A CHARACTERIZATION OF NORMALITY

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0. Summary. A probability distribution on an n-dimensional Euclidean space can be specified by giving either the joint distribution of the usual rectangular coordinates or by the joint distribution of the polar coordinates. The object of this note is to obtain a characterization of normality in terms of independence of these coordinates under suitable regularity conditions. Further, if we slightly relax these conditions we get a new class of distributions which includes the normal distribution.

1. Introduction. When we consider only the absolutely continuous distributions the joint density function $p_1(x_1, \cdots, x_n)$ of the rectangular coordinates $X_1, \cdots, X_n$ and the joint density function $p_2(r, \theta_1, \cdots, \theta_{n-1})$ of the polar coordinates $R, \Theta_1, \cdots, \Theta_{n-1}$ are related by

$$p_1(x_1, \cdots, x_n)r^{n-1} \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \cdots \sin \theta_{n-2} = p_2(r, \theta_1, \cdots, \theta_{n-1}), \tag{1.1}$$

where the coordinates are related as

$$\begin{aligned} x_1 &= r \cos \theta_1, \\ x_2 &= r \sin \theta_1 \cos \theta_2, \\ x_3 &= r \sin \theta_1 \sin \theta_2 \cos \theta_3, \\ &\vdots \\ x_{n-1} &= r \sin \theta_1 \cdots \sin \theta_{n-2} \cos \theta_{n-1}, \\ x_n &= r \sin \theta_1 \cdots \sin \theta_{n-2} \sin \theta_{n-1}; \end{aligned} \tag{1.2}$$

$-\infty < x_i < \infty$ for $i = 1, \cdots, n$; $0 \leqq r < \infty$ and $0 \leqq \theta_1 < 2\pi$ for $i = 1, \cdots, n - 1$.

We note that $r^2 = \sum_{i=1}^n x_i^2$.

2. Characterization. We now make the following assumption:

Assumption (A). $p_2(r, \theta_1, \cdots, \theta_{n-1})/r^{n-1} \sin^{n-2} \theta_1 \cdots \sin \theta_{n-2}$ is well-defined and nonzero everywhere, continuous in r, and equal to $p_1(x_1, \cdots, x_n)$ which is then continuous in each x_i.

Then we get the following interesting result.

Theorem 2.1. *Under Assumption* (A), $X_1, \cdots, X_n$ *are mutually independent and* R *is independent of* $\Theta_1, \cdots, \Theta_{n-1}$ *if and only if* $X_1, \cdots, X_n$ *are distributed independently normally with zero means and equal variances.*

Proof. The 'if' part of the theorem is obvious. The 'only if' part can be proved as follows:

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Let the marginal density functions of X_i be f_i, $i = 1, \cdots, n$, the marginal density function of R be g_1 and the joint marginal density function of $(\Theta_1, \cdots, \Theta_{n-1})$ be g_2 so that in view of the assumed independence we have

$$p_1(x_1, \cdots, x_n) = \prod_{i=1}^n f_i(x_i)$$

and

$$p_2(r, \theta_1, \cdots, \theta_{n-1}) = g_i(r) g_2(\theta_1, \cdots, \theta_{n-1})$$

which we substitute in (1.1) to get

$$\prod_{i=1}^n f_i(x_i) = (g_1(r)/r^{n-1}) \cdot g_2(\theta_1, \cdots, \theta_{n-1})/\sin^{n-2}\theta_1 \cdots \sin\theta_{n-2}.$$

Writing the two factors on the right-hand side as $h_1(r)$ and $h_2(\theta_1, \cdots, \theta_{n-1})$ we get

$$(2.1) \qquad \prod_{i=1}^n f_i(x_i) = h_1(r) h_2(\theta_1, \cdots, \theta_{n-1})$$

so that letting $r \to 0$ we have in view of continuity in (A)

$$(2.2) \qquad \prod_{i=1}^n f_i(0) = h_1(0) h_2(\theta_1, \cdots, \theta_{n-1})$$

where, in view of (A) $f_i(0) \neq 0$, $i = 1, \cdots, n$, and $h_1(0) \neq 0$.

Dividing (2.1) by (2.2) and writing $f_i^*(y) = f_i(y)/f_i(0)$ and $h(r) = h_1(r)/h_1(0)$ we get

$$(2.3) \qquad \prod_{i=1}^n f_i^*(x_i) = h(r).$$

Putting $x_i = 0$ for all $i \neq j$ and $x_j = y$ in (2.3) we get $f_j^*(y) = h(|y|)$, $j = 1, 2, \cdots, n$, because $f_i(0) = h(0) = 1$.

Substituting in (2.3) we get $\prod_{i=1}^n h(|x_i|) = h(r)$, which we write as

$$(2.4) \qquad \prod_{i=1}^n t(x_i^2) = t(r^2),$$

where $t(z^2) = h(z)$, $z \geqq 0$.

Letting $x_i^2 = u_i$ in (2.4) we get

$$(2.5) \qquad \prod_{i=1}^n t(u_i) = t\left(\sum_{i=1}^n u_i\right), \qquad u_i \geqq 0.$$

Putting $u_3 = \cdots = u_n = 0$ in (2.5) and noting that $t(0) = 1$ we get a well-known functional equation

$$t(u_1)t(u_2) = t(u_1 + u_2) \qquad u_1, u_2 \geqq 0.$$

Here in view of Assumption (A) t is continuous and nonzero so that the general solution of the equation is $t(z) = \exp(cz)$. (See [1], p. 45.)

Hence we have $f_i(z) = f_i(0) f_i^*(z) = f_i(0) h(|z|) = f_i(0) t(z^2) = f_i(0) \exp(cz^2)$ $= b \exp(cz^2)$, say.

The fact that $f_i(z)$ is a density function leads us to conclude that c must be negative, say $-1/2\sigma^2$, so that b must be $1/(2\pi)^{\frac{1}{2}}\sigma$, in view of the assumption that $f_i(z) > 0$ for all z.

Thus X_i has the density function $((2\pi)^{\frac{1}{2}}\sigma)^{-1} \exp(-x_i^2/2\sigma^2)$ which was to be proved.

If we remove the restriction that $p_1(x_1, \cdots, x_n)$ be nonzero everywhere and replace it by a weaker one that it should be nonzero at the origin then we still can prove that $f_i(x_i)$ is of the form $b \exp(cx_i^2)$ and then, noting that the ranges of X_i's should be independent and also that the range of R should be independent of the ranges of Θ's, we can deduce that for each X_i the range is either $[0, \infty)$, $(-\infty, 0]$ or $(-\infty, \infty)$ and, hence, that each X_i is distributed as $|Y|$, $-|Y|$ or Y respectively where Y is normal with zero mean.

If we also drop the restriction that $p_1(x_1, \cdots, x_n)$ be nonzero at the origin then it is conjectured that the distribution of X_i will belong to a wider class whose typical density function is $ax^b \exp(cx^2)$. But this conjecture is proved here only with some additional conditions on the density functions.

Let $\mathfrak{C}$ be the class of all density functions $f(z)$ for which $zf'(z)/f(z)$ is well-defined and continuous within the range. Let also $\mathfrak{J}$ be the class of all density functions $f(z)$ expressible as $az^b \exp(cz^2)$, within the range, where the range is either $[0, \infty)$, $(-\infty, 0]$ or $(-\infty, \infty)$. It can be verified that $\mathfrak{J} \subset \mathfrak{C}$.

It may be noted that in $\mathfrak{J}$, c must be negative and b greater than -1. If the range is $(-\infty, 0]$ or $(-\infty, \infty)$ then b should further be rational and expressible with even numerator and odd denominator.

THEOREM 2.2. *If the density function of the marginal distribution of* X_i *is* f_i, $i = 1, 2, \cdots, n$, *of* R *is* g_1, *where* f_i *and* g_1 *belong to* $\mathfrak{C}$ *and* $R, \Theta_1, \cdots, \Theta_{n-1}$ *are related to* $X_1, \cdots, X_n$ *by* (1.2) *then* $X_1, \cdots, X_n$ *are mutually independent and* R *is independent of* $\Theta_1, \cdots, \Theta_{n-1}$ *if and only if* f_i *belongs to* $\mathfrak{J}$.

PROOF. Here again the 'if' part is obvious. For the 'only if' part we have as in the previous case ((2.1))

$$\prod_{i=1}^n f_i(x_i) = h_1(r)h_2(\theta_1, \cdots, \theta_{n-1})$$

where $h_1(r) = g_1(r)r^{-(n-1)}$ and $h_1 \varepsilon \mathfrak{C}$ because $g_1 \varepsilon \mathfrak{C}$.

Now differentiating logarithmically (2.1) partially wrt r we get

$$\sum_{i=1}^n [f_i'(x_i)/f_i(x_i)](\partial x_i/\partial r) = h_1'(r)/h_1(r) \text{ for } x_i \text{ for which } f_i > 0, \quad i = 1, \cdots, n.$$

Noting that $r(\partial x_i/\partial r) = x_i$ this becomes

$$\sum_{i=1}^n x_i f_i'(x_i)/f_i(x_i) = rh_1'(r)/h_1(r)$$

which, we can say, is true for all x_i because $h_1 \varepsilon \mathfrak{C}$ and $f_i \varepsilon \mathfrak{C}$, $i = 1, \cdots, n$.

Putting $f_i^{**}(z) = zf_i'(z)/f(z)$ and $k(z) = zh_1'(z)/h_1(z)$ we write this as

$$\sum_{i=1}^n f_i^{**}(x_i) = k(r) \quad \text{for all } x_i. \tag{2.6}$$

Now letting $x_i \to 0$ for $i \neq j$ and $x_j \to y$ in (2.6) we get $f_j^{**}(y) = k(|y|) - \sum_{i \neq j} f_i^{**}(0)$ which gives, by substituting back in (2.6),

$$\sum_{i=1}^n k(|x_i|) = k(r) + (n-1)\sum_{i=1}^n f_i^{**}(0). \tag{2.7}$$

Here we may write $k(z) = s(z^2) + \sum_{i=1}^n f_i^{**}(0)$ for $z \geqq 0$ and get a functional equation, satisfied by s, as $\sum_{i=1}^n s(x_i^2) = s(r^2)$.

Now, as in the previous case, we let $x_i^2 = u_i$ so that $r^2 = \sum_{i=1}^n u_i$ and

$$\sum_{i=1}^n s(u_i) = s(\sum_{i=1}^n u_i), \qquad u_i \geqq 0. \tag{2.8}$$

Since $h_1 \varepsilon \mathfrak{C}$, s is continuous. Also $s(0) = k(0) - \sum_{i=1}^n f_i^{**}(0) = 0$ so that the general solution of (2.8) can be written as $s(z) = 2cz$. (See [1], p. 45.)

This gives us $zf_j'(z)/f_j(z) = f_j^{**}(z) = k(|z|) - \sum_{i \neq j} f_i^{**}(0) = s(z^2) + g_j(0) = 2cz^2 + b_j$, say.

Thus $f_j(z)$ is a solution of the differential equation $f'(z) = f(z)(2cz + b_j/z)$ so that $f_j(z) = a_j z^{b_j} \exp(cz^2)$ i.e., $f_j \varepsilon \mathfrak{J}$ which was to be proved.

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REFERENCE

[1] Wilson, E. B. (1912). *Advanced Calculus.* Ginn, Boston.