ON HITTING PLACES FOR STABLE PROCESSES

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1. Introduction. Throughout this paper $X(t)$ will denote a drift free stable process on R^d (d-dimensional Euclidean space) having transition density $p(t, x)$ and paths which are normalized to be right continuous with left hand limits at every point. The familiar fact that all so normalized processes are strong Markov processes will be used without further explicit mention. For a compact subset $B \subset R^d$, let

$$T_B = \inf\{t > 0: X(t) \varepsilon B\} (= \infty \quad \text{if} \quad X(t) \not\varepsilon B \quad \text{for all} \quad t > 0).$$

be the first hitting time of B. Our purpose in this paper will be to investigate the asymptotic behavior, for large t, of the quantity

$$(1.1) \qquad F(t, x) = \int_B P_x(t < T_B < \infty, \quad X(T_B) \varepsilon dy) f(y),$$

where f is a continuous function on B. Previously this quantity was investigated for planar Brownian motion by Hunt [2], and in the special case of $f \equiv 1$ for general stable processes by the author in [4] and [5]. The results we obtain here will be extensions of those for the case $f \equiv 1$ to the case of an arbitrary f, and the proofs of these results will be dependent on the results for the case $f \equiv 1$. In essence, our technique will be to show that the general case can be reduced to the case $f \equiv 1$.

In order to state our results it will be necessary to recall some concepts and notation from [4] and [5]. Here we shall be brief referring the reader to the above cited papers for fuller details.

The measure $P_x(T_B > t, X(t) \varepsilon dy)$ has an upper semi-continuous density $g_B(t, x, y)$ which satisfies the well-known first passage relation:

$$(1.2) \quad p(t, y - x) - \int_B \int_0^t P_x(T_B \varepsilon ds, \quad X(s) \varepsilon dz) p(t - s, y - z) = g_B(t, x, y).$$

Let

$$H_B(x, dy) = P_x(T_B < \infty, X(T_B) \varepsilon dy)$$

and

$$g_B(x, y) = \int_0^\infty g_B(t, x, y)\, dt.$$

We must now discuss the case of recurrent and transient processes separately.

In the recurrent case we assume that $P_x(T_B < \infty) \equiv 1$. Then we know (see [5]) that $g_B(x, y) < \infty$ for $x \neq y$ and $y \not\varepsilon B$. [Actually more is true, but this is all we shall need]. Moreover, except for linear Brownian motion, the limits

$$\lim_{|y| \to \infty} g_B(x, y) = g_B(x, \infty),$$

$$\lim_{|x| \to \infty} \int_B H_B(x, dy) f(y) = \int_B H_B(\infty, dy) f(y)$$

Received 23 February 1967.

exist when f is continuous on B. If α is the exponent of the stable characteristic function of $X(1) - X(0)$, then in the recurrent case, either $2 \geqq \alpha > d = 1$ or $\alpha = d$, $d = 1$ or 2. [For $\alpha = d = 2$ we have planar Brownian motion, for $\alpha = d = 1$ the symmetric Cauchy process.] Set

$$(1.3) \qquad \begin{aligned} h(t) &= p(1, 0)\Gamma(1 - 1/\alpha)\Gamma(1/\alpha)t^{1-1/\alpha}, && \alpha > d = 1, \\ &= p(1, 0) \log t, && \alpha = d. \end{aligned}$$

Our first result is the following

THEOREM 1. *Assume $X(t)$ is recurrent and $P_x(T_B < \infty) \equiv 1$. Let f be continuous on B and let $F(t, x)$ be given by (1.1). Then except for linear Brownian motion*

$$\lim_{t\to\infty} h(t)F(t, x) = g_B(x, \infty) \int_B H_B(\infty, dy)f(y).$$

REMARK. For the standard linear Brownian motion it easily follows from the continuity of the paths that

$$\begin{aligned} \lim_{t\to\infty} (\pi t/2)^{\frac{1}{2}}F(t, x) &= |x - p_2| f(p_2) && x > p_2 \\ &= 0 && p_1 \leqq x \leqq p_2 \\ &= |x - p_1| f(p_1) && x < p_1 \end{aligned}$$

where p_1 = glb of B and p_2 = lub. of B.

We now turn our attention to the transient processes. Here we shall only deal with symmetric processes, i.e. those processes such that $X(1) - X(0)$ has log characteristic function $-|\theta|^\alpha$ with $\alpha < d$. It is well-known that the potential kernel density for these processes is just the Riesz kernel

$$(1.4) \qquad g(x) = \Gamma((d - \alpha)/2)/[2^\alpha \pi^{d/2}(\alpha/2)]^{-1} |x|^{\alpha-d}.$$

From (1.2) we easily derive that

$$(1.5) \qquad g(y - x) - \int_B H_B(x, dz)g(y - z) = g_B(x, y),$$

and that $g_B(x, y) < \infty$ whenever $x \neq y$. Also (Prop. 18.4) of [3]) there is a unique measure π_B having support contained in B (called the capacitary measure of B) such that

$$(1.6) \qquad P_x(T_B < \infty) = \int_B g(y - x)\pi_B(dy).$$

The finite total mass of π_B, $\pi_B(B) = C(B)$, is called the capacity of B.

Our first result is to show that the conditional hitting distribution of B from ∞ is just the normalized capacitary measure of B. The analogue of this result for recurrent processes can be found in [5].

THEOREM 2. *Assume $C(B) > 0$. Then for any continuous function f on B,*

$$\lim_{|x|\to\infty} \int_B P_x(X(T_B) \in dy \mid T_B < \infty)f(y) = \int_B (\pi_B(dy)/C(B))f(y).$$

The result for transient processes corresponding to Theorem 1 is as follows.

THEOREM 3. *Under the same assumptions as Theorem 2*

$$\lim_{t\to\infty} t^{(d/\alpha)-1}[(d/\alpha) - 1]p(1, 0)^{-1}F(t, x) = P_x(T_B = \infty) \int_B f(y)\pi_B(dy),$$

where

$$p(1, 0) = 2\Gamma(d/\alpha)[\alpha\Gamma(d/2)(4\pi)^{d/2}]^{-1}.$$

We conclude this section by showing how the above results for the case of Brownian motion yield interesting facts about the exterior Dirichlet problem for B. For planar Brownian motion this was done by Hunt [2], so we will consider only the case $d > 2$.

Hunt [2] showed that for Brownian motion on R^d, $d \geqq 2$,

$$K(t, x) = \int_B P_x(T_B \leqq t, X(T_B) \varepsilon dy)f(y)$$

was the unique bounded solution of the exterior Dirichlet problem for B for the heat equation,

$$\partial K/\partial t = \nabla^2 K,$$

with initial value 0 on $R^d - B$ and boundary function f on B. It is also a familiar fact that if $d > 2$, then the corresponding problem for Laplace's equation has a unique bounded solution φ satisfying the condition

$$\varphi(x) = O(g(x)), \qquad |x| \to \infty. \tag{1.7}$$

Now it is well-known that $H_B f(x)$ is a bounded solution of the exterior Dirichlet problem for B for Laplace's equation. Moreover (See (2.1))

$$\lim_{|x| \to \infty} H_B f(x)/g(x) = \int_B f(y)\pi_B(dy).$$

Thus $H_B f(x)$ is the unique solution satisfying (1.7). Clearly

$$K(t, x) \uparrow H_B f(x)$$

and

$$F(t, x) = H_B f(x) - K(t, x).$$

We see therefore that $F(t, x)$ is just the discrepancy between the time independent solution and this time dependent solution. It follows from Theorem 18.9 of [3] that $P_x(T_B < \infty) = \varphi_{\pi_B}(x)$ is just the classical capacitary potential of B. Consequently, Theorem 3 translates into the following purely analytic result on the Dirichlet problem.

COROLLARY 1. *Let B be a compact subset of R^d, $d > 2$, having positive Newtonian capacity. If $K(t, x)$ and $\varphi(x)$ are as above, then*

$$\lim_{t\to\infty} ((d/2) - 1)(4\pi)^{d/2} |\varphi(x) - K(t, x)| t^{(d/2)-1} = [1 - \varphi_{\pi_B}(x)] \int_B f(s)\pi_B(ds) \tag{1.8}$$

where $\pi_B(ds)$ *is the capacitary measure of B, and φ_{π_B} is the corresponding capacitary potential.*

Actually, arguing as in Hunt [2] Section 6.6 we can establish a slightly more general version of the above result.

THEOREM 4. *Let B, φ_{π_B}, and π_B be as in Corollary 1. Assume $L(x)$ is bounded* *and continuous on* $R^d - B$, *and such that* $L(x)/g(x) \to L^*$, $|x| \to \infty$. *Let* $K_L(t, x)$ *denote the bounded solution of the heat equation on* $R^d - B$ *with initial values* $L(x)$ *on* $R^d - B$ *and boundary function* f *on* B. *If* $\varphi(x)$ *is the bounded solution of Laplace's equation on* $R^d - B$ *with boundary function* f *on* B *satisfying* (1.7), then

$$\lim_{t\to\infty} ((d/2) - 1)(4\pi)^{d/2} |K_L(t, x) - \varphi(x)| t^{(d/2)-1} = [1 - \varphi_{\pi_B}(x)]|L^* - \int_B f(s)\pi_B(ds)|.$$

2. Proofs. We first establish Theorem 1. Let $M = \sup_{x\epsilon B} |f(x)|$, and set $a = \int_B H_B(\infty, dy) f(y)$. Then

$$h(t)F(t, x) = h(t) \int_{R^d} g_B(t, x, y) H_B f(y)\, dy$$
$$= h(t) \int_{R^d} g_B(t, x, y)[H_B f(y) - a]\, dy + aP_x(T_B > t)h(t).$$

Now by Theorems 2 and 4 of [5], we know that the 2nd term on the right converges to $ag_B(x, \infty)$, and thus to complete the proof we need only show that the first term converges to 0. Let $\epsilon > 0$ be given and choose r such that $|H_B f(y) - a| < \epsilon$ whenever $|y| > r$. Then using the scaling property, $p(t, x) = t^{-(d/\alpha)} p(1, xt^{-1/\alpha})$ we see that

$$|h(t) \int_{R^d} g_B(t, x, y)[H_B f(y) - a]\, dy|$$
$$\leqq h(t)[P_x(T_B > t)\epsilon + 2M \int_{|y| \leqq r} g_B(t, x, y)\, dy]$$
$$\leqq \epsilon h(t) P_x(T_B > t) + O(h(t) t^{-d/\alpha}).$$

The desired result now follows at once from the above estimate.

PROOF OF THEOREM 2. From (1.6) and the fact that $g(x + y) \sim g(x)$, $|x| \to \infty$, uniformly in y on compacts, it is clear that Theorem 2 is equivalent to showing

$$\lim_{|x|\to\infty} \int_B H_B(x, dy) f(y)/g(x) = \int_B f(y)\pi_B(dy). \tag{2.1}$$

To establish this we proceed as follows. Define measures

$$\gamma_x(dy) = g(x)^{-1} \mathrm{H}_B(x, dy).$$

Then from (1.6) we see that

$$\lim_{|x|\to\infty} \gamma_x(B) = \lim_{|x|\to\infty} \int_B (g(y - x)/g(x))\pi_B(dy) = \pi_B(B) = C(B) < \infty, \tag{2.2}$$

and thus there is a sequence $\{x_n\}$, $|x_n| \to \infty$ such that $\gamma_{x_n}(dy)$ converges weakly to some measure $\gamma(dy)$ on B having total mass $C(B)$. From (1.5) we see that

$$\lim_{|y|\to\infty} g_B(x, y)/g(y) = 1 - H_B(x, B). \tag{2.3}$$

But Hunt's duality results (see Section 17 of [3]) show that $g_B(x, y) = g_B(y, x)$ for our symmetric processes. It now easily follows from (1.5) and (2.3) that

$$\lim_{|x|\to\infty} \int_B \gamma_x(dz) g(y - z) = P_y(T_B < \infty).$$

Thus we see that

$$\varphi_\gamma(y) = \int_B \gamma(dz)g(y - z) \leqq P_y(T_B < \infty). \tag{2.4}$$

We will now show that $\gamma = \pi_B$. To this end let $M(B)$ be the set of all measures λ with support in B whose potential

$$\varphi_\lambda(x) = \int_B g(y - x)\lambda(dy) \leqq 1.$$

If $\lambda \in M(B)$, then $\lambda(A) = 0$ whenever A is polar. Indeed, if A is polar and $\lambda(A) > 0$, then there is a compact polar set $K \subset A$ such that $\lambda(K) > 0$. But then if ν is the restriction of λ to K, we see that

$$0 < \varphi_\nu(x) \leqq \varphi_\lambda(x) \leqq 1.$$

By Theorem 18.9 of [3]

$$P_x(T_K < \infty) = \sup_{\lambda \in M(K)} \varphi_\lambda(x)$$

and thus $P_x(T_K < \infty) > 0$, a contradiction. The symmetric processes considered here satisfy Hunt's hypothesis H ([3] p. 193) with the consequence that

$$\{x \in B: P_x(T_B < \infty) < 1\}$$

is polar. For measures λ_1, λ_2 having support in B, define

$$(\lambda_1, \lambda_2) = \int_B \int_B g(y - x)\lambda_1(dy)\lambda_2(dx).$$

Then the Schwartz inequality holds:

$$(\lambda_1, \lambda_2)^2 \leqq (\lambda_1, \lambda_1)(\lambda_2, \lambda_2).$$

Since $\gamma(B) = \pi_B(B) = C(B)$ and $\gamma, \pi_B \in M(B)$ we see that

$$C(B)^2 = (\gamma, \pi_B)^2 \leqq (\gamma, \gamma)C(B).$$

Thus

$$(\gamma, \gamma) \geqq C(B).$$

On the other hand (2.4) implies that

$$(\gamma, \gamma) \leqq (\pi_B, \gamma) = C(B).$$

Thus $(\gamma, \gamma) = (\pi_B, \pi_B)$ and thus $\gamma = \pi_B$ (see, e.g. Lemma 4.1 of [1]).

If there were another subsequence $\{x_n\}$, $|x_n| \to \infty$ such that $\gamma_{x_n}(dy)$ converged weakly to a measure $\gamma(dy)$, then (2.2) would again show $\gamma(B) = C(B)$, and the same argument as used above would show $\gamma = \pi_B$. Thus (2.1) holds. This completes the proof.

Proof of Theorem 3. Set $\varphi(z) = P_z(T_B < \infty)$ and set $A = \int_B f(x)\pi_B(dx)$. Then we may write

$$F(t, x) = \int_{R^d} g_B(t, x, z)\varphi(z)[H_B f(z)/\varphi(z) - A/C(B)]\, dz + (A/C(B))P_x(t < T_B < \infty).$$

Now by Theorem 1 of [4] we know that

$$\lim_{t\to\infty} t^{(d/\alpha)-1}P_x(t < T_B < \infty) = [(d/\alpha) - 1)]^{-1}p(1,0)C(B)P_x(T_B = \infty).$$

To complete the proof we must now show that

$$(2.5) \quad \lim_{t\to\infty} t^{(d/\alpha)-1} \int_{R^d} g_B(t, x, z)\varphi(z)[H_Bf(z)/\varphi(z) - A/C(B)]\, dz = 0.$$

Choose $\epsilon > 0$. Then by Theorem 2 above, there is a δ such that whenever $|z| > \delta$,

$$|H_Bf(z)/\varphi(z) - A/C(B)| < \epsilon.$$

Also

$$\int_{|x|\leqq\delta} g_B(t, x, z)\, dz \leqq \int_{|z|\leqq\delta} p(t, z - x)\, dz = O(t^{-d/\alpha}).$$

Equation (2.5) now easily follows from these two facts. This completes the proof.

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