ABSTRACTS OF PAPERS

(Abstracts of papers to be presented at the Annual meeting, Washington, D. C., December 27–30, 1967. Additional abstracts appeared in the June, August and October issues and will appear in the February issue.)

9. Optimally robust linear estimators of location. ALLAN BIRNBAUM and EUGENE M. LASKA, Courant Institute of Mathematical Sciences, New York University, and Rockland State Hospital.

The approach to optimal efficiency-robust estimation outlined in Birnbaum (these abstracts, **32** (1961): 622), and developed in "A General Theory of Robustness", E. Laska ((1962), NYU Ph.D. thesis), is applied to determine admissible and maximin-efficient linear unbiased estimators (ALUEs and MLUEs) of location, and their efficiencies, for ordered samples of sizes 5, 10, 15, and 20, from the normal, logistic, double exponential, Cauchy, long-tailed (Tukey), and rectangular distributions. It is shown that the results and interpretations may be summarized relatively compactly because of the striking tendency of these distributions to admit a simple ordering such that the MLUE over any subset of the distributions is just the MLUE over the extreme pair of distributions in the ordered subset. MLUEs based (a) on uniformly spaced sample quantiles, and (b) on optimally spaced sample quantiles, are determined and compared with those based on complete samples. Relations to other methods and results are discussed. Extended tables of moments of order statistics of Tukey's long-tailed distribution, computer for this investigation, are given. (Received 28 August 1967.)

10. Asymptotic distribution of the sample size for a sequential probability ratio test. K. C. CHANDA, University of Florida.

Let $X_1, X_2, \cdots$ be a sequence of mutually independent and identically distributed random variables with a common pdf $f_\theta(x)$ (wrt a measure $\mu(x)$). Consider the standard sequential probability ratio test (SPRT) for $H: \theta = \theta_0$ against the alternative $K: \theta = \theta_1$. Let $\Delta = \theta_1 - \theta_0 > 0$, and let n denote the number of observations required to complete the SPRT. Then we accept H if $\sum_{j=1}^n Z_j \leqq \log B$ and reject H if $\sum_{j=1}^n Z_j \geqq \log A$, where $Z_j = \log f_{\theta_1}(x_j) - \log f_{\theta_0}(X_j)$. Write $\log A = a$, $\log B = b$, $\mu = E_\theta(Z_1)$, $\sigma^2 = V_\theta(Z_1)$ and assume that a and b are finite preassigned quantities with $b < 0 < a$. Further let $\mu/\Delta \neq 0$. Then it is proved, under some mild regularity conditions, that as $\Delta \to 0$, the following results hold: (i) If $\mu/\Delta > 0$ then the distribution of $n^* = (n - a/\mu)/c_1$ where $c_1 = \sigma(a/\mu^3)^{\frac{1}{2}}$, tends to $N(0, 1)$. (ii) If $\mu/\Delta < 0$, the distribution of $n^{**} = (n - b/\mu)/c_2$ where $c_2 = (b/\mu^3)^{\frac{1}{2}}$, tends to $N(0, 1)$. (Received 7 August 1967.)

11. The two-armed bandit problem with finite memory. THOMAS M. COVER, Stanford University.

Robbins has proposed a finite memory constraint on the two-armed bandit problem in which the coin to be tossed at each stage may depend on the history of the previous tosses only through the outcomes of the last r tosses. Letting the choice of coin depend on the time at which the toss is made, we exhibit a deterministic rule with memory $r = 2$, the description of which is independent of the coin biases p_1 and p_2, which achieves, with probability one, a limiting proportion of heads equal to $\max\{p_1, p_2\}$. Thus, this rule is asymptotically uniformly best among the class of time-varying finite memory rules. (Received 28 August 1967.)

12. Bayesian zero-mean uniform discrimination. SEYMOUR GEISSER and D. M. MAHAMUNULU, State University of New York at Buffalo.

A Bayesian approach to the problem of assigning an individual to one of two particular multivariate normal populations π_i ($i = 1, 2$), is developed. It is assumed that the mean vector of each of the populations is the null vector and that the structure of the covariance matrices is uniform (equal variances and covariances). We obtain here an analysis of this problem for varying assumptions involving the uniform covariance structure. In each case the predictive density of a future observation is obtained, given the available data, and is used to assign the observation to the most appropriate population. Whenever a predictive linear discriminant was obtainable we also present the associated errors of misclassification. (Received 6 September 1967.)

13. Inference on discrimination coefficients. SOMESH DAS GUPTA, University of Minnesota.

The exact means and covariances of the coefficients of the sample linear discrimination function are computed under the assumption that the parent distribution is normal. Moreover, the large-sample distribution of these coefficients are obtained under the normality assumption and also for the general case under some mild assumptions. Analogous results are obtained for the coefficients in the two-sample case. Some optimum tests are derived for some hypothesis-testing problems concerning discrimination coefficients and a few properties of some standard tests are discussed. In particular, it is shown that the power function of the step-down test for testing that all the discrimination coefficients are zero has a "weak" monotonicity property and does not have the usual monotonicity property. However, given two sets of parameters such that the components in one set is greater than the corresponding components of the other set there exists a step-down test whose power will be greater for the first set of parameters than for the second set. (Received 5 October 1967.)

14. On pooling means when variance is unknown. CHIEN-PAI HAN and T. A. BANCROFT, Iowa State University.

In estimating the mean μ_1 of a normal population with variance unknown, a sample of size n_1 is taken. However a second sample of size n_2 is suspected to have come from the same population. It is assumed that the population variance of the second sample is the same as the first population, but the mean μ_2 may differ from μ_1. Whether we pool the second sample for estimating μ_1 depends on the outcome of a preliminary test (a t-test) on the hypothesis $\mu_2 = \mu_1$. If the hypothesis is accepted, the estimator is $(n_1\bar{x}_1 + n_2\bar{x}_2)/(n_1 + n_2)$, otherwise, $\bar{x}_1$ would be used. The bias and the mean square error of the sometimes-pool estimator are obtained which are of the form of finite sums. [Kitagawa (1963), University of California publications in statistics **3** 143–186, gave the bias and mean square error in the form of infinite sums.] Relative efficiency of the sometimes-pool estimator to the never-pool estimator is given. By observing the tables, we noted the relative efficiency is bigger than one for certain regions of $(\mu_2 - \mu_1)/\sigma$. In case *a priori* information is available for the distance of $\mu_2 - \mu_1$, in particular, if $(\mu_2 - \mu_1)/\sigma$ is distributed as $N(0, a^2)$, the estimator obtained by the maximum likelihood pooling has smaller mean square error than the sometimes-pool estimator. (Received 9 October 1967.)

15. Certain uncorrelated nonparametric test statistics. MYLES HOLLANDER, Florida State University.

Let $(X, Y) = (X_1, \cdots, X_m, Y_1, \cdots, Y_n)$ be an $m + n$ dimensional random vector having the cumulative distribution function $H(x, y) = P(X \leqq x, Y \leqq y) = P(X_1 \leqq x_1,$

$\cdots, X_m \leqq x_m, Y_1 \leqq y_1, \cdots, Y_n \leqq y_n)$ for $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_n)$. For a scalar b, let $b_{(k)}$ denote the k-vector $(b, b, \cdots, b)$. The statistic $U(X, Y)$ is said to be an odd translation invariant two sample statistic if for all m-vectors x, n-vectors y, and scalars b, $U(x, y) = U(x + b_{(m)}, y + b_{(n)}) = -U(-x, -y)$. The statistic $W(X, Y)$ is said to be an even translation invariant two sample statistic if for all m-vectors x, n-vectors y, and scalars b, $W(x, y) = W(x + b_{(m)}, y + b_{(n)}) = W(-x, -y)$. We say the distribution function of (X, Y) is equally symmetric if for some scalar μ, the distribution function of $(X - \mu_{(m)}, Y - \mu_{(n)})$ is the same as that of $(\mu_{(m)} - X, \mu_{(n)} - Y)$. It is shown that odd translation invariant two sample statistics are uncorrelated with even translation invariant two sample statistics when the distribution function of the combined sample is equally symmetric. This result is used to establish conditions under which certain dispersion test statistics (including ones proposed by Lehmann, Mood, Ansari-Bradley-Freund, Klotz, and Moses) are uncorrelated with, and asymptotically independent of, certain location test statistics (including the Mann-Whitney-Wilcoxon and normal scores). (Received 5 September 1967.)

16. Likelihood intervals, 1: introduction, and the binomial case.
D. J. HUDSON, Bell Telephone Laboratories, Inc.

"Maximum likelihood" is a standard method of obtaining a point estimate $\hat{\theta}$ of a parameter θ from the likelihood function $l(\theta \mid \mathrm{x}) = c \cdot \Pr(\mathrm{x}; \theta)$. It is not so well known that the likelihood function can, in many instances, be successfully used to provide an interval estimate of θ. Let $I = \{\theta: \theta \varepsilon \Omega \;\&\; l(\theta) \geqq e^{-2} \cdot l(\hat{\theta})\}$, where Ω is the parameter space. This "likelihood interval" is examined primarily for its usefulness as a confidence interval. Exact values of I and the relevant confidence coefficients $\alpha(\theta)$ will be given for the cases of positive and negative binomial sampling, where $\alpha(\theta) = \Pr_\theta\{\theta \varepsilon I\}$. Note also that (i) asymptotically, $\alpha(\theta) \equiv .954$ almost everywhere in Ω for all regular estimation problems; (ii) the interval I [and, hence, the coefficient $\alpha(\theta)$] is invariant under monotone transformations of the parameter; (iii) the method can be widely extended to cover other cases such as censored data, multidimensional θ, and estimation in the presence of nuisance parameters. (Received 24 August 1967.)

17. Nonparametric tests for detection of shift at an unknown time point.
R. A. JOHNSON, University of Wisconsin.

Let the independent random variables $X_1, \cdots, X_N$ with the continuous cdf's $F_1, \cdots, F_N$ correspond to observations on a process at N consecutive time points. We consider the problem of testing $H_0 : F_1 = F_2 = \cdots = F_N$ against the alternative $H_1 : F_1(x) = \cdots = F_m(x) \geqq F_{m+1}(x) = \cdots = F_N(x)$; m unknown, $F_m \neq F_{m+1}$. That is, under H_1, an upward shift has occured at some unknown intermediate time. Two versions of the testing problem are studied. In the first, the initial process level is assumed to be known and is characterized by the known point of symmetry of the distribution prior to the shift. The second version deals with the case of unknown initial process level and here the distributions prior to the shift are not restricted to be symmetric. For both the problems, optimal invariant tests are derived for certain local translation alternatives. The optimality criterion employed is the maximization of local average power where the average is over the space of the nuisance parameter m with respect to an arbitrary weighting $\{q_i, i = 1, 2, \cdots, N: q_i \geqq 0, \sum_{i=1}^n q_i = 1\}$. The tests are shown to be unbiased for general classes of shift alternatives and for all possible points of shift. The test statistics are distribution-free under H_0 and are asymptotically normal. Their limiting distributions under local translation alternatives are also obtained and the Pitman asymptotic efficiencies are investigated. The effect of the choice of a weighing function $\{q_i\}$ on the Pitman efficiency is studied. For a specific test, some small sample power computations are performed for translation alternatives in a normal distribution. (Received 5 October 1967.)

18. Distribution free sufficiency in sampling form finite populations. B. K. KALE, University of Manitoba.

Godambe [A new approach to sampling from finite populations, II: Distribution free sufficiency, *J. Roy. Statist. Soc. Ser. B* **28** 320–328] introduced the concept of distribution free sufficiency for the estimation of population total $T = \sum_{i=1}^{n} x_i$, and showed that an estimator $e(s, \mathbf{x}) = k_1(s)\sum_{i\varepsilon s} x_i + k_2(s)$ is DF-sufficient for T. In this note it is proved that any linear estimator $e(s, \mathbf{x}) = \sum_{i\varepsilon s} b(s, i)x_i$ such that $b(s, i) \neq 0$ for any $i \varepsilon s$, is DF-sufficient for T. This result thus indicates that the concept of DF-sufficiency for T, may not be very helpful to formulate a reduction principal. (Received 3 October 1967.)

19. Contributions to robust estimation. VALERIE MIKÉ, New York University and Bell Telephone Laboratories, Inc.

For the problem of efficiency-robust estimation of the location parameter θ of a family of symmetric pdf's $f(x - \theta \mid \lambda), \lambda \varepsilon \Lambda = \{1, \cdots, m\}, \theta \varepsilon \Theta = \{\theta \mid -\infty < \theta < \infty\}$, the method of "mixture models" of Birnbaum (*Ann. Math. Statist.* **32** (1961) 622) is applied to determine generalized Pitman estimators, which are shown to be admissible, with squared error loss function, under broad regularity conditions. With increasing sample size, these estimators are proved to be fully efficient (i.e., asymptotically equivalent, for each value of λ, to the maximum likelihood estimator which would be appropriate if the true value of λ were known). Computationally tractable analogous estimators based on k sample quantiles are defined in the context of the model representing their asymptotic normal distributions. It is shown that with increasing k these approach equivalence to the fully efficient estimators based on complete samples. Equivalent estimators are given also for the case of unknown scale parameter. Efficiency-robust linear unbiased estimators based on sample quantiles are derived and the optimal spacing of quantiles is discussed. (Received 10 August 1967.)

20. Connectivity in random graphs and digraphs. JOSEPH I. NAUS and LARRY RABINOWITZ, The City College of New York and Rutgers—The State University.

A random graph is constructed from a null graph with N labelled vertices by drawing an edge between each of the $\binom{N}{2}$ pairs of vertices, independently and with probability p $(0 < p \leqq 1)$. Similarly, by drawing a directed edge between each of the $2\binom{N}{2}$ pairs of ordered vertices with probability p, a random digraph is formed. Heap (*Numerische Math.* (1966) 114–122) has shown that for N large, Pr (random digraph is strongly connected) $\sim 1 - 2N(1 - p)^{N-1}$. In this paper, we obtain an expression for Pr (diameter of random digraph $\leqq K$) by using the properties of an adjacency matrix. For $K = N - 1$, this reduces to Pr (random digraph is strongly connected). These results are extended to allow for parallel lines. We also prove that P_2 = Pr (diameter of random graph $\leqq 2) \sim 1 - \binom{N}{2} \cdot (1 - p)(1 - p^2)^{N-2}$, for N large. Thus $P_2 \to 1$ as $N \to \infty$, which extends the work of Gilbert (*Ann. Math. Stat.* (1959) 1141–1144) who shows that Pr (random graph is connected) $\sim 1 - N(1 - p)^{N-1} \to 1$. (Received 3 August 1967.)

21. Minimization of the eigenvalues of a matrix and optimality of principal components. M. OKAMOTO and M. KANZAWA, Iowa State University and Osaka University.

Let x be a random p-vector with mean zero and variance matrix $E(xx') = \Sigma$. Let $\lambda_1 \geqq \cdots \geqq \lambda_p$ be the eigenvalues of Σ and $v_1, \cdots, v_p$ be the corresponding orthonormal eigenvectors. The linear combination $\xi_i = v_i'x$ is called by Hotelling the ith principal component of x. Principal components have several optimal properties due to Hotelling, Rao, Darroch, etc., but in this paper we give a fairly general one which includes some ofthem as special cases. Our result is as follows: *Let A be any $p \times k$ ($k \leqq p$) matrix and y be any random k-vector, then the eigenvalues of the matrix $E(x - Ay)(x - Ay)'$ are minimized simultaneously when and only when $Ay = v_1\xi_1 + \cdots + v_k\xi_k$*. This theorem is based on the following algebraic lemma: *For any real symmetric matrix A let $\lambda_i(A)$ denote the ith largest eigenvalue of A. If B and $A - B$ are non-negative definite and if B is at most of rank k, then $\lambda_i(A - B) \geqq \lambda_{k+1}(A)$ for each i.* A necessary and sufficient condition that all equality sign hold simultaneously is that $B = \lambda_1(A)v_1v_1' + \cdots + \lambda_k(A)v_kv_k'$, where $v_1, \cdots, v_k$ are orthonormal eigenvectors of A corresponding to $\lambda_1(A), \cdots, \lambda_k(A)$. (Received 23 August 1967.)

22. On a characterization of symmetric stable processes with finite mean. B. L. S. PRAKASA RAO, University of California, Berkeley.

Laha [*Ann. Math. Statist.* **27** (1956) 187–195] gave a characterization of symmetric stable laws through regression properties. We extend his result to symmetric stable processes with finite mean. We prove the following theorem. THEOREM. *Let $\{X(t), t \in [0, 1]\}$ be a continuous homogeneous stochastic process with independent increments with $X(0) = 0$. Further suppose that the increments have symmetric distributions and $E[X(t)] = 0$ for all t. Define $Y = \int_0^1 a(t)\, dX(t)$ and $Z = \int_0^1 t\, dX(t)$. These stochastic integrals are well defined in the sense of convergence in probability for any continuous function $a(t)$ on [0, 1]. Then the process $\{X(t), t \in [0, 1]\}$ is a symmetric stable process with finite mean if and only if for some $\lambda > 1$ $E(Y \mid Z) = \beta Z$* a.e *where $\beta = (\lambda + 1)\int_0^1 a(t)t^{\lambda-1}\, dt$, for all infinitely differentiable functions $a(t)$ on* [0, 1]. (Received 13 August 1967.)

23. Some new results in storage theory. N. U. PRABHU, Cornell University.

In the storage model proposed by D. G. Kendall ("Some problems in the theory of dams," *J. Roy. Statist. Soc. Ser. B* **19** (1957), 207–212), the input is assumed to be a stochastic process with stationary independent increments, and the release is continuous and is at a unit rate. The storage process arising from this model is known to reach a steady state if the expected net input per unit time is negative. In this paper we consider the situation where this expected net input is non-negative and derive the limiting distributions for the wet period, total dry period during a time interval, and the dam content. (Received 19 September 1967.)

24. Extreme nth moments for distributions on [0, 1] and the inverse of a moment space map to the n-dimensional unit cube. MORRIS SKIBINSKY, Brookhaven National Laboratory.

Let n be a positive integer. $M_n = \{(c_1, c_2, \cdots, c_n): c_i = \int_I x^i\, dQ(x),\ i = 1, 2, \cdots, n,\ Q \in \mathfrak{Q}\}$, where $\mathfrak{Q}$ denotes the class of all probability measures on the unit interval $I =$ [0, 1]. For $j = 1, 2$, let $p_j = (\nu_j - \nu_j^-)/(\nu_j^+ - \nu_j^-)$, whenever the denominator on the right hand side is positive and take $q_j = 1 - p_j$, where ν_j is the jth coordinate function on M_n, and for each point $(c_1, \cdots, c_{j-1})$ in M_{j-1},

$$\nu_j^+(c_1, \cdots, c_{j-1}) = \max\{c: (c_1, \cdots, c_{j-1}, c) \in M_j\},$$

$$\nu_j^-(c_1, \cdots, c_{j-1}) = \min\{c: (c_1, \cdots, c_{j-1}, c) \in M_j\}.$$

THEOREM. *At each point $(c_1, \cdots, c_n)$ of M_n such that $(c_1, \cdots, c_{n-1})$ is interior to M_{n-1}, $\nu_n = \sum_{j=0}^{[n/2]} S_{j,n-j}^2 \prod_{k=1}^{n-2j} \xi_k$ where $\xi_k = q_{k-1}p_k$, $k = 1, 2, \cdots$, $q_0 = 1$; $S_{jk} = \sum_{i=j}^{k} \xi_{i-j+1}S_{j-1,i}$,*

$j = 0, 1, \cdots, k, k = 1, 2, \cdots, S_{0k} \equiv 1$; *and* $[n/2]$ *is the integral part of* $n/2$. ν_n^- is the above sum with its initial term (index $j = 0$) deleted. $\nu_n^+ = \nu_n^- + \prod_{k=1}^{n-1} p_k q_k$. This simplifies to the *Theorem*. At each point $(c_1, \cdots, c_n)$ of M_n such that $(c_1, \cdots, c_{n-1})$ is interior to M_{n-1}, $\nu_n = S_{nn}$. (Received 4 October 1967.)

25. Partially balanced arrays with $\mu_2 = 2$ and 3.

J. N. SRIVASTAVA and D. V. CHOPRA, Colorado State University and Wichita State University.

A matrix T $(m \times N)$, with elements 0 and 1, is called a partially balanced array (PBA), (see Chakravarti, *Ann. Math. Statist.* **32** 1181) with parameters $(m, N, 2, t)$ and index set $(\mu_0, \mu_1, \cdots, \mu_t)$, if in every $(t \times N)$ submatrix, every vector of weight i appears μ_i times. The following main results on the existence of such arrays are established here. THEOREM. *Suppose there exists an array* T *with* $t = 4$, *and* $m \geqq 7$. *Then the following relations hold.* (1) $\mu_0 + \mu_4 \geqq \mu_1 + \mu_3$, (2) $\mu_1 + \mu_3 > 2\mu_2$, (3) $N < m(m-1)$ *implies* $\min(\mu_1, \mu_3) > 1$. **(4) $\mu_0 + \mu_4 > \mu_1 + \mu_3$, *and* $m > 7$ *implies* $\mu_0 \geqq \mu_1$, $\mu_4 \geqq \mu_3$, *(5) $\mu_0 + \mu_4 < \mu_1 + \mu_3$, *and* $m > 7$, *implies* $\mu_1 \geqq \mu_0$, $\mu_3 \geqq \mu_4$, *(6) $\mu_1 = \mu_0$ *and* $\mu_3 > \mu_4$ *implies* $\mu_3 = \mu_4 + 1$. *(7) $\mu_0 + \mu_4 = \mu_1 + \mu_3$ *implies either* $\mu_0 + \mu_1$, $\mu_4 = \mu_3$, *or* $\mu_0 = 1 + \mu_1$, $\mu_4 = -1 + \mu_3$, *or* $\mu_0 = -1 + \mu_1$, $\mu_4 = 1 + \mu_3$. (8) $\mu_0 + \mu_4 = \mu_1 + \mu_3$ *with* $m \geqq 7$, $\mu_0 > \mu_1$ *and* $\mu_4 < \mu_3$ *implies that* T *does not exist and that for* $\mu_0 = \mu_1$ *and* $\mu_4 = \mu_3$ *along with the existence of* T *implies that* $N \geqq 42$. In the above, two stars mean the result holds for $\mu_2 = 2$ and 3 both. Similarly, one star stands for $\mu_2 = 3$, and no star for $\mu_2 = 2$. (Received 18 August 1967.)

26. A Bayesian analysis of grouped response times: the exponential distribution case.

THOMAS H. STARKS, Southern Illinois University.

Consider an experiment with a completely randomized design in which t treatments are applied to $\sum_{j=1}^{t} N_j$ experimental units. At $(k-1)$ times, $t_i = i$ $(i = 1, 2, \cdots, k-1)$, after the start of the experiment, each of the units is inspected to determine the number of units in which a particular type of response has occurred. The time to response T_j of a unit receiving treatment j has an exponential distribution with parameter θ_j. In this paper, Bayesian methods are developed that provide estimators of the θ_j's and multiple comparison tests of the homogeneity of the response-time distributions of subsets of the set of all treatments in the experiment. In addition, a procedure for approximating the appropriate number of units to be employed under each treatment is discussed. (Received 5 September 1967.)

27. Unbiasedness of some test criteria for the equality of one or two covariance matrices.

NARIAKI SUGIURA and HISAO NAGAO, University of North Carolina and Hiroshima University.

Let p variate normal population have mean μ and covariance matrix $\Sigma(\Sigma_1, \Sigma_2)$ in one sample (two sample) case. We shall prove the unbiasedness of the following test criteria based on a random sample of arbitrary size. (1) For testing the hypothesis (H) $\Sigma = \Sigma_0$ against the alternatives (K) $\Sigma \neq \Sigma_0$, the modified likelihood ratio (LR) test is unbiased. (2) For testing H: $\Sigma_1 = \Sigma_2$ against K: $\Sigma_1 \neq \Sigma_2$ the modified LR test is unbiased. (3) For testing H: $\Sigma = \sigma^2 I$ (where σ^2 is unknown) against K: $\Sigma \neq \sigma^2 I$, the LR test is unbiased. (4) For testing H: $\mu = \mu_0$ and $\Sigma = \Sigma_0$ against K: $\mu \neq \mu_0$ or $\Sigma \neq \Sigma_0$, the LR test is unbiased. The problem (2) is stated by Anderson and Gupta (*Ann. Math. Statist.* **35** (1964) 1059–1063). The result of (3) is already obtained by Gleser (*Ann. Math. Statist.* **37** (1966) 464–467), but our method of proof is somewhat different from his. (Received 4 October 1967.)

28. Inverse cumulative approximation to simulation and order statistic moment evaluation. MICHAEL E. TARTER, University of Michigan.

The general problem of fitting the inverse cumulative distribution function $F^{-1}(y)$ by a polynomial expansion in terms of a more tractable function $G^{-1}(y)$ is considered. It is shown that the computation of the coefficients of this expansion need not rely upon the evaluation of $F^{-1}(y)$ for specific values of y, but instead can be based on the evaluation of the cumulative $F(x)$. If $G^{-1}(y)$ is chosen to be $-\log(1 - y)$, the solution in this particular case is shown to be based upon a generalization of the Laguerre polynomials. The orthogonal polynomials associated with the logistic distribution are calculated and used to find approximations in terms of the logit function for the inverse Gaussian cumulative. Applications of the results of the above methods are described for such problems as: random number generation, order statistic moment and product moment calculation, as well as the smoothing of the sample cumulative. (Received 6 September 1967.)

29. Optimal stopping in a Markov process. HOWARD M. TAYLOR, Cornell University.

Let $(X(t), t \geqq 0)$ be a Markov process, $g(\cdot)$ a nonnegative function, and T a Markov time or stopping time. Dynkin (*Soviet Mathematics* **4** 627–629) showed that $f(x) = \sup_T E^x g(X(T))$ is the excessive majorant function to $g(\cdot)$, and used this relation to study the existence and characterization of optimal and nearly optimal stopping times. We give a simpler version of Dynkin's development, valid for a Markov process induced by a Feller transition function. We also study several other criteria, including a form of long-run time average return. Finally some techniques for implementing the approach in a variety of situations are given, along with examples of their use. (Received 18 September 1967.)

(Abstracts of papers not connected with any meeting of the Institute.)

1. Tests for monotone failure rate based on normalized spacings. PETER J. BICKEL and KJELL A. DOKSUM, University of California, Berkeley.

Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics of a random sample from a population with density f and distribution function F such that $F(0) = 0$. Let $q(t) = f(t)\,(1 - F(t))^{-1}$ be the failure rate of F. In testing $H_0 : q(t) = \lambda$ vs. $H_1 : q(t) \uparrow$, Proschan and Pyke (*Fifth Berkeley Symp.*) considered certain statistics based on $R_1, \cdots, R_n$, the ranks of the normalized sample spacings $D_i = (n - i + 1)(X_{(i)} - X_{(i-1)})$, $1 \leqq i \leqq n$, $X_{(0)} = 0$. They show that these statistics are asymptotically normal for fixed F and compute the efficacy of one of these statistics for selected distributions. We show that asymptotic normality holds also for sequences of alternatives approaching H_0 as $n \to \infty$ and conclude that the above efficacies yield Pitman efficiencies. The statistic $V = -\sum iR_i$ is asymptotically equivalent to the one considered by Proschan and Pyke. If $S = -\sum i \log [1 - R_i(n + 1)^{-1}]$ then the Pitman efficiency of V to S is $\frac{3}{4}$. If $\{f_{\theta_n}\}$ is a sequence of alternative densities, let $h(t) = [\partial \log f_\theta(t)/\partial\theta]_{\theta=0}$, $c_i = [\int_{a_i}^{\infty} h'(t) \exp(-t)\,dt]\,[1 - i(n + 1)^{-1}]^{-1}$, $a_i = -\log\{1 - i(n + 1)^{-1}\}$. Then $\sum c_i D_i$ is asymptotically most powerful for $\{f_{\theta_n}\}$. However, the statistic S is nowhere asymptotically most powerful, although it is the asymptotically most powerful rank statistic for a suitable parametric family $\{f_{\theta_n}\}$. A comparative study of these and other statistics is given in terms of Pitman efficiencies, and Monte Carlo power. (Received 18 September 1967.)

2. Limit theorems for the multi-urn Ehrenfest model. DONALD L. IGLEHART, Cornell University.

In the multi-urn Ehrenfest model N balls are distributed among $d + 1$ $(d \geqq 2)$ urns. If we label the urns $0, 1, \cdots, d$, then the system is said to be in state $\mathbf{i} = (i_1, i_2, \cdots, i_d)$ when there are i_j balls in urn j $(j = 1, 2, \cdots, d)$ and $N - \mathbf{1}\cdot\mathbf{i}$ balls in urn 0. At discrete epochs a ball is chosen at random from one of the $d + 1$ urns; each of the N balls has probability $1/N$ of being selected. The ball chosen is removed from its urn and placed in urn i $(i = 0, 1, \cdots, d)$ with probability p^i, where the p^i's are elements of a given vector, $(p^0, \mathbf{p})$, satisfying $p^i > 0$ and $\sum_{i=0}^{d} p^i = 1$. We shall let $\mathbf{X}_N(k)$ denote the state of the system after the kth such rearrangement of balls, and define the processes

$$\{\mathbf{Y}_N(k): k = 0, \cdots, N\}$$

for $N = 1, 2, \cdots$ by the relation $\mathbf{Y}_N(k) = (\mathbf{X}_N(k) - N\mathbf{p})/N^{\frac{1}{2}}$. Next we introduce a sequence of stochastic processes $\{\mathbf{y}_N(t): 0 \leqq t \leqq 1\}$ which are continuous, linear on the intervals $((k-1)N^{-1}, kN^{-1})$, and satisfy $\mathbf{y}_N(kN^{-1}) = \mathbf{y}_N(k)$ for $k = 0, 1, \cdots, N$. If we let the ith component of $\mathbf{X}_N(0)$ be $[N^{\frac{1}{2}}y_0^i + Np^i]$, where $\mathbf{y}_0 = (y_1, \cdots, y_d)$, then the process $\{\mathbf{y}_N(t): 0 \leqq t \leqq 1\}$ induces a measure $\mu_N(\cdot; \mathbf{y}_0)$ on the Borel sets of $C_d[0, 1]$, the product space of d copies of $C[0, 1]$. The principal result is that the measures $\mu_N(\cdot; \mathbf{y}_0)$ converge weakly as $N \to \infty$ to $\mu(\cdot; \mathbf{y}_0)$, which is the measure corresponding to a d-dimension Ornstein-Uhlenbeck process. (Received 28 August 1967.)

3. One-sided problems in multivariate analysis. MICHAEL D. PERLMAN, Stanford University.

Given a sample from a p-dimensional normal distribution $N(\mu, \Sigma)$ with Σ unknown, two problems are studied: I. Test $H: \mu = 0$ against the one-sided alternative that μ lies in a cone C with vertex 0. The likelihood ratio test (LRT) is obtained, and also the MLE of μ under the restriction μ in C. (Nuesch (*Ann. Math. Statist.* **37** 116–119) considered the special case $C = \{\mu: \mu_i \geqq 0, i = 1, \cdots, p\}$ but made an error.) Sharp bounds (independent of C) for the distribution of the LRT statistic under H as Σ varies are derived, providing the exact value of the level α cutoff point. It is shown that the power of the LRT approaches one uniformly as $\mu'\Sigma^{-1}\mu \to \infty$, μ in C, whereas the power of the (nonunique) most stringent somewhere most powerful similar test (Schaafsma and Smid, *Ann. Math. Statist.* **37** 1161–1172) does not approach one if C contains two half lines. For the case

$$C = \{\mu: \mu_i \geqq 0, i = 1, \cdots, p\}$$

the exact distribution of the LRT statistic under H is obtained, and a family of conditional tests studied. The above results are extended to the problem of testing $H: \mu_1 = 0, \mu_2 = 0$ vs. $K: \mu_1 = 0, \mu_2$ in C, where $\mu' = (\mu_1', \mu_2', \mu_3')$. II. For testing $H: \xi_1 = 0, \xi_2 = 0$ vs. $K: \xi_1 = 0, \xi_2$ in C, where $\xi = \Sigma^{-1}\mu$ and $\xi' = (\xi_1', \xi_2', \xi_3')$, analogous results are obtained. This problem is a one-sided analog of that treated by Giri (*Ann. Math. Statist.* **35** 181–189 and **36** 1061–1065). (Received 3 August 1967.)

4. Optimal stopping for functions of Markov chains. ALBERTO RUIZ-MONCAYO, University of California, Berkeley.

Let $X_1, X_2, \cdots$ be a stationary Markov chain with countable state space I, stationary initial distribution $\{\pi(j)\}_{j\varepsilon I}$ and with finite second moments for the time of return to each state. f and g are real valued functions defined on the state space I such that

$\sup_{j \in I} |f(j)| < \infty$, g is positive and is bounded away from zero and $\sum_{j \in I} \pi(j)g(j) < \infty$. Let t denote a stopping rule. A finite stopping rule exists which maximizes

$$E[(f(X_t) + \cdots + f(X_1))/(g(X_1) + \cdots + g(X_t))].$$

Let $X_1, X_2, \cdots$ be a sequence of independent random variables with $E(X_n) = 0$, $E(X_n) = \sigma_n^2$, $n = 1, 2, \cdots$, and the σ_n's are uniformly bounded away from zero and from infinity. Also

$$L[(X_1 + \cdots + X_n)/(\sigma_1^2 + \cdots + \sigma_n^2)] \to N(0, 1). \quad (1)$$

A finite stopping rule exists which maximizes $E(S_t/t)$. If (1) is not satisfied then the assertion is not necessarily true. These results constitute a generalization of work done by A. Dvoretzky appearing in *Fifth Berkeley Symp. Math. Statist. Prob.* (Received 28 September 1967.)

5. Admissibility of the maximum likelihood estimator in the regression of two predictands upon one stochastic predictor.
STANLEY L. SCLOVE, Stanford University.

Given a random sample of N observations from the joint normal distribution of two predictands and one predictor with known means, the problem is to estimate the two coefficients in the regression of the predictands upon the predictor. The loss function is the square of the distance between the parameter vector and its estimate in the metric of the residual covariance matrix. The maximum likelihood estimator is shown to be admissible when $N \geqq 5$ by applying Stein's theorem regarding admissibility for estimating two parameters (Lemma 2 in "Multiple Regression" by C. Stein, pp. 424–443 in *Contributions to Probability and Statistics—Essays in Honor of Harold Hotelling*, ed. by I. Olkin, Stanford University Press, Stanford, California, 1960). (Received 14 September 1967.)

6. Statistical inference in random coefficient regression models and its application in economic analysis.
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In this paper we consider the linear model $y(i) = X(i)\beta(i) + u(i)$ $(i = 1, 2, \cdots N)$ where $y(i)$ is a $T \times 1$ vector of observations, $X(i)$ is a $T \times \Lambda$ matrix of observations with rank Λ on Λ fixed regressors, $\beta(i)$ is a $\Lambda \times 1$ vector of regression coefficients, and $u(i)$ is a $T \times 1$ vector of disturbances for the ith unit in a sample of N units. We assume that $\beta(i)$ is distributed across units with mean $\bar{\beta}$ and variance-covariance matrix Δ. $u(i)$ is distributed with mean zero and variance-covariance matrix $\sigma_{ii}I$. $\beta(i)$ and $u(i)$ are assumed to be uncorrelated vectors. Under these assumptions we suggest a consistent and asymptotically efficient estimator of $\bar{\beta}$ and unbiased estimators of Δ and σ_{ii}. Assuming β(i) and $u(i)$ are normally distributed we develop a large sample procedure for testing a linear hypothesis on $\bar{\beta}$. We apply these procedures in the analysis of investment demand function. (Received 10 October 1967.)