NONPARAMETRIC PROCEDURES FOR SELECTING A SUBSET CONTAINING THE POPULATION WITH THE LARGEST α-QUANTILE[1]

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1. Introduction and summary. A nonparametric solution is given for the problem of selecting a subset of k populations which with high probability contains the one with the largest α-quantile, $0 < \alpha < 1$. It is assumed that each population has associated with it a continuous cdf $F_i(x)$ $(i = 1, 2, \cdots, k)$, that samples of equal sizes are taken from each population and that observations from the same or different populations are all independent. We use the notation of the companion paper [7] without redefining all the symbols and, in particular, it is convenient to identify each population by its cdf.

A procedure for the problem, based on order statistics is proposed in Section 2 and limitations on its feasibility in terms of a bound P_1 on the possible guarantee probability P^* are examined in Section 3. An asymptotic expression in terms of tabled functions is also given for P_1. The expected size of the selected subset is examined exactly in Section 4 and asymptotically in Section 5; the latter also contains an asymptotic formula for the smallest sample size n required to meet the P^*-guarantee. Asymptotic relative efficiency evaluations of the proposed procedure are carried out in Section 5. Section 6 treats the dual problem of selecting the smallest α-quantile; Section 7 verifies a monotonicity property. Extensive tables giving P_1 and the integer constant defining the procedure are provided for the case $\alpha = \frac{1}{2}$.

2. Formulation of the problem. A common fixed (i.e., given) number of observations are taken from each of the k populations. A constant $P^* > 1/k$ is preassigned and we are required to find a procedure R_1 for selecting a nonempty subset of k populations which contains the one with the largest α-quantile with probability at least P^*, i.e., a procedure R_1 such that

$$P\{\text{CS} \mid R_1\} \geqq P^*, \tag{2.1}$$

for all possible k-tuples $(F_1, F_2, \cdots, F_k)$ for which

$$\min_{1 \leqq j \leqq k} F_{[j]}(y) = F_{[k]}(y) \quad \text{for all} \quad y. \tag{2.2}$$

We denote this set of k-tuples $(F_1, F_2, \cdots, F_k)$ by Ω_1.

We shall assume that

$$1 \leqq (n + 1)\alpha \leqq n \tag{2.3}$$

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(for any $\alpha \neq 0, 1$ this is satisfied for n sufficiently large) and we define a positive integer r by the inequalities

$$r \leqq (n + 1)\alpha < r + 1. \tag{2.4}$$

It follows that $1 \leqq r \leqq n$.

We now define a procedure $R_1 = R_1(c)$ in terms of a positive integer c ($1 \leqq c \leqq r - 1$) and the order statistics $Y_{j,i}$ where $Y_{j,i}$ denotes the jth order statistic from the population $F_i(y)$ ($j = 1, 2, \cdots, n$; $i = 1, 2, \cdots, k$).

PROCEDURE R_1. Put F_i in the selected subset if and only if

$$Y_{r,i} \geqq \max_{1 \leqq j \leqq k} Y_{r-c,j} \tag{2.5}$$

where c is the smallest integer with $1 \leqq c \leqq r - 1$ for which (2.1) is satisfied.

We shall show that for any given α and k a value of $c \leqq r - 1$ may not exist for some pairs (n, P^*) but if P^* is chosen not greater than some function $P_1 = P_1(n, \alpha, k)$ where $1/k < P_1 < 1$, then a value of $c \leqq r - 1$ does exist that satisfies (2.1). The function P_1 will be derived and evaluated; in particular we shall be interested to see for fixed k how rapidly it approaches 1 as n increases.

[If we allow $c = r$ and define $Y_{0,j} = -\infty$ then any value of $P^* \leqq 1$ can be obtained by putting all the populations in the selected subset. Thus P_1 represents the largest P^* for which the (non-randomized) problem is non-degenerate. If $P_1 < P^* < 1$ then randomization between the rule (2.5) with $c = r - 1$ and the degenerate rule ($c = r$) is desirable, but we shall not consider any such rules in the discussion below. It should be pointed out however that all the results below remain valid for both $c = r$ and $c = 0$.]

3. Probability of a correct selection for R_1. Using the notation of Sections 1 and 2 of [7], the probability of a correct selection $P\{\text{CS} \mid R_1\}$ is easily seen to be

$$P\{\text{CS} \mid R_1\} = \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} H_{r-c,i}(y)\, dH_{r,k}(y) \tag{3.1}$$

where

$$H_{r,i}(y) = \sum_{j=r}^{n} \binom{n}{j} F_{[i]}^{j}(y)(1 - F_{[i]}(y))^{n-j}. \tag{3.2}$$

To find the infimum of (3.1) in Ω_1, we note that $H_{r-c,i}(y)$ depends on y only through $F_{[i]}(y)$ and that it is an increasing function of $F_{[i]}(y)$. Hence we obtain the infimum by setting $F_{[i]}(y) = F_{[k]}(y)$ for $i = 1, 2, \cdots, k - 1$. Making the transformation $u = F_{[k]}(y)$ then gives

$$\inf_{\Omega_1} P\{\text{CS} \mid R_1\} = \int_{-\infty}^{\infty} H_{r-c,k}^{k-1}(y)\, dH_{r,k}(y) = \int_0^1 G_{r-c}^{k-1}(u)\, dG_r(u) \tag{3.3}$$

where $G_r(u) = I_u(r, n - r + 1)$ is the standard incomplete beta function. When $c = 0$ the last member of (3.3) is clearly $1/k$ and we wish to show that $G_{r-c}(u)$ is an increasing function of c ($c = 1, 2, \cdots, r - 1$) for fixed r and u or that $G_r(u)$ is decreasing in r for fixed u. Integrating $G_r(u)$ by parts gives

$$G_r(u) - G_{r+1}(u) = -\binom{n}{r} u^r (1 - u)^{n-r} \tag{3.4}$$

which shows the required monotonicity. Hence the maximum value of P^* for which a value of c satisfying (2.1) exists is obtained by setting $c = r - 1$ in (3.3) and this easily reduces to

$$(3.5) \qquad P_1 = r\binom{n}{r}\int_0^1(1-v^n)^{k-1}(1-v)^{r-1}v^{n-r}dv$$
$$= \binom{n}{r}\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}/\binom{n(i+1)}{r}.$$

Hence for any $P^* < P_1(n, \alpha, k) = P_1$ a value of $c \leqq r - 1$ exists and it is unique so that the procedure R_1 is well defined for $P^* < P_1$. A short table (Table 2) of P_1-values for $\alpha = \frac{1}{2}$, so that $r = (n+1)/2$, and $k = 2(1)10$ is given; the integral in (3.5) was evaluated by Gaussian quadrature in the preparation of Table 2. It can be shown that $P_1 \to 1$ exponentially fast as $n \to \infty$. In fact, we can write for large n (with $r/n \to \alpha$)

$$(3.6) \quad P_1 \approx 1 - (k-1)\binom{n}{r}/\binom{2n}{r} \approx 1 - (k-1)[(1+\alpha')^{1+\alpha'}/4(\alpha')^{\alpha'}]^n C_{\alpha'}$$

where $\alpha' = 1 - \alpha$ and $C_{\alpha'}$ does not depend on k or n; a detailed derivation of (3.6) is omitted. Since $1 + \alpha' > 0$ and $[(1+\alpha')/\alpha']^{\alpha'} > 0$ are both increasing in α' $(0 < \alpha' < 1)$ their product is also and hence the maximum value of the quantity in square brackets in (3.6) is one; this shows that $P_1 \to 1$ exponentially fast.

Of course, the same phenomenon described above can be interpreted from another point of view, namely, that for a given $P^* > 1/k$ there is a minimum common number n of observations required to satisfy (2.1). Table 2 immediately provides the smallest common odd integer n for which $P_1(n) \geqq P^*$ for $\alpha = \frac{1}{2}$; for instance, if $k = 2$, $P^* = .750$, this smallest value of n such that $P_1(n) \geqq .750$ is 3.

For small values of k and any α satisfying (2.3) the integral in (3.3) can be written more explicitly and, in particular, for $k = 2$ we obtain after algebraic simplification the three equivalent expressions

$$(3.7) \qquad \begin{aligned} \inf_{\Omega_1} P\{\text{CS} \mid R_1, k=2\} &= \binom{2n}{n}^{-1}\sum_{i=r-c}^{n}\binom{r-1+i}{r-1}\binom{2n-r-i}{n-r} \\ &= \tfrac{1}{2} + \binom{2n}{n}^{-1}\sum_{i=r-c}^{r-1}\binom{r-1+i}{r-1}\binom{2n-r-i}{n-r} \\ &= 1 - \binom{2n}{n}^{-1}\sum_{i=0}^{r-c-1}\binom{r-1+i}{r-1}\binom{2n-r-i}{n-r}. \end{aligned}$$

The last expression is more useful for computing if r is small and the middle one is more useful if c is small. A table (Table 3) of $r - c$ values satisfying (2.1) with $\alpha = \frac{1}{2}$, so that $r = (n+1)/2$, and selected values of n, k and P^* is given. Gaussian quadrature was also used to evaluate (3.3) in the preparation of Table 3 and (3.7) served as a check for $k = 2$.

4. Expected size of the selected subset. The size S of the subset selected by procedure R_1 is, of course, a random variable and its expected value may be taken as a measure of the efficiency of R_1, with small values corresponding to high efficiency. In this section we derive an expression for $E\{S \mid R_1\}$ and show that the maximum value in Ω_1 is attained when the cdf's are all identical.

It is easily seen that for any c $(1 \leqq c \leqq r - 1)$

$$E\{S \mid R_1\} = \sum_{i=1}^{k} P\{F_{[i]} \text{ is included in the subset} \mid R_1\} = \sum_{i=1}^{k} \int_{-\infty}^{\infty} \prod_{j=1, j\neq i}^{k} H_{r-c,j}(y)\, dH_{r,i}(y). \tag{4.1}$$

For the special configuration (which we call the P-configuration)

$$F_{[1]}(y) = F_{[2]}(y) = \cdots = F_{[k-1]}(y), \tag{4.2}$$

this reduces to

$$E\{S \mid R_1, P\} = \int_{-\infty}^{\infty} H_{r-c,1}^{k-1}(y)\, dH_{r,k}(y) + (k-1) \int_{-\infty}^{\infty} H_{r-c,k}(y) H_{r-c,1}^{k-2}(y)\, dH_{r,1}(y) \tag{4.3}$$

and, if we set all of the $F_{[i]}$'s equal (the W-configuration), we obtain

$$E\{S \mid R_1, W\} = k \int_{-\infty}^{\infty} H_{r-c,k}^{k-1}(y)\, dH_{r,k}(y); \tag{4.4}$$

for c equal to the integer c-value satisfying (2.1) the right side of (4.4) is approximately kP^*.

To prove that $E\{S \mid R_1\}$ is a maximum when $F_{[1]}(y) = F_{[2]}(y) = \cdots = F_{[k]}(y)$ for all y, we first prove the

LEMMA. *For any two* cdf's $F_1(y)$ *and* $F_2(y)$ *such that* $F_1(y) \geqq F_2(y)$ *for all* y *we have* $F_i(y) = F_{[i]}(y)$ $(i = 1, 2)$ *and for any integer* c *with* $1 \leqq c \leqq r - 1$

$$H_{r-c,1}(y) H_{r,2}(y) - H_{r,1}(y) H_{r-c,2}(y) = Q \text{ (say)} \tag{4.5}$$

is non-increasing in $F_1(y)$ *for any fixed* $F_2(y)$.

PROOF. Differentiating Q with respect to $F_1(y)$, it is sufficient to show that

$$Q_1 = H_{r,2}(y)\, dH_{r-c,1}(y) - H_{r-c,2}(y)\, dH_{r,1}(y) \leqq 0. \tag{4.6}$$

We show (4.6) by letting $F_2(y) \to 0$ for all y, so that $Q_1 \to 0$, and showing that 0 is the maximum value of Q_1 for $F_1(y) \geqq F_2(y)$. For this purpose we differentiate Q_1 with respect to $F_2(y)$ and obtain (neglecting common factors)

$$\begin{aligned} Q_{12} &= F_1^{r-c-1}(y)[1 - F_1(y)]^{n-r+c} F_2^{r-1}(y)[1 - F_2(y)]^{n-r} \\ &\quad - F_1^{r-1}(y)[1 - F_1(y)]^{n-r} F_2^{r-c-1}(y)[1 - F_2(y)]^{n-r+c} \\ &= F_1^{r-c-1}(y)[1 - F_1(y)]^{n-r} F_2^{r-c-1}(y)[1 - F_2(y)]^{n-r} \\ &\quad \cdot \{[1 - F_1(y)]^c F_2^c(y) - F_1^c(y)[1 - F_2(y)]^c\}. \end{aligned} \tag{4.7}$$

It follows from (4.7) that $Q_{12} \leqq 0$ whenever $F_2(y) \leqq F_1(y)$ for all y and this proves the lemma.

THEOREM. $E\{S\}$ *is a maximum in* Ω_1 *when*

$$F_{[1]}(y) = F_{[2]}(y) = \cdots = F_{[k]}(y) \quad \textit{for all} \quad y. \tag{4.8}$$

PROOF. Consider the pair $F_{[i]}$ and $F_{[k]}$ for any particular $i \neq k$; we wish to show that if $F_{[i]}(y)$ decreases on any subset of y values without crossing $F_{[i+1]}(y)$, then $E\{S\}$ is not decreased. Integrating by parts in the general expression (4.1) for $E\{S\}$ gives

$$
\begin{aligned}
E\{S\} &= \sum_{j=1}^{k-1} \int_{-\infty}^{\infty} \prod_{\alpha=1,\,\alpha\neq j}^{k} H_{r-c,\,\alpha}(y)\, dH_{r,j}(y) \\
&\qquad\qquad + \int_{-\infty}^{\infty} \prod_{\alpha=1}^{k-1} H_{r-c,\,\alpha}(y)\, dH_{r,k}(y) \\
&= 1 - \sum_{j=1}^{k-1} \int_{-\infty}^{\infty} \prod_{\alpha=1,\,\alpha\neq j}^{k-1} H_{r-c,\,\alpha}(y)[H_{r,k}(y)\, dH_{r-c,j}(y) \\
&\qquad\qquad - H_{r-c,k}(y)\, dH_{r,j}(y)].
\end{aligned} \tag{4.9}
$$

For fixed $F_{[k]}(y)$ we can disregard all terms except $j = i$ since in the jth term $(j \neq i)$ if $F_{[i]}(y)$ decreases then $H_{r-c,i}(y)$ decreases and hence $E\{S\}$ increases. Hence we need only show that

$$\int_{-\infty}^{\infty} \prod_{\alpha=1,\alpha\neq i}^{k-1} H_{r-c,\alpha}(y)[H_{r,k}(y)\, dH_{r-c,i}(y) - H_{r-c,k}(y)\, dH_{r,i}(y)] \tag{4.10}$$

is a non-decreasing function of $F_{[i]}(y)$ for fixed $F_{[k]}(y)$ in Ω_1. Integrating by parts again we find that it is sufficient to show that both of the functions

$$H_{r-c,i}(y)H_{r,k}(y) - H_{r,i}(y)H_{r-c,k}(y), \tag{4.11}$$

$$H_{r-c,i}(y)\, dH_{r,k}(y) - H_{r,i}(y)\, dH_{r-c,k}(y), \tag{4.12}$$

and non-increasing functions of $F_{[i]}(y)$ for fixed $F_{[k]}(y)$ in Ω_1. This was shown for (4.11) in the lemma; to show this for (4.12) we differentiate (4.12) with respect to $F_{[i]}(y)$ and the proof is exactly the same as in (4.7). This completes the proof of the theorem.

5. Asymptotic results and asymptotic relative efficiency (ARE). Asymptotic $(n \to \infty)$ expressions for $\inf_{\Omega_1} P\{\text{CS} \mid R_1\}$ and $E\{S \mid R_1\}$ will be obtained for fixed P^*. A definition is given for the asymptotic relative efficiency ARE $(R', R''; A)$ of a procedure R' relative to another procedure R'' under the alternatives A. Our procedure R_1 for $\alpha = \frac{1}{2}$ is then compared to others, e.g. the one based on the sample means, for different translation alternatives including normal shift alternatives.

We now hold k and P^* fixed and let $n \to \infty$ with $r/n \to \alpha$ where $0 < \alpha < 1$; let $\beta = c/(n + 1) = \gamma/n^{\frac{1}{2}}$ (only γ will be fixed below). Let $N(\mu, \sigma^2)$ denote a normal chance variable with mean μ and variance σ^2, let $\Phi(x)$ denote the standard normal cdf and let $\bar{\alpha} = 1 - \alpha$. We shall use a simple consequence of a well-known result on the asymptotic distribution of sample quantiles which we state as

Lemma 1. *If U_n has the distribution $G_{r-c}(u)$ with parameters $r - c \approx n(\alpha - \beta) = \alpha n - \gamma n^{\frac{1}{2}}$ and $n - r + 1 \approx (1 - \alpha)n + \gamma n^{\frac{1}{2}}$ then $Y_n = n^{\frac{1}{2}}(U_n - \alpha)$ is asymptotically $N(-\gamma, \alpha\bar{\alpha})$.*

If we set $c = \gamma = \beta = 0$ in Lemma 1 then we obtain

Corollary 1. *If the distribution $G_r(u)$ has parameters $r \approx n\alpha$ and $n - r + 1 \approx n(1 - \alpha)$ then*

$$G_r(u) \approx \Phi((u - \alpha)n^{\frac{1}{2}}/(\alpha\bar{\alpha})^{\frac{1}{2}}). \tag{5.1}$$

PROOF OF LEMMA 1. The standard proof that holds for fixed values of $\alpha - \beta$ also shows that for $\alpha - \beta = \alpha - \gamma/n^{\frac{1}{2}} \to \alpha$ $(0 < \alpha < 1)$

$$W_n = n^{\frac{1}{2}}(U_n - \alpha + \gamma/n^{\frac{1}{2}})/(\alpha\bar{\alpha})^{\frac{1}{2}} = (Y_n + \gamma)/(\alpha\bar{\alpha})^{\frac{1}{2}}$$

is asymptotically $N(0, 1)$ and it follows that Y_n is asymptotically $N(-\gamma, \alpha\bar{\alpha})$.

Letting $u = \alpha + y(\alpha\bar{\alpha}/n)^{\frac{1}{2}}$, we now use the above lemma and corollary to obtain

$$G_{r-c}(u) = P\{U_n \leqq u\} = P\{Y_n \leqq y(\alpha\bar{\alpha})^{\frac{1}{2}}\} \approx \Phi(y + \gamma/(\alpha\bar{\alpha})^{\frac{1}{2}})$$

and since $G_r(u) \approx \Phi(y)$ we have from (3.3)

$$(5.2) \qquad \lim_{n\to\infty} \inf_{\Omega_1} P\{\mathrm{CS} \mid R_1\} = \int_{-\infty}^{\infty} \Phi^{k-1}(y + \gamma/(\alpha\bar{\alpha})^{\frac{1}{2}})\, d\Phi(y).$$

Since this quantity is set equal to the fixed constant $P^* < 1$, we now see why $\gamma = \beta n^{\frac{1}{2}}$ is a fixed constant so that $\beta \to 0$ (like $n^{-\frac{1}{2}}$) and $c \to \infty$ (like $n^{\frac{1}{2}}$). Moreover if $s = s(k, P^*)$ is the (unique) root of

$$(5.3) \qquad \int_{-\infty}^{\infty} \Phi^{k-1}(y + s)\, d\Phi(y) = P^*,$$

(s or $H = s/2^{\frac{1}{2}}$ has been tabulated by Bechhofer [2], Gupta [4] and Milton [6]) then the rate at which $\beta \to 0$ and $c \to \infty$ is given by

$$(5.4) \qquad \beta n^{\frac{1}{2}} = cn^{\frac{1}{2}}/(n + 1) \approx s(\alpha\bar{\alpha})^{\frac{1}{2}} \qquad (n \to \infty).$$

Treating the latter part of (5.4) as an equality and solving for c we can make the procedure R_1 explicit and (2.1) will be satisfied for large n. A small table of c-values based on (5.4) is numerically compared below with the corresponding exact results based on (3.7) and excerpted from Table 3. Here the approximate c-values are given to one decimal and one would use the largest integer contained in this decimal to be conservative; the asymptotic approximations are quite good (and appear to be equally good) for each of the P^*-values investigated, $P^* = .90, .95$ and $.99$; in these empirical results the approximation never differs from the corresponding exact values by more than one and, when there is a difference of one, it is always on the conservative side.

It is useful to note that for each value of P^* in Table 1 and in Table 3 the entries $g(n)$ seem to approach an arithmetic progression as n increases. Setting $\beta \approx c/n$ and $r \approx n/2$ we obtain for $g(n) = r - c$ asymptotically $(n \to \infty,$ $r/n \to \frac{1}{2})$

$$(5.5) \qquad g(n) \approx \tfrac{1}{2}n(1 - s/n^{\frac{1}{2}})$$

where $s = s(k, P^*)$ is given by (5.3). Hence if we let Δ_0 denote the common difference in the n-values in Table 3, we have asymptotically $(n \to \infty, r/n \to \frac{1}{2})$

$$(5.6) \quad g(n + \Delta_0) - g(n) \approx \tfrac{1}{2}\Delta_0 - \tfrac{1}{2}s[(n + \Delta_0)^{\frac{1}{2}} - n^{\frac{1}{2}}] \approx \tfrac{1}{2}\Delta_0(1 - s/2n^{\frac{1}{2}}) \approx \tfrac{1}{2}\Delta_0$$

and this result is independent of both k and P^*.

We now examine $E\{S \mid R_1\}$ under the same limiting operation, assuming that the $F_{[i]}$'s differ only in a location parameter θ. Particular subclasses of this

TABLE 1

Comparison of Approximate and Exact Values of $r - c$*

The 3 entries in each cell correspond to $P^* = .90, .95$ and $.99$ respectively.
(Based on (5.4) and (3.3) with $\alpha = \frac{1}{2}$ and $r = (n+1)/2$)

n	$k = 2$	$k = 3$	$k = 4$	$k = 6$	$k = 8$	$k = 10$
45	16.8 (16) 15.0 (15) 11.7 (12)	15.4 (15) 13.7 (14) 10.6 (11)	14.6 (14) 13.0 (13) 10.0 (10)	13.7 (14) 12.2 (12) 9.2 (10)	13.2 (13) 11.7 (12) 8.8 (9)	12.8 (13) 11.3 (11) 8.4 (9)
95	39.1 (39) 36.5 (36) 31.8 (32)	37.0 (37) 34.6 (34) 30.2 (30)	35.9 (36) 33.6 (33) 29.3 (29)	34.7 (34) 32.4 (32) 28.2 (28)	33.9 (34) 31.7 (32) 27.6 (28)	33.3 (33) 31.2 (31) 27.1 (27)
145	62.0 (62) 58.9 (59) 53.1 (53)	59.5 (59) 56.6 (56) 51.1 (51)	58.1 (58) 55.3 (55) 50.0 (50)	56.6 (56) 53.8 (54) 48.7 (49)	55.6 (55) 52.9 (53) 47.9 (48)	54.9 (55) 52.3 (52) 47.3 (47)
195	85.3 (85) 81.7 (81) 74.9 (75)	82.3 (82) 79.0 (79) 72.6 (72)	80.8 (80) 77.5 (77) 71.3 (71)	79.0 (79) 75.8 (76) 69.8 (70)	77.9 (78) 74.8 (75) 68.9 (69)	77.1 (77) 74.0 (74) 68.2 (68)
295	132.4 (132) 128.0 (128) 119.7 (119)	128.8 (128) 124.6 (124) 116.8 (117)	126.9 (126) 122.9 (123) 115.3 (115)	124.6 (124) 120.8 (120) 113.4 (113)	123.3 (123) 119.5 (119) 112.3 (112)	122.3 (122) 118.5 (118) 111.4 (111)
395	179.9 (180) 174.8 (174) 165.2 (165)	175.8 (175) 171.0 (171) 162.0 (162)	173.6 (173) 168.9 (169) 160.2 (160)	171.0 (171) 166.5 (166) 158.0 (158)	169.4 (169) 165.0 (165) 156.7 (157)	168.3 (168) 163.9 (164) 155.7 (156)
495	227.8 (227) 222.1 (222) 211.3 (211)	223.1 (223) 217.8 (217) 207.7 (207)	220.7 (220) 215.5 (215) 205.7 (205)	217.8 (218) 212.8 (213) 203.3 (203)	216.0 (216) 211.1 (211) 201.8 (201)	214.8 (215) 209.9 (210) 200.7 (200)

* The exact values are in parentheses.

class, e.g., normal shift alternatives (NA), will be considered in the ARE discussions below. We consider the P-configuration in which $\theta_{[1]} = \theta_{[2]} = \cdots = \theta_{[k-1]} = -\theta/n^{\frac{1}{2}}$ (say) and, for convenience, we set $\theta_{[k]} = 0$, i.e., we set

$$\begin{aligned} F_{[i]}(x) &= F(x) && \text{for} \quad i = k \\ &= F_1(x) = F(x + \theta/n^{\frac{1}{2}}) && \text{for} \quad i < k. \end{aligned}$$

We assume that F has a continuous derivative $f > 0$ at the (unique) point λ for which $F(\lambda) = \alpha$. If $\lambda_i = \lambda_i(\alpha)$ is the αth quantile of $F_{[i]}$ then clearly $\lambda_k = \lambda$, $\lambda_i = \lambda_1 = \lambda - \theta/n^{\frac{1}{2}}$ for $i < k$ and $f_i(\lambda_i) = f(\lambda)$. We shall also make use of

LEMMA 2. *If X_n is the $(\alpha - \gamma/n^{\frac{1}{2}})$th sample quantile in a random sample of size n from a population with distribution function F having a continuous derivative $f > 0$ at the (unique) point $\lambda = \lambda(\alpha)$ for which $F(\lambda) = \alpha$ $(0 < \alpha < 1)$, then $Y_n' = n^{\frac{1}{2}}f(\lambda)(X_n - \lambda)$ is asymptotically $N(-\gamma, \alpha\bar{\alpha})$.*

PROOF. For convenience we shall write $\lambda(\alpha)$ as λ_α in this proof since there is no danger of confusion. It is well known that

$$W_n' = f(\lambda_{\alpha-\beta})[n/(\alpha - \beta)(\overline{\alpha - \beta})]^{\frac{1}{2}}(X_n - \lambda_{\alpha-\beta}) \tag{5.7}$$

is asymptotically $N(0, 1)$; the standard proof that holds for $\alpha - \beta$ fixed also holds for $\alpha - \beta \to \alpha$ $(0 < \alpha < 1)$. Since $f(t)$ is continuous at $\lambda(\alpha)$

$$\alpha - \beta = \int_{-\infty}^{\lambda_{\alpha-\beta}} f(t)\,dt = \alpha - \int_{\lambda_{\alpha-\beta}}^{\lambda_\alpha} f(t)\,dt \approx \alpha - (\lambda_\alpha - \lambda_{\alpha-\beta})f(\lambda_\alpha)$$

and since $\beta \to 0$ as $n \to \infty$ we can now write

$$\lambda_{\alpha-\beta} = \lambda_\alpha - \beta/f(\lambda_\alpha) = \lambda_\alpha - \gamma/f(\lambda_\alpha)n^{\frac{1}{2}}.$$

Substituting this in (5.7) and using again the continuity of $f(x)$ at $x = \lambda_\alpha$ and the fact that $\beta \to 0$, we obtain the asymptotic $(n \to \infty, \beta \to 0)$ equivalence

$$W_n' \approx (Y_n' + \gamma)/(\alpha\bar{\alpha})^{\frac{1}{2}},$$

which yields the desired result that Y_n' is asymptotically $N(-\gamma, \alpha\bar{\alpha})$.

Applying Lemma 2 to $F_{[i]}$ in (4.3) we have for the sample quantile $X_{n,i}$ with cdf $H_{r-c,i}$ for each $i < k$

$$\begin{aligned} H_{r-c,i}(w/n^{\frac{1}{2}} + \lambda) &= P\{n^{\frac{1}{2}}(X_{n,i} - \lambda_i)f(\lambda) \leqq (w + \theta)f(\lambda)\} \\ &\approx \Phi(((w + \theta)f(\lambda) + \gamma)/(\alpha\bar{\alpha})^{\frac{1}{2}}) = \Phi(z + d + s) \end{aligned} \tag{5.8}$$

where $z = wf(\lambda)/(\alpha\bar{\alpha})^{\frac{1}{2}}$, $d = \theta f(\lambda)/(\alpha\bar{\alpha})^{\frac{1}{2}}$ and, as above $s = \gamma/(\alpha\bar{\alpha})^{\frac{1}{2}}$. Likewise for each $i < k$, $H_{r,i}(w/n^{\frac{1}{2}} + \lambda) \approx \Phi(z + d)$. For $i = k$ the same results are valid with $\theta = d = 0$. Hence setting $y = \lambda + w/n^{\frac{1}{2}}$ in (4.3), writing the first integral in terms of z and the second in terms of $z' = z + d$ we obtain (after dropping primes)

$$\begin{aligned} \lim_{n\to\infty} E\{S \mid R_1, P\} &= \int_{-\infty}^{\infty} \Phi^{k-1}(z + d + s)\,d\Phi(z) \\ &\quad + (k - 1)\int_{-\infty}^{\infty} \Phi(z - d + s)\Phi^{k-2}(z + s)\,d\Phi(z). \end{aligned} \tag{5.9}$$

For the W-configuration, in accordance with the remark after (4.4)

$$\lim_{n\to\infty} E\{S \mid R_1, W\} = kP^*.$$

Denote the right side of (5.9) by $K(d)$; we see that it is a decreasing function of d or θ (by differentiation) and that $K(\infty) = 1$. We now seek a simple explicit expression for d_ϵ such that $K(d_\epsilon) = 1 + \epsilon$ where $\epsilon > 0$ is small. Then, setting $\Delta = \theta/n^{\frac{1}{2}}$ in the definition of d above, we use the resulting

$$n_1(\epsilon) = d_\epsilon^2\alpha\bar{\alpha}/\Delta^2 f^2(\lambda) \tag{5.10}$$

as an approximation to the sample size required to satisfy

$$E\{S \mid R_1, P_\Delta\} \leqq 1 + \epsilon \tag{5.11}$$

where P_Δ is the P-configuration with $\theta_{[k]} - \theta_{[i]} = \Delta$ $(i = 1, 2, \cdots, k - 1)$. It should be noted that $K(d)$ is an asymptotic formula for $E\{S \mid R_1, P\}$ obtained by holding d (and θ) fixed and we are now evaluating it for large d.

From (5.9) and (5.11) we derive (5.12) below by using twice (on the first line and the fourth line) the fact that $(1 - x)^k \approx 1 - kx$ for x small and also using twice (in deriving the second line and the last expression on the fifth line) the first term of the Feller-Laplace (or Mill's ratio) normal approximation $\Phi(-z) \approx \varphi(z)/z$ for large z; here $\varphi(z)$ denotes the standard normal density. Thus we have (using $\approx$ to denote an approximation) for large d

$$\begin{aligned} K(d) - 1 &\approx (k-1)\{\textstyle\int_{-\infty}^{\infty} \Phi(y - d + s)\Phi^{k-2}(y+s)\, d\Phi(y) \\ &\qquad - \Phi((-s-d)/2^{\frac{1}{2}})\} \\ &\approx (k-1)\{\textstyle\int_{-\infty}^{\infty} \Phi^{k-2}(y+s)[\varphi(y-d+s)/d]\, d\Phi(y) \\ &\qquad - \Phi((-s-d)/2^{\frac{1}{2}})\} \\ &\approx [(k-1)/d2^{\frac{1}{2}}]\{\varphi((s-d)/2^{\frac{1}{2}}) \textstyle\int_{-\infty}^{\infty} \Phi^{k-2}(x/2^{\frac{1}{2}} \\ &\qquad + (s+d)/2)\, d\Phi(x) - d2^{\frac{1}{2}}\Phi((-s-d)/2^{\frac{1}{2}})\} \\ &\approx [(k-1)/d2^{\frac{1}{2}}]\{\varphi((s-d)/2^{\frac{1}{2}})[1 - (k-2)\Phi((-s-d)/6^{\frac{1}{2}})] \\ &\qquad - d2^{\frac{1}{2}}\Phi((-s-d)/2^{\frac{1}{2}})\} \\ &\approx [(k-1)/d2^{\frac{1}{2}}]\varphi((s-d)/2^{\frac{1}{2}}) \approx \tfrac{1}{2}(k-1)\Phi((s-d)/2^{\frac{1}{2}}). \end{aligned} \tag{5.12}$$

If we set this equal to $\epsilon > 0$ then the approximate value $n_1(\epsilon)$ of n required by procedure R_1 (for ϵ small) to satisfy (5.11) is given by

$$n_1(\epsilon) \approx \alpha\bar{\alpha}(s - \lambda_{\epsilon'}^{(0)}2^{\frac{1}{2}})^2/\Delta^2 f^2(\lambda) \tag{5.13}$$

where $\epsilon' = 2\epsilon/(k-1)$ and $\lambda_{\epsilon'}^{(0)}$ is the ϵ'-quantile of Φ.

We now apply the same analysis to the procedure R_2 based on sample means (see Gupta [3] for a detailed development and a fairly complete set of references). We assume that the $F_{[i]}$ have a finite, known and (for simplicity) common variance σ^2. To ensure that procedure R_2 is accomplishing the same goal as R_1, we assume that the sample mean $\bar{x}_{(i)}$ from the population with cdf $F_{[i]}$ has an asymptotic normal distribution with mean $\theta_{[i]}$ where $\theta_{[1]} \leqq \theta_{[2]} \leqq \cdots \leqq \theta_{[k]}$; this certainly holds if we are dealing with a location parameter.

The procedure R_2 puts the ith population (which gave rise to $\bar{x}_i$) in the selected subset iff

$$\bar{x}_i < \max_{1 \leqq j \leqq k} \bar{x}_j - \delta$$

where $\delta > 0$ is chosen to satisfy the P^*-condition (2.1). For specified $P^* < 1$ we find that $\delta = s\sigma/n^{\frac{1}{2}}$ where $s = s(k, P^*)$ is defined by (5.3). For the P-configuration with $\theta_{[k]} = 0$ and $\theta > 0$ we obtain as in (5.9) and (5.12), letting $d' = \theta/\sigma > 0$,

$$\begin{aligned} \lim_{n\to\infty} E\{S \mid R_2, P\} &= \textstyle\int_{-\infty}^{\infty} \Phi^{k-1}(y + d' + s)\, d\Phi(y) \\ &\quad + (k-1)\textstyle\int_{-\infty}^{\infty} \Phi(y - d' + s)\Phi^{k-2}(y+s)\, d\Phi(y), \end{aligned} \tag{5.14}$$

$$E\{S \mid R_2, P\} - 1 \approx \tfrac{1}{2}(k-1)\Phi((s-d')/2^{\frac{1}{2}}) \approx K(d') - 1 \tag{5.15}$$

As before we let $\Delta = \theta/n_2^{\frac{1}{2}} = \sigma d'/n_2^{\frac{1}{2}}$, set d' equal to the value d_ϵ' such that

$K(d') = 1 + \epsilon$ and solve for $n_2 = n_2(\epsilon)$; for ϵ small we obtain

$$n_2(\epsilon) \approx (s - \lambda_\epsilon^{(0)} 2^{\frac{1}{2}})^2 \sigma^2 / \Delta^2. \tag{5.16}$$

For translation alternatives (TA) we define the asymptotic relative efficiency ARE $(R', R''; \text{TA})$ of R' relative to R'' to be the limit as $\epsilon \to 0$ of the ratio of $n''(\epsilon)$ to $n'(\epsilon)$; hence for R_1 and R_2 we have

$$\text{ARE}\,(R_1, R_2; \text{TA}) = \lim_{\epsilon \to 0} [n_2(\epsilon)/n_1(\epsilon)] = \sigma^2 f^2(\lambda)/\alpha\bar{\alpha}, \tag{5.17}$$

which is independent of θ for any translation alternatives.

For $\alpha = \frac{1}{2}$ and normal shift alternatives (NA) with $\sigma = 1$ we obtain from (5.17)

$$\text{ARE}\,(R_1, R_2; \text{NA}) = 2/\pi. \tag{5.18}$$

For $\alpha = \frac{1}{2}$ and two sided exponential shift alternatives (TEA) with continuous symmetric densities about the median value θ_i, we obtain

$$\text{ARE}\,(R_1, R_2; \text{TEA}) = 2; \tag{5.19}$$

i. e., R_1 is asymtotically twice as efficient as the procedure R_2 under TEA.

We can also compare R_1 with both non-randomized and randomized rank sum (or score) procedures which have been applied to the problem of selecting a subset containing the best population by Bartlett [1]. The basic procedures are defined in Lehmann [5] in terms of a function $J(x)$ defined for $0 \leqq x \leqq 1$, whose inverse $J^{(-1)}(x) = H(x)$ is a cdf. For some fixed function $J(x)$ we can define the score for the ith population by

$$S'_{N,i} = n^{-1} \sum_{j=1}^{n} J(R_{ij}/(N + 1)) \qquad (i = 1, 2, \cdots, k), \tag{5.20}$$

where $R_{i,j}$ denotes the rank of the jth observation $X_{i,j}$ from the ith population $\prod_i$ in the combined sample of N observations; we are assuming that there is a common number from each population so that $N = kn$. Then the non-randomized procedure $R' = R'(J)$ based on these scores is to put $\prod_i$ in the selected subset iff

$$S'_{N,i} > \max_{1 \leqq j \leqq k} S'_{N,j} - c', \tag{5.21}$$

where c' is a positive number chosen to satisfy the P^*-condition (2.1). The randomized procedure R'' is similarly defined in terms of randomized scores but, since the asymptotic properties of R'' are equivalent to those of R', we need not define R'' here. The cdf's $F_i(x)$ are again assumed to be of the form $F(x - \theta_i)$, so that we are dealing with the selection problem for a location parameter. We shall use the result from [1] that for asymptotically small d^* (our Δ), the common (large) sample size n' that would be needed for R' (and also for R'') to satisfy both (2.1) and (5.11) is

$$n' \approx [A\, d_\epsilon/\Delta \int_{-\infty}^{\infty} (d/dx)\{J(F(x))\}\, dF(x)]^2, \tag{5.22}$$

where d_ϵ in (5.22) is the root in d of the equation obtained by setting the right side of (5.9) equal to $1 + \epsilon$, and where

$$A^2 = \int_0^1 J^2(u)\,du - \left(\int_0^1 J(u)\,du\right)^2. \tag{5.23}$$

It should be noted that n' in (5.22) represents the value of n that would be needed if one knew that the true configuration were given by $\theta_{[k]} - \theta_{[i]} = \Delta$ ($i = 1, 2, \cdots, k$) for some fixed Δ, not depending on n. In [1] the limiting process on n is equivalent to ours with fixed $\epsilon > 0$ and fixed $\theta = d_\epsilon(\alpha\bar{\alpha})^{\frac{1}{2}}/f(\lambda)$ as in (5.8). The value $n_1(\epsilon)$ of n_1 required by procedure R_1 to satisfy (5.11) for the P_Δ configuration is given by (5.10) with d_ϵ the same as in (5.22). Hence for any translation alternatives (TA) we have from (5.10) and (5.22)

$$\text{ARE}\,(R_1, R'(J); \text{TA}) = A^2 f(\lambda)/\alpha\bar{\alpha}\left[\int_{-\infty}^{\infty} (d/dx)\{J(F(x))\}\,dF(x)\right]^2 \tag{5.24}$$

and it is interesting to note that this is independent of ϵ.

If we take $J^{(-1)} = H = \Phi$ and consider normal alternatives (NA) then (5.24) with $\alpha = \frac{1}{2}$ gives the same result as in (5.18)

$$\text{ARE}\,(R_1, R'(\Phi); \text{NA}) = 2/\pi. \tag{5.25}$$

For the same H if we consider TEA then (5.24) with $\alpha = \frac{1}{2}$ gives, after using symmetry and straightforward methods of integration,

$$\text{ARE}\,(R_1, R'(\Phi); \text{TEA}) = (\tfrac{1}{2})^2/\tfrac{1}{4}(2/\pi) = \pi/2. \tag{5.26}$$

If we take $J^{(-1)} = H = U$, the standard uniform distribution, and consider logistic alternatives (LA) then (5.24) with $\alpha = \frac{1}{2}$ gives

$$\text{ARE}\,(R_1, R'(U); \text{LA}) = \tfrac{1}{12}(\tfrac{1}{4}\sigma_1)^2/\tfrac{1}{4}(\tfrac{1}{6}\sigma_1)^2 = \tfrac{3}{4}; \tag{5.27}$$

for the LA it is known that this "score statistic" with $J^{(-1)} = H = U$ reduces to the Wilcoxon statistic and gives the locally most powerful test. For the same H if we consider the (TEA) then (5.24) with $\alpha = \frac{1}{2}$ gives

$$\text{ARE}\,(R_1, R'(U); \text{TEA}) = \tfrac{1}{12}(\tfrac{1}{2})^2/\tfrac{1}{4}(\tfrac{1}{4})^2 = \tfrac{4}{3}. \tag{5.28}$$

We note with curiosity (but without explanation) that the last two numbers are reciprocals and the two numbers before that are also reciprocals.

It can be shown (we omit the proof) that the ARE in both (5.17) and (5.24), considered as a function of α for $0 \leqq \alpha \leqq 1$, is a maximum for $\alpha = \frac{1}{2}$ for all three of the alternatives considered above, i.e., for NA, LA and TEA. A sufficient condition for this is that $f(x)$ be symmetric about $x = 0$ and $f^2(x)/[F(x)(1 - F(x))]$ be decreasing for $x \geqq 0$; these properties hold in each of the three different alternatives considered above.

6. Dual problem. Consider the dual problem of selecting a subset containing the population with the smallest α-quantile. For specified $P^* > 1/k$ we now require a procedure R_1' that satisfies

$$P\{\text{CS} \mid R_1'\} \geqq P^* \tag{6.1}$$

for all possible k-tuples $(F_1, F_2, \cdots, F_k)$ for which

$$\max_{1 \leq j \leq k} F_{[j]}(y) = F_{[1]}(y), \quad \text{for all} \quad y; \tag{6.2}$$

denote the set of all such k-tuples by Ω_1'. We again assume (2.3) and define r by (2.4). The procedure dual to procedure R_1 of section 2 is

PROCEDURE R_1'. Put F_i in the selected subset if and only if

$$Y_{r,i} \leqq \min_{1 \leq j \leq k} Y_{r+c,j} \tag{6.3}$$

where c is the smallest integer with $1 \leqq c \leqq n - r$ for which (6.1) is satisfied.

It will be convenient to use also the notation $H_r(F_{[i]}(y))$ for $H_{r,i}(y)$. For this problem Equation (3.1) is replaced by

$$\begin{aligned} P\{\text{CS} \mid R_1'\} &= \int_{-\infty}^{\infty} \prod_{j=2}^{k} \{1 - H_{r+c,j}(y)\}\, dH_{r,1}(y) \\ &= \int_{-\infty}^{\infty} \prod_{j=2}^{k} H_{n-r+1-c}(1 - F_{[j]}(y))\, dH_{n-r+1}(1 - F_{[1]}(y)). \end{aligned} \tag{6.4}$$

If the $F_i(y)$ are all symmetric about $y = 0$ then for any k, by making the transformation $s = -y$, we obtain

$$P\{\text{CS} \mid R_1'\} = \int_{-\infty}^{\infty} \prod_{j=2}^{k} H_{n-r+1-c}(F_{[j]}(x))\, dH_{n-r+1}(F_{[1]}(x)) \tag{6.5}$$

and we note that the problem is equivalent to selecting a subset containing the largest $(1 - \alpha)$-quantile $(\alpha \neq \frac{1}{2})$.

Equation (3.3) is replaced by

$$\begin{aligned} \inf_{\Omega_1'} P\{\text{CS} \mid R_1'\} &= \int_0^1 [1 - G_{r+c}(u)]^{k-1}\, dG_r(u) \\ &= \int_0^1 G_{n-r+1-c}^{k-1}(u)\, dG_{n-r+1}(u). \end{aligned} \tag{6.6}$$

If $\alpha = \frac{1}{2}$ then $n - r + 1 = r$ and hence for any k we note that (6.6) reduces to (3.3). Hence for $\alpha = \frac{1}{2}$ the tables for determining c are the same for both problems; this is not true in general for $\alpha \neq \frac{1}{2}$. In particular, the tables we have computed for $\alpha = \frac{1}{2}$ can also be used for the problem of selecting a subset containing the population with the smallest median.

Other discussions for this problem are similar to those of the dual problem and are omitted.

7. Property of monotonicity of unbiasedness. Let p_i denote the probability that R_1 retains $F_{[i]}$ in the selected subset.

THEOREM. *For any two* cdf's $F_i(\cdot)$ *and* $F_j(\cdot)$ *such that*

$$F_i(y) = F_{[i]}(y) \geqq F_{[j]}(y) = F_j(y)$$

for all y we have

$$p_i \leqq p_j. \tag{7.1}$$

PROOF. Letting M denote the set $\{m : m = 1, 2, \cdots, k; m \neq i, m \neq j\}$ we use the fact that $H_{r,i}(y)$ is an increasing function of $F_{[i]}(y)$ and then integrate by parts to obtain

$$
\begin{aligned}
p_i - p_j &= \int_{-\infty}^{\infty} [\prod_{m \varepsilon M} H_{r-c,m}(y)]\{H_{r-c,j}(y)\, dH_{r,i}(y) \\
&\quad - H_{r-c,i}(y)\, dH_{r,j}(y)\} \\
&\leqq \int_{-\infty}^{\infty} [\prod_{m \varepsilon M} H_{r-c,m}(y)]\{H_{r-c,i}(y)\, dH_{r,i}(y) \\
&\quad - H_{r-c,i}(y)\, dH_{r,j}(y)\} \\
&\leqq \int_{-\infty}^{\infty} [\prod_{m \varepsilon M} H_{r-c,m}(y)]\{H_{r,j}(y) - H_{r,i}(y)\}\, dH_{r-c,i}(y) \\
&\quad + \sum_{m' \varepsilon M} \int_{-\infty}^{\infty} [\prod_{m \varepsilon M, m \neq m'} H_{r-c,m}(y)] H_{r-c,i}(y) \{H_{r,j}(y) \\
&\quad - H_{r,i}(y)\}\, dH_{r-c,m'}(y).
\end{aligned} \tag{7.2}
$$

Since $F_{[i]}(y) \geqq F_{[j]}(y)$ for all y it follows that $H_{r,i}(y) \geqq H_{r,j}(y)$ for all y and every term in the last member of (7.2) is non-positive; this proves the theorem.

It follows that under our assumption (2.2) the probability of including $F_{[k]}$ in the selected subset is not less than the probability of including any other $F_{[j]}$ in the selected subset, i.e., the procedure R_1 is unbiased.

A similar result holds for the dual problem; the proof is similar to the above proof and is omitted.

TABLE 2

Values† of $P_1(n)$ for $\alpha = \frac{1}{2}$ as a function of k and n

n	k								
	2	3	4	5	6	7	8	9	10
1	.50000	.33333	.25000	.20000	.16667	.14286	.12500	.11111	.10000
3	.80000	.68333	.60455	.54675	.50204	.46612	.43646	.41143	.38995
5	.91667	.85531	.80716	.76779	.73467	.70620	.68132	.65929	.63959
7	.96503	.93592	.91094	.88906	.86959	.85206	.83611	.82150	.80801
9	.98529	.97215	.96023	.94931	.93921	.92982	.92103	.91276	.90496
11	.99381	.98803	.98261	.97749	.97263	.96801	.96359	.95936	.95530
13	.99739	.99489	.99250	.99019	.98796	.98580	.98371	.98168	.97970
15	.99890	.99783	.99679	.99577	.99478	.99381	.99285	.99192	.99101
17	.99954	.99908	.99863	.99819	.99776	.99733	.99691	.99649	.99608
19	.99980	.99961	.99942	.99923	.99904	.99886	.99867	.99849	.99831
21	.99992	.99984	.99975	.99967	.99959	.99951	.99943	.99936	.99928
23	.99997	.99993	.99990	.99986	.99983	.99979	.99976	.99973	.99969
25	.99999	.99997	.99996	.99994	.99993	.99991	.99990	.99988	.99987
27		.99999	.99998	.99998	.99997	.99996	.99996	.99995	.99994
29			.99999	.99999	.99999	.99998	.99998	.99998	.99998
31						.99999	.99999	.99999	.99999

For example, for $k = 2$ and $P^* = .99$ it follows from the above table that we require at least $n = 11$ observations from each of $k = 2$ populations to be able to find a value of c or $r - c$ to satisfy the basic requirement (2.1).

† Based on (3.5) with $r = (n + 1)/2$ for $\alpha = \frac{1}{2}$ by (2.4).

TABLE 3

Largest values† of r − c for which $P\{CS \mid R_1\} \geqq P^*$ *for* $\alpha = \frac{1}{2}$

n	r	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$
		$P^* = .750$								
5	3	1	1	1	1	0※	0※	0※	0※	0※
15	8	6	5	4	4	4	4	3	3	3
25	13	10	9	8	8	8	7	7	7	7
35	18	15	13	13	12	12	11	11	11	11
45	23	19	18	17	16	16	16	15	15	15
55	28	24	22	21	21	20	20	20	19	19
65	33	29	27	26	25	25	24	24	24	23
75	38	33	31	30	30	29	29	28	28	28
85	43	38	36	35	34	33	33	33	32	32
95	48	43	41	39	39	38	37	37	37	37
145	73	67	64	62	61	61	60	60	59	59
195	98	91	87	86	85	84	83	83	82	82
245	123	115	111	109	108	107	106	106	105	105
295	148	139	135	133	132	131	130	129	129	128
345	173	164	159	157	155	154	153	153	152	152
395	198	188	183	181	179	178	177	176	176	175
445	223	212	207	205	203	202	201	200	199	199
495	248	237	231	229	227	225	224	224	223	222
		$P^* = .900$								
5	3	1	0※	0※	0※	0※	0※	0※	0※	0※
15	8	4	3	3	3	3	2	2	2	2
25	13	8	7	7	6	6	6	6	5	5
35	18	12	11	10	10	10	9	9	9	9
45	23	16	15	14	14	14	13	13	13	13
55	28	21	19	19	18	18	17	17	17	17
65	33	25	24	23	22	22	21	21	21	21
75	38	30	28	27	26	26	26	25	25	25
85	43	34	32	31	31	30	30	29	29	29
95	48	39	37	36	35	34	34	34	33	33
145	73	62	59	58	57	56	56	55	55	55
195	98	85	82	80	79	79	78	78	77	77
245	123	108	105	103	102	101	101	100	100	99
295	148	132	128	126	125	124	124	123	122	122
345	173	156	152	150	148	147	147	146	145	145
395	198	180	175	173	172	171	170	169	169	168
445	223	203	199	197	195	194	193	192	192	191
495	248	227	223	220	219	218	217	216	215	215
		$P^* = .950$								
5	3	0※	0※	0※	0※	0※	0※	0※	0※	0※
15	8	3	3	2	2	2	2	2	2	2
25	13	7	6	6	5	5	5	5	5	4
35	18	11	10	9	9	8	8	8	8	8
45	23	15	14	13	13	12	12	12	12	11
55	28	19	18	17	16	16	16	16	15	15
65	33	23	22	21	20	20	20	19	19	19
75	38	28	26	25	24	24	24	23	23	23
85	43	32	30	29	29	28	28	27	27	27

TABLE 3—*Continued*

n	r	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$
		$P^* = .950$								
95	48	36	34	33	33	32	32	32	31	31
145	73	59	56	55	54	54	53	53	52	52
195	98	81	79	77	76	76	75	75	74	74
245	123	104	101	100	99	98	97	97	96	96
295	148	128	124	123	121	120	120	119	119	118
345	173	151	147	145	144	143	142	142	141	141
395	198	174	171	169	167	166	165	165	164	164
445	223	198	194	192	190	189	188	188	187	186
495	248	222	217	215	214	213	212	211	210	210
		$P^* = .975$								
5	3	0#	0#	0#	0#	0#	0#	0#	0#	0#
15	8	2	2	2	1	1	1	1	1	1
25	13	6	5	5	4	4	4	4	4	4
35	18	10	9	8	8	7	7	7	7	7
45	23	13	12	12	11	11	11	11	10	10
55	28	17	16	16	15	15	14	14	14	14
65	33	22	20	19	19	19	18	18	18	18
75	38	26	24	23	23	22	22	22	22	21
85	43	30	28	27	27	26	26	26	26	25
95	48	34	32	32	31	30	30	30	30	29
145	73	56	54	53	52	51	51	51	50	50
195	98	78	76	74	74	73	72	72	72	71
245	123	101	98	97	96	95	94	94	93	93
295	148	124	121	119	118	117	117	116	116	115
345	173	147	144	142	141	140	139	138	138	137
395	198	170	166	165	163	162	162	161	160	160
445	223	193	190	188	186	185	184	184	183	182
495	248	217	213	211	209	208	207	206	206	205
		$P^* = .990$								
5	3	0#	0#	0#	0#	0#	0#	0#	0#	0#
15	8	2	1	1	1	1	1	1	1	1
25	13	5	4	4	3	3	3	3	3	3
35	18	8	7	7	7	6	6	6	6	6
45	23	12	11	10	10	10	9	9	9	9
55	28	16	15	14	14	13	13	13	13	12
65	33	20	18	18	17	17	17	16	16	16
75	38	24	22	21	21	21	20	20	20	20
85	43	28	26	25	25	24	24	24	24	23
95	48	32	30	29	29	28	28	28	28	27
145	73	53	51	50	49	49	48	48	48	47
195	98	75	72	71	70	70	69	69	69	68
245	123	97	94	93	92	91	91	90	90	90
295	148	119	117	115	114	113	113	112	112	111
345	173	142	139	137	136	135	135	134	134	133
395	198	165	162	160	159	158	157	157	156	156
445	223	188	184	183	181	180	180	179	178	178
495	248	211	207	205	204	203	202	201	201	200

Degenerate cases in which all the populations go into the selected subset with probability one.

† Based on (3.3) with $r = (n + 1)/2$.

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