REDUCED GROUP DIVISIBLE PAIRED COMPARISON DESIGNS

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1. Introduction. Suppose that t treatments $1, 2, \cdots, t$ are to be compared in pairs. It will often be impractical to make all possible $\frac{1}{2}t(t-1)$ pairings. Designs are therefore required that reduce the number of comparisons without serious imbalance, that is, give estimates of treatment contrasts with variances as low and as equal as possible.

Incomplete block designs with two treatments per block, i.e. paired comparison designs, given previously fall into two groups. Firstly, the partially balanced incomplete block (PBIB) designs with two associate classes which include the group divisible, the triangular and the square designs. These designs have been completely enumerated by Clatworthy (1955) for $t \leqq 20$ and $2 \leqq r \leqq 10$. Secondly, the cyclic designs. The structure and enumeration of these designs have been given by David (1963) and David (1965).

The purpose of this paper is to produce a class of designs that, in many cases, give more efficient designs than the PBIB or cyclic designs. An efficiency factor will be obtained for each design so enabling comparison to be made with the designs given by Clatworthy (1955) and David (1963). Simplicity of analysis is much less important in the days of electronic computers and, although some of the designs proposed here possess a high degree of symmetry that makes analysis simple, designs have not been constructed with ease of analysis in mind.

The measure of efficiency of paired comparison designs will be obtained from the covariance matrix of the estimates of treatment parameters. The efficiency, E, will be defined as the ratio of the average between treatment variance for the full design to the average between treatment variance for the incomplete design. This is the same measure as used by David (1963), but Clatworthy's figures need to be multiplied by $2(t-1)/t$ to convert them to the values of E.

The designs considered will be of three types. The first two types, A and B, will have the t treatments divisible into m groups of n members each. Let θ_{ip} be the ith treatment in the pth group ($i = 1, 2, \cdots n; p = 1, 2, \cdots, m$). Blocks will be of the form $(\theta_{ip}, \theta_{jq})$ where $p > q$, i.e. pairings or comparisons will be made between groups. For Type A designs, blocks are chosen so as to include *particular* combinations of i and j for *all* combinations of p and q ($p > q$). If all combinations of i and j are included the resulting design is a group divisible design. For Type B designs, blocks are chosen so as to include *all* combinations of i and j for *particular* combinations of p and q ($p > q$). Type C designs will be defined later.

2. Type A designs. Two classes of Type A designs will be discussed, namely designs with $r = (m-1)(n-1)$ and designs with $m = 2$. It is possible to

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construct designs for $m > 2$ and $r < (m - 1)(n - 1)$ but these will be relatively inefficient and will not, therefore, be considered.

2.1. *Balanced Type* A *designs*. The between group comparisons are set out in $\frac{1}{2}m(m - 1)$ $n \times n$ squares. A balanced Type A design will be given by omitting the leading diagonal comparisons of each square from the full group divisible design.

For example, nine treatments in three groups of three

$$1 \quad 2 \quad 3 \qquad 4 \quad 5 \quad 6 \qquad 7 \quad 8 \quad 9$$

will give the design.

$$\begin{array}{cccccc} \multicolumn{2}{c}{(1, 5)(1, 6)} & \multicolumn{2}{c}{(1, 8)(1, 9)} & \multicolumn{2}{c}{(4, 8)(4, 9)} \\ (2, 4) & (2, 6) & (2, 7) & (2, 9) & (5, 7) & (5, 9) \\ \multicolumn{2}{c}{(3, 4)(3, 5)} & \multicolumn{2}{c}{(3, 7)(3, 8)} & \multicolumn{2}{c}{(6, 7)(6, 8)} \end{array}$$

The number of blocks will be $b = \frac{1}{2}mn(m - 1)(n - 1)$ with each treatment replicated $r = (m - 1)(n - 1)$ times. Balanced Type A designs belong to the class of PBIB designs with three associate classes, and can also be obtained by the method given by Vartak (1959). Hence, it is not difficult to show that the efficiency of these designs is given by

$$E = (t - 1)^2/[r(t - 1) + tx(m - 1) + ty(n - 1)]$$

where $x = 1 + 1/m(r - 1)$ and $y = 1 + 1/n(r - 1)$.

For $m = n$ the designs become the square PBIB designs given by Clatworthy (1955). The designs are also members of the Type F designs given by Pearce (1963).

2.2. *Type* A *designs for two groups*. For these designs $m = 2$. The between group comparisons are set out in an $n \times n$ square such that the ijth element, denoted by $(i\, v\, j)$, is the comparison $(i, n + j)$. To obtain the Type A designs an incomplete block design with n treatments numbered $1, 2, \cdots, n$, n blocks and k units per block is constructed. The elements $(i\, v\, j)$ are then chosen such that, for a given value of i, the values of j comprise the ith block of the incomplete block design. Each treatment will therefore be replicated k times in the design.

The Type A design for two groups will be classified according to the type of incomplete block design used. The types considered are (i) Type A(B)—balanced incomplete block designs, (ii) Type A(P)—PBIB designs with two associate classes, (iii) Type A(C)—cyclic designs.

(i) Type A(B) designs. These designs exist only if balanced incomplete block (BIB) designs exist for n treatments and n blocks, that is if

$$r(r - 1)/(n - 1) = \lambda, \quad \text{say}$$

where λ is a positive integer. When $\lambda = r$ the designs are group divisible, and when $\lambda = r - 1$ the designs are balanced Type A. Hence, only designs for $\lambda < r - 1$ will be considered. Designs for $\lambda = 1$ have been given by Quenouille (1953).

TABLE 1
Balanced Type A Designs $6 \leqq t \leqq 16$

t	r	m	E	t	r	m	E
6	2	2	0.714	12	6	3	0.935
8	3	2	0.845	14	6	2	0.940
9	4	3	0.889	15	8	3	0.956
10	4	2	0.897	16	7	2	0.951
12	5	2	0.924	16	9	4	0.962

From the method of construction it can be seen that Type A(B) designs belong to the class of PBIB designs with three associate classes, and association scheme $n_1 = n - 1$, $n_2 = r$ and $n_3 = n - r$ and

$$P_1 = \begin{bmatrix} n-2 & 0 & 0 \\ & \lambda & r-\lambda \\ & & n-2r+\lambda \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & r-1 & n-r \\ & 0 & 0 \\ & & 0 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & r & n-r-1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}.$$

Hence it can be shown that the efficiency of these designs is given by $E = (t-1)^2(r-1)/[4r(n-1)^2 + n(r-1)]$.

Given the BIB design the construction of the Type A(B) design is straightforward. For example, to construct the design for $t = 14$ and $r = 3$ the BIB will be

(1, 2, 3) (1, 4, 5) (1, 6, 7) (2, 4, 6)
(2, 5, 7) (3, 4, 7) (3, 5, 6)

and, taking $(i\ v\ j)$ to be the comparison $(i, n + j)$ the 21 blocks of the Type A(B) design are

(1, 8)	(1, 9)	(1, 10)
(2, 8)	(2, 11)	(2, 12)
(3, 8)	(3, 13)	(3, 14)
(4, 9)	(4, 11)	(4, 13)
(5, 9)	(5, 12)	(5, 14)
(6, 10)	(6, 11)	(6, 14)
(7, 10)	(7, 12)	(7, 13)

(ii) Type A(P) designs. PBIB designs with two associate classes have been tabulated by Bose, Clatworthy and Shriklande (1954) and Clatworthy (1956) for $r \leqq 10$. For the Type A(P) designs in Table 2 below the number in the last column refers to the designs given by Bose, et. al. and (with asterisks) by Clatworthy. The analysis of these designs does not take a simple form, so that the efficiency factor has been obtained by computer. The construction of the designs follows from that of the Type A(B) designs.

(iii) Type A(C) designs. Cyclic incomplete block designs ($k \geqq 3$) have been given by David and Wolock (1965) and John (1966). For the Type A(C) designs in Table 2 below the first block of the cyclic design used is given in the last column. Again the efficiency has been obtained by computer, and the construction follows from the Type A(B) designs.

3. Type B designs. Again the between group comparisons are set out in $\frac{1}{2}m(m - 1)$ $n \times n$ squares. A Type B design will consist of all the comparisons from a fraction of the $\frac{1}{2}m(m - 1)$ squares. Hence, if p squares are chosen the design will consist of pn^2 blocks. The squares will be chosen such that each treatment will be replicated r times in the design. The best way of constructing these designs can be seen from a consideration of the between-treatment variance of the three different types of treatment comparisons, namely the within group, the between group included in the design and the between group excluded from the design. For the within group comparisons the variance will depend upon the number of replications of each treatment. The variance of the between group comparisons included in the design will be close to the variance of the full design of $\frac{1}{2}t(t - 1)$ blocks. The variance of the remaining between group comparisons will depend on the design itself. These comparisons will be made via the other groups and, hence, the stronger the interconnection between groups the smaller will be their variance. Therefore it is necessary to arrange the m groups into pairs so that each group occurs, say, s times and so that the groups are connected together as efficiently as possible. This is equivalent to choosing the best paired comparison design for m treatments and s replications. The resulting Type B design will have $r = ns$ and $b = \frac{1}{2}mn^2s$.

For example, the best, or most efficient, paired comparison design for $m = 5$

TABLE 2

Type A Designs for Two Groups $6 \leqq t \leqq 16$

t	r	E	Type	Ref.	t	r	E	Type	Ref.
8	2	0.583	A(P)		16	3	0.678	A(C)	(1, 2, 5)
10	2	0.491	A(P)		16	4	0.819	A(P)	SR7
10	3	0.785	A(P)		16	4	0.828	A(C)	(1, 2, 3, 6)
12	3	0.761	A(P)	R1	16	4	0.828	A(C)	(1, 2, 3, 5)
12	3	0.723	A(C)	(1, 2, 3)	16	5	0.888	A(P)	R108*
12	4	0.862	A(P)	R2	16	5	0.872	A(P)	R109*
12	4	0.852	A(P)	S2	16	5	0.884	A(C)	(1, 2, 3, 5, 6)
12	4	0.862	A(C)	(1, 2, 3, 4)	16	5	0.879	A(C)	(1, 2, 3, 4, 5)
14	3	0.758	A(B)		16	6	0.922	A(P)	S9
14	4	0.849	A(B)		16	6	0.924	A(C)	(1, 2, 3, 4, 5, 7)
14	5	0.901	A(C)	(1, 2, 3, 4, 5)	16	6	0.924	A(C)	(1, 2, 3, 4, 5, 6)
16	3	0.730	A(P)	R5					

and $s = 2$ is

$$(1, 3) \quad (1, 4) \quad (2, 4) \quad (2, 5) \quad (3, 5).$$

Hence, for $n = 3$, the Type B design for $t = 15$ and $b = 45$ will be

(1, 7)	(1, 8)	(1, 9)	(1, 10)	(1, 11)	(1, 12)
(2, 7)	(2, 8)	(2, 9)	(2, 10)	(2, 11)	(2, 12)
(3, 7)	(3, 8)	(3, 9)	(3, 10)	(3, 11)	(3, 12)
(4, 10)	(4, 11)	(4, 12)	(4, 13)	(4, 14)	(4, 15)
(5, 10)	(5, 11)	(5, 12)	(5, 13)	(5, 14)	(5, 15)
(6, 10)	(6, 11)	(6, 12)	(6, 13)	(6, 14)	(6, 15)
(7, 13)	(7, 14)	(7, 15)			
(8, 13)	(8, 14)	(8, 15)			
(9, 13)	(9, 14)	(9, 15)			

The paired comparison designs used in the construction of the Type B designs are given in the last column of Table 3 below. Two special groups of Type B designs arise from the way these designs are constructed. Firstly, if the design used to choose the squares is a group divisible (GD) design then the resulting Type B design will also be a GD design. Thus the designs for $m = 6$, $s = 4$, $m = 2s$ and $m = s + 1$ are GD designs. Secondly, if the design used is a PBIB design with k associate classes the resulting Type B design will be a PBIB design with $k + 1$ associate classes. In particular, since the designs for $m = 5$, $s = 2$ and $m = 10$, $s = 3$ are PBIB designs with two associate classes the resulting Type B designs will give another set of designs belonging to the class of PBIB designs with three associate classes.

4. Type C designs. The Type A and Type B designs given in the last two sections were constructed for a set of treatments that were divisible into m groups each of n members. By using a number of redundant or dummy treatments designs will now be constructed for $t \neq mn$ treatments. These designs will be called Type C designs. The dummy treatments are introduced so that the construction follows from the designs of the previous sections. In the final paired comparison design blocks containing dummy treatments will be omitted.

TABLE 3
Type B Designs $6 \leqq t \leqq 16$

t	r	m	s	E	Ref.	t	r	m	s	E	Ref.
8	4	4	2	0.942	GD	14	8	7	4	0.965	Type C
10	4	5	2	0.900	Cyclic	15	6	5	2	0.933	Cyclic
12	4	6	2	0.852	Balanced A	16	4	8	2	0.760	Cyclic
12	6	6	3	0.960	GD	16	6	8	3	0.926	Cyclic
12	6	4	2	0.960	GD	16	8	8	4	0.970	GD
12	8	6	4	0.976	GD	16	8	4	2	0.970	GD
14	4	7	2	0.805	Cyclic						

For these designs the total number of actual and dummy treatments, T, will be divided into m groups of n treatments each. The construction of Type C designs will now be in two stages. Firstly, a GD, Type A or Type B design is chosen for the T treatments. In this design a number of actual treatments will be paired with the dummy treatments and, consequently, these treatments will be replicated less than r time in the final design. The second stage will, therefore, pair these treatments together in such a way that each treatment will now be replicated r times in the final design. This can best be done by pairing the treatments most inefficiently linked together as a result of the first stage of construction.

For example, the Type B design for ten treatments and five groups is

(1, 5)(1, 6)	(1, 7)(1, 8)	(3, 7)(3, 8)
(2, 5)(2, 6)	(2, 7)(2, 8)	(4, 7)(4, 8)
(3, 9)(3, 10)	(5, 9)(5, 10)	
(4, 9)(4, 10)	(6, 9)(6, 10)	

If treatment 10 is taken as the dummy treatment then in the above design treatments 3, 4, 5 and 6 are each paired once with the dummy treatment. The comparison of treatments 3 with 4, and 5 with 6, is already strong, and hence the best way of joining these four treatments together will be (3, 5) (4, 6) or (3, 6) (4, 5). Due to the symmetry either of these two blocks can be chosen. The Type C design for $t = 9$ and $b = 18$ will be

(1, 5)(1, 6)		(1, 7)(1, 8)	(3, 7)(3, 8)
(2, 5)(2, 6)		(2, 7)(2, 8)	(4, 7)(4, 8)
(3, 9)	(5, 9)	(3, 5)	
(4, 9)	(6, 9)	(4, 6)	

TABLE 4

Type C Designs $6 \leqq t \leqq 16$

t	r	m	s	E	Ref.	t	r	m	s	E	Ref.
7	2	2	4	0.643	A(C)	13	2	2	7	0.396	A(C)
7	4	2	4	0.934	GD	13	4	2	7	0.848	A(B)
7	4	3	3	0.934	Balanced A	13	4	7	2	0.823	B
9	2	2	5	0.533	A(C)	13	6	2	7	0.936	Balanced A
9	4	2	5	0.893	Balanced A	13	6	5	3	0.944	B
9	4	5	2	0.909	B	13	8	3	5	0.961	Balanced A
11	2	2	6	0.455	A(C)	13	8	7	2	0.968	B
11	4	2	6	0.872	A(C)	15	2	2	8	0.350	A(C)
11	4	2	6	0.860	A(P)	15	4	2	8	0.832	A(C)
11	4	6	2	0.868	B	15	4	8	2	0.779	B
11	6	2	6	0.955	GD	15	6	2	8	0.924	A(C)
11	6	3	4	0.943	Balanced A	15	6	8	2	0.927	B
11	8	3	4	0.978	GD	15	8	2	8	0.967	GD
11	8	6	2	0.978	B						

The designs given in Table 4 below are for t odd. The number of dummy treatments used will be $d = mn - t$, where $d < n$. The last column of Table 4 gives the design type used in the first stage of the construction.

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