1}{2}}.$$

Here $E^r\{\cdots\}$ denotes expectation on the Gaussian process with covariance function r.

Theorem 2. *Let $f(t)$ be positive and continuous on $[a, b]$ and let $r(s, t)$, $D(\lambda)$, $K_\lambda(s, t)$ and λ_1 be as above. If $\lambda < 1/\lambda_1$ and if we let $\rho_\lambda(s, t) = K_\lambda(s, t)/[f(s)f(t)]^{\frac{1}{2}}$, then $\rho_\lambda(s, t)$ is a covariance function, P_{ρ_λ} is equivalent to P_r and*

$$(dP_{\rho_\lambda}/dP_r)(x) = [D(\lambda)]^{\frac{1}{2}}\exp[\tfrac{1}{2}\lambda\int_a^b f(t)x^2(t)\,dt]. \tag{1.1}$$

Theorem 3 (zero-one law). *Let $f(t)$ be measurable on $[a, b]$ and let $r(s, t)$ be as above. The set of x's for which $f(t)x^2(t) \in L^1(a, b)$ is either of probability (P_r measure) one or zero, and these alternatives occur according as $f(t)r(t, t)$ is or is not in $L^1(a, b)$.*

If r is the covariance function of a stationary Gaussian process (i.e., $r(s, t) = p(|s - t|)$) so that $r(t, t)$ is a positive constant, we have the particularly simple

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COROLLARY. *In the stationary case, $f(t)x^2(t)\ \varepsilon\ L^1(a, b)$ with probability one or zero according as $f(t)$ is or is not in $L_1(a, b)$.*

2. Proof of Theorem 1.

CASE 1. f is positive and continuous. We make use of techniques and a representation of Gaussian processes due to Kac and Siegert [1]. On some probability space Ω, let $\alpha_1(\omega), \alpha_2(\omega), \cdots$ be a sequence of independent identically distributed normal random variables each with mean 0 and variance 1. Let $\{\lambda_k\}$ be the characteristic values (including multiplicities) and $\{\phi_k\}$ the corresponding set of normalized characteristic functions of the integral equation

$$\lambda\phi(t) = \int_a^b K(s, t)\phi(s)\, ds.$$

Noting that $f(t) > 0$, we define

$$x_\omega(t) = [f(t)]^{-\frac{1}{2}} \sum_{k=1}^{\infty} \lambda_k^{\frac{1}{2}} \alpha_k(\omega)\phi_k(t). \tag{2.1}$$

For each t, this series converges with probability one (i.e., for almost all $\omega\ \varepsilon\ \Omega$) and hence determines a Gaussian process $\{x_\omega(t), a \leqq t \leqq b\}$ which moreover has $r(s, t)$ as its covariance function. Furthermore with probability one, the series (2.1) determines a real-valued function $x_\omega(\cdot)$ which belongs to $L^2(a, b)$ and to which the series converges both for almost all t and in the mean. For these facts one may see [1] and [2] or for more details [6].

Using the representation (2.1), Parseval's equation, monotone convergence and the independence and normality of the α_k's, we get

$$\begin{aligned} E^r\{\exp [\tfrac{1}{2}\lambda \textstyle\int_a^b f(t)x^2(t)\, dt]\} &= E\{\exp [\tfrac{1}{2}\lambda \textstyle\sum_{k=1}^{\infty} \lambda_k \alpha_k^2(\omega)]\} \\ &= \lim_{n\to\infty} E\{\exp [\tfrac{1}{2}\lambda \textstyle\sum_{k=1}^{n} \lambda_k \alpha_k^2(\omega)]\} \\ &= \lim_{n\to\infty} \textstyle\prod_{k=1}^{n} E\{\exp [\lambda\lambda_k \alpha_k^2(\omega)/2]\} \\ &= \lim_{n\to\infty} \textstyle\prod_{k=1}^{n} (1 - \lambda\lambda_k)^{-\frac{1}{2}} = [D(\lambda)]^{-\frac{1}{2}}. \end{aligned}$$

CASE 2. f is nonnegative, bounded and upper semi-continuous. Then there is a uniformly bounded sequence $\{f_n\}$ of positive continuous functions decreasing monotonically to f (see [5], pp. 284–292 and pp. 314, 315 for a discussion of upper and lower semi-continuous functions and their relation to measurable functions). Moreover using the obvious notation,

$$\begin{aligned} E^r\{\exp [\tfrac{1}{2}\lambda \textstyle\int_a^b f(t)x^2(t)\, dt]\} &= \lim_{n\to\infty} E^r\{\exp [\tfrac{1}{2}\lambda \textstyle\int_a^b f_n(t)x^2(t)\, dt]\} \\ &= \lim_{n\to\infty} [D_n(\lambda)]^{-\frac{1}{2}} = [D(\lambda)]^{-\frac{1}{2}}. \end{aligned}$$

The first equality follows by monotone convergence. The second is a consequence of Case 1 and the easily demonstrated fact that $\lambda_{1n} \to \lambda_1$ so that $\lambda < 1/\lambda_1$ implies that $\lambda < 1/\lambda_{1n}$ for large n. The third equality is a little more tedious to substantiate. We note first that [3], p. 173,

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} (-\lambda)^m c_m/m!$$

where

$$c_m = \int_a^b \cdots \int_a^b \begin{vmatrix} r(t_1, t_1) & \cdots & r(t_1, t_m) \\ \vdots & & \vdots \\ r(t_m, t_1) & \cdots & r(t_m, t_m) \end{vmatrix} f(t_1) \cdots f(t_m)\, dt_1 \cdots dt_m .$$

Also $D_n(\lambda) = 1 + \sum_{m=1}^{\infty} (-\lambda)^m c_{mn}/m!$ where c_{mn} is the same as c_m except that f is replaced by f_n.

Now consider a fixed λ and let $\epsilon > 0$ be given. Using the fact that f is bounded, that the f_n's are uniformly bounded and Hadamard's lemma for determinants, it is easy to show that there exist M (depending on λ and ϵ but not n) such that

$$|\textstyle\sum_{m=M}^{\infty}(-\lambda)^m c_m/m!| < \epsilon/3 \quad \text{and} \quad |\textstyle\sum_{m=M}^{\infty}(-\lambda)^m c_{mn}/m!| < \epsilon/3.$$

On the other hand, $\lim_{n\to\infty} c_{mn} = c_m$ so that we may choose N such that for $n \geqq N$

$$|\textstyle\sum_{m=1}^{M-1} (-\lambda)^m (c_m - c_{mn})/m!| < \epsilon/3.$$

These inequalities clearly imply that for $n \geqq N$, $|D(\lambda) - D_n(\lambda)| < \epsilon$.

CASE 3. f is nonnegative, bounded and measurable. Then there is a uniformly bounded sequence $\{f_n\}$ of nonnegative upper semicontinuous functions increasing monotonically to f almost everywhere. The proof now proceeds as above.

3. Proof of Theorem 2. By Theorem 1, $[D(\lambda)]^{\frac{1}{2}} \exp [\frac{1}{2}\lambda \int_a^b f(t)x^2(t)\, dt]$ is integrable and has expectation one. We may therefore introduce a probability measure P_* satisfying (1.1) by defining

$$P_*(M) = [D(\lambda)]^{\frac{1}{2}} E^r\{\chi_M(x) \ \exp\ [\tfrac{1}{2}\lambda \textstyle\int_a^b f(t)x^2(t)\, dt]\},$$

χ_M being the indicator of the set M. We need only show that P_* is a Gaussian measure with covariance function $\rho_\lambda(s, t)$. To do this we will show that P_* has the right n-dimensional characteristic function. Let $\{t_j\}$ and $\{\xi_j\}$, $j = 1, 2, \cdots, m$, be sequences of real numbers with $t_j \in [a, b]$. Then using the representation of section 2, we have

$$\begin{aligned}
&[D(\lambda)]^{-\frac{1}{2}} E^*\{\exp\, [i \textstyle\sum_{j=1}^m \xi_j x(t_j)]\} \\
&= E^r\{\exp\, [i \textstyle\sum_{j=1}^m \xi_j x(t_j) + \frac{1}{2}\lambda \int_a^b f(t) x^2(t)\, dt]\} \\
&= E\{\exp\, [i \textstyle\sum_{j=1}^m \xi_j \sum_{k=1}^{\infty} \lambda_k^{\frac{1}{2}} \alpha_k(\omega) \phi_k(t_j) (f(t_j))^{-\frac{1}{2}} + \frac{1}{2}\lambda \sum_{k=1}^{\infty} \lambda_k \alpha_k^2(\omega)]\} \\
&= \lim_{n\to\infty} E\{\exp\, [i \textstyle\sum_{j=1}^m \xi_j \sum_{k=1}^{n} \lambda_k^{\frac{1}{2}} \alpha_k(\omega) \phi_k(t_j) (f(t_j))^{-\frac{1}{2}} + \frac{1}{2}\lambda \sum_{k=1}^{n} \lambda_k \alpha_k^2(\omega)]\} \\
&\qquad\qquad\qquad \text{(by dominated convergence)} \\
&= \lim_{n\to\infty} \textstyle\prod_{k=1}^n E\{\exp\, [i\alpha_k(\omega) A_k + \lambda\lambda_k \alpha_k^2(\omega)/2]\} \\
&\qquad\qquad\qquad \text{where } A_k = \lambda_k^{\frac{1}{2}} \textstyle\sum_{j=1}^m \xi_j \phi_k(t_j)(f(t_j))^{-\frac{1}{2}} \\
&= \lim_{n\to\infty} \textstyle\prod_{k=1}^n (1 - \lambda\lambda_k)^{-\frac{1}{2}} \exp\, [-A_k^2/2(1 - \lambda\lambda_k)] \\
&= [D(\lambda)]^{-\frac{1}{2}} \exp\, [-\tfrac{1}{2} \textstyle\sum_{i,j=1}^m [\xi_i \xi_j / [f(t_i) f(t_j)]^{\frac{1}{2}}] \\
&\quad \cdot \textstyle\sum_{k=1}^{\infty} [\lambda_k/(1 - \lambda\lambda_k)] \phi_k(t_i) \phi_k(t_j)].
\end{aligned}$$

But this implies

$$E^*\{\exp[i\textstyle\sum_{j=1}^m \xi_j x(t_j)]\} = \exp[-\frac{1}{2}\sum_{i,j=1}^m \xi_i\xi_j\rho_\lambda(t_i, t_j)]$$

as desired.

In these calculations we have used the Mercer expansions [3], p. 245, for the kernels K and K_λ and the product expansion of $D(\lambda)$.

4. Proof of Theorem 3.

CASE 1. $f(t)r(t, t) \not\in L^1(a, b)$. Let us suppose first that $f(t)$ is nonnegative. It is then almost everywhere the monotone limit of a sequence of nonnegative simple functions $\{f_n(t)\}$. From Theorem 1 and the series expansion of $D(\lambda)$ [3], p. 173,

$$\begin{aligned} E^r\{\exp[-\tfrac{1}{2}\textstyle\int_a^b f_n(t)x^2(t)\,dt]\} &= [D_n(-1)]^{-\frac{1}{2}} \\ &= [1 + \textstyle\int_a^b f_n(t)r(t, t)\,dt + \text{positive terms}]^{-\frac{1}{2}}. \end{aligned}$$

Since $f(t)r(t, t) \not\in L^1(a, b)$, the latter approaches zero as $n \to \infty$ and so

$$E^r\{\exp[-\tfrac{1}{2}\textstyle\int_a^b f(t)x^2(t)\,dt]\} = 0.$$

But this means that $\int_a^b f(t)x^2(t)\,dt = +\infty$ with probability one.

For a general f, we write $f(t) = f^+(t) - f^-(t)$ and note that either $f^+(t)r(t, t) \not\in L^1(a, b)$ or $f^-(t)r(t, t) \not\in L^1(a, b)$. The conclusion follows easily.

CASE 2. $f(t)r(t, t) \in L^1(a, b)$. Then both $f^+(t)r(t, t)$ and $f^-(t)r(t, t)$ are in $L^1(a, b)$ and by Fubini's theorem,

$$\begin{aligned} \textstyle\int_a^b f(t)r(t, t)\,dt &= \textstyle\int_a^b f^+(t)r(t, t)\,dt - \int_a^b f^-(t)r(t, t)\,dt \\ &= E^r\{\textstyle\int_a^b f^+(t)x^2(t)\,dt\} - E^r\{\int_a^b f^-(t)x^2(t)\,dt\} \\ &= E^r\{\textstyle\int_a^b f(t)x^2(t)\,dt\}. \end{aligned}$$

From this it follows that $f(t)x^2(t) \in L^1(a, b)$ with probability one.

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