# PREFERENCE-BASED DEFINITIONS OF SUBJECTIVE PROBABILITY[1]

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**1. Introduction.** Two main approaches have been used in defining subjective (personal) probability. In the intuitive approach, used by Koopman [13], [14], Kraft, Pratt, and Seidenberg [15], Scott [22], Good [9], Villegas [26], and to some extent by de Finetti [6], the axioms apply a comparative probability relation "is not more probable than" to a set of events or propositions.

The second main approach bases the axioms on a comparative preference-indifference relation $\precsim$ ("is not preferred to"): representatives include Ramsey [20], Savage [21], Suppes [23], Davidson and Suppes [3], Anscombe and Aumann [2], and Pratt, Raiffa, and Schlaifer [19]. Each axiomatization in this approach permits the derivation of a probability measure and a utility function that combines with the probabilities to yield a subjective expected utility model consistent with $\precsim$.

This paper presents two related axiomatizations, each of which leads to a unique probability distribution on a set of $n$ states in the context of an expected utility model. (The extension to more general sets of states is not pursued here.) The method used in each axiomatization involves three steps.

Step 1. Axioms are first given to obtain an expected-utility model in which the utilities are holistic (involving all $n$ states) and the probabilities are associated with events that may have no direct connection with the $n$ states.

Step 2. An additional axiom, when necessary, is then used to render each holistic utility equal to a sum of state-dependent utilities: symbolically, $u = u_1 + \cdots + u_n$, where $u$ is the holistic utility function and $u_i$ is a utility function associated with the $i$th state, $s_i$.

Step 3. Each $u_i$ is defined on the same set in a given theory. The assumption that each $u_i$ (for nonnull $s_i$) has the same ordering on this set then leads to the definition of the subjective state probabilities and an expected utility model over the states.

Each preference-based axiomatization of subjective probability-utility has its own special characteristics and, depending upon the critic of the moment, its own merits and demerits. Ramsey [20], Suppes [23], and Davidson and Suppes [3] hypothesize an even-chance event (pr. $=\frac{1}{2}$) on the basis of $\precsim$, use this to scale utilities, then use the utilities to measure (other) subjective probabilities. Our second axiomatization, based on Debreu's even-chance theory [4], [5], uses this idea. It also assumes an infinite set of consequences, which ties in closely with

Received 28 June 1966; revised 7 June 1967.

[1] Final revisions on this paper were made during the author's leave from RAC as Visiting Professor at The Institute of Mathematical Statistics and Operations Research, The Technical University of Denmark.

Ramsey and Suppes: Davidson and Suppes use a finite set of consequences assumed to be equally spaced in utility.

Pratt, Raiffa, and Schlaifer [19] employ a canonical experiment of equally-likely outcomes for scaling utilities and subjective probabilities. The canonical experiment is an operational measurement device. In a related manner Anscombe and Aumann [2] use "known" probabilities, associated with the outcomes of symmetric gambling devices such as roulette wheels, to measure utilities via the von Neumann-Morgenstern expected utility theory. The subjective probabilities for the states (typified by the outcomes of a horse race) are then derived from the utilities.

Our first axiomatization also uses the von Neumann-Morgenstern theory, applying it to $n$-tuples of probability distributions. This application is presented in Section 3 following a brief summary of the basic theory in the next section.

After comparing our two theories with other theories (Sections 4 and 6) we comment briefly on two prominent concerns of the preference-based approach to defining subjective probabilities in the decision-under-uncertainty context: the question of open and pure axiomatizations (Section 7) and the problem of constant acts (Section 8).

**2. The von Neumann-Morgenstern theory: outline.** A *mixture set* is a set $\mathcal{A} = \{A, B, C, \cdots\}$ and an operation $\alpha A + (1 - \alpha)B$ associating an element of $\mathcal{A}$ with each $\alpha \varepsilon [0, 1]$ and each ordered pair $(A, B) \varepsilon \mathcal{A}^2$ such that if $A, B \varepsilon \mathcal{A}$ and $\alpha, \beta \varepsilon [0, 1]$ then

1. $1A + 0B = A$,
2. $\alpha A + (1 - \alpha)B = (1 - \alpha)B + \alpha A$,
3. $\alpha[\beta A + (1 - \beta)B] + (1 - \alpha)B = \alpha\beta A + (1 - \alpha\beta)B$.

This is identical to the definition given by Herstein and Milnor [10]. Properties 1, 2, and 3 imply that $\alpha A + (1 - \alpha)A = A$ and that $\alpha[\beta A + (1 - \beta) B] + (1 - \alpha)[\gamma A + (1 - \gamma)B] = [\alpha\beta + (1 - \alpha) \gamma]A + [1 - \alpha\beta - (1 - \alpha) \gamma]B$. Both results are useful in proving Theorem 1: Luce and Suppes [18], p. 288, give a proof of the latter result.

With $\lesssim$ a binary relation on $\mathcal{A}$, let $A < B = [A \lesssim B$ and not $B \lesssim A]$, and $A \sim B = [A \lesssim B$ and $B \lesssim A]$. $\lesssim$ on $\mathcal{A}$ is a *weak order* if it is transitive and connected (or complete).

Axiom 0 (Structure). *$\mathcal{A}$ is a mixture set.*

Axiom 1 (Order). *$\lesssim$ on $\mathcal{A}$ is a weak order.*

Axiom 2 (Independence for Convex Combinations). *If $A, B, C \varepsilon \mathcal{A}$, $\alpha \varepsilon (0, 1)$ and $A < B$ then $\alpha A + (1 - \alpha)C < \alpha B + (1 - \alpha)C$.*

Axiom 3 (Archimedean). *If $A, B, C \varepsilon \mathcal{A}$, $A < B$ and $B < C$ then $\alpha A + (1 - \alpha) C < B$ and $B < \beta A + (1 - \beta)C$ for some $\alpha, \beta \varepsilon (0, 1)$.*

These axioms are fairly standard. We also require, for the proof of Theorem 1, that $A \sim B$ *and* $\alpha \varepsilon [0, 1]$ *imply* $\alpha A + (1 - \alpha)C \sim \alpha B + (1 - \alpha)C$: Jensen [11] proves that this is a consequence of Axioms 0, 1, 2, and 3. With the given results the proof of Theorem 1 is essentially that given by von Neumann and Morgenstern [27], Appendix, Savage [21], Chapter 5, and others.

THEOREM 1. *Given Axiom* 0, *Axioms* 1, 2, *and* 3 *hold if and only if there is a real-valued function u on* $\mathcal{A}$ *such that*

$$(1) \qquad A \preceq B \quad \text{if and only if} \quad u(A) \leqq u(B), \quad \text{for all} \quad A, B \varepsilon \mathcal{A};$$

$$(2) \quad u(\alpha A + (1 - \alpha)B) = \alpha u(A) + (1 - \alpha)u(B), \quad \text{for all} \quad A, B \varepsilon \mathcal{A} \quad \text{and} \quad \alpha \varepsilon [0, 1].$$

*If u and v on* $\mathcal{A}$ *each satisfy* (1) *and* (2) *then there are numbers a, b with* $a > 0$ *such that*

$$(3) \qquad v(A) = au(A) + b \quad \text{for all} \quad A \varepsilon \mathcal{A}.$$

**3. Distribution product sets.** Throughout, $S = \{s_1, \cdots, s_n\}$ is the set of $n$ states and $X$ is a set of consequences. A probability distribution on a set is *simple* if some finite subset has probability 1 under the distribution.

Let $\mathcal{R}$ be the set of all simple probability distributions on $X$ and let $\mathcal{H} = \mathcal{R}^n$. Each $\mathbf{P} \varepsilon \mathcal{H}$ is an $n$-tuple of distributions of the form $\mathbf{P} = (P_1, \cdots, P_n)$. $P_i$ in $(P_1, \cdots, P_n)$ is associated with $s_i$, the interpretation being that if $\mathbf{P} \varepsilon \mathcal{H}$ is selected and $s_i$ is the true state then the resulting consequence in $X$ will be chosen using $P_i$.

We define $\alpha\mathbf{P} + (1 - \alpha)\mathbf{Q}$ in the natural scalar-vector manner as $\alpha\mathbf{P} + (1 - \alpha)\mathbf{Q} = (\alpha P_1 + (1 - \alpha)Q_1, \cdots, \alpha P_n + (1 - \alpha)Q_n)$. $\alpha P_i + (1 - \alpha)Q_i$ is the usual convex combination of probability distributions. Axiom 0 is easily seen to hold when $\mathcal{H} = \mathcal{A}$.

With $\mathcal{A} = \mathcal{H}$ the theory of Section 2 is directly applicable to $\mathcal{H}$: the axioms apply to $n$-tuples of probability distributions. In this setting Axiom 2 seems reasonable provided that the states are formulated consistent with the notion that exactly one is in fact the true state (or obtains). In a different context, when $i$ denotes time and each $P_i$ in $(P_1, \cdots, P_n)$ selects an $x_i \varepsilon X$, resulting in a vector consequence $(x_1, \cdots, x_n)$ over the $n$ time periods, simple examples show that Axiom 2 has little rational appeal. [Similar remarks apply for Axiom 5 in this section and Axiom 5* in Section 5.]

With $\mathcal{A} = \mathcal{H}$, Step 2 of Section 1 requires no new axioms.

THEOREM 2. *Axioms* 1, 2, *and* 3 *with* $\mathcal{A} = \mathcal{H} = \mathcal{R}^n$ *imply that, with u on* $\mathcal{H}$ *satisfying* (1) *and* (2), *there are real-valued functions* $u_1, \cdots, u_n$ *on* $\mathcal{R}$ *such that*

$$(4) \qquad u(P_1, \cdots, P_n) = \textstyle\sum_{i=1}^n u_i(P_i) \quad \text{for all} \quad (P_1, \cdots, P_n) \varepsilon \mathcal{H},$$

*and when* (4) *holds, for each i*

$$(5) \quad u_i(\alpha R + (1 - \alpha)R') = \alpha u_i(R) + (1 - \alpha)u_i(R') \quad \text{for all} \quad R, R' \varepsilon \mathcal{R}, \quad \alpha \varepsilon [0, 1].$$

*If u and the* $u_i$ *satisfy* (1), (2) *and* (4), *and if real-valued functions v on* $\mathcal{H}$ *and* $v_i$ *on* $\mathcal{R}$ *for* $i = 1, \cdots, n$ *satisfy* (1), (2), *and* (4) *then there are numbers a,* $b_1, \cdots, b_n$ *with* $a > 0$ *such that* (3) *holds with* $b = b_1 + \cdots + b_n$ *and*

$$(6) \qquad v_i(R) = au_i(R) + b_i \quad \text{for all} \quad R \varepsilon \mathcal{R}; i = 1, \cdots, n.$$

PROOF. Given (1) and (2) let $\mathbf{P}^0 = (P_0, \cdots, P_0) \varepsilon \mathcal{H}$ be fixed and define $u_i(P_i) = u(P_0, \cdots, P_0, P_i, P_0, \cdots, P_0) - u(\mathbf{P}^0)(n-1)/n$ for all $P_i \varepsilon \mathcal{R}$, $i = 1, \cdots, n$. Summed over $i$ this gives $\sum_i u_i(P_i) = \sum_i u(P_0, \cdots, P_0, P_i, P_0, \cdots, P_0) - (n-1)u(\mathbf{P}^0)$. But $(1/n)\mathbf{P} + [(n-1)/n]\mathbf{P}^0 = \sum_i (1/n)(P_0, \cdots, P_0, P_i, P_0, \cdots, P_0)$ when $\mathbf{P} = (P_1, \cdots, P_n)$, since the two convex combinations are identical. Hence, by (1), the two convex combinations have equal utilities; application of (2) to each utility then gives $u(\mathbf{P}) = \sum_i u(P_0, \cdots, P_0, P_i, P_0, \cdots, P_0) - (n-1)u(\mathbf{P}^0)$. By definition, the right side of this equals $\sum_i u_i(P_i)$, so that $u(\mathbf{P}) = \sum_i u_i(P_i)$.

To verify (5) let $\mathbf{P} = (P_0, \cdots, P_0, R, P_0, \cdots, P_0)$ and $\mathbf{Q} = (P_0, \cdots, P_0, R', P_0, \cdots, P_0)$. Then, by (4), $u(\alpha\mathbf{P} + (1-\alpha)\mathbf{Q}) = \sum_{j\neq i} u_j(P_0) + u_i(\alpha R + (1-\alpha)R')$, and by (2) and (4) $u(\alpha\mathbf{P} + (1-\alpha)\mathbf{Q}) = \sum_{j\neq i} u_j(P_0) + \alpha u_i(R) + (1-\alpha)u_i(R')$, which together give $u_i(\alpha R + (1-\alpha)R') = \alpha u_i(R) + (1-\alpha)u_i(R')$.

For the final part of the theorem, let $u, u_1, \cdots, u_n$ and $v, v_1, \cdots, v_n$ both satisfy (1), (2), and (4). By Theorem 1 there are numbers $a > 0$ and $b$ such that (3) holds. With $P_0 \varepsilon \mathcal{R}$ fixed, let $b_i = v_i(P_0) - au_i(P_0)$, so that by (3) and (4), $\sum b_i = \sum v_i(P_0) - a\sum u_i(P_0) = v(\mathbf{P}^0) - au(\mathbf{P}^0) = b$. We also have $v(P_0, \cdots, P_0, R, P_0, \cdots, P_0) - v(\mathbf{P}^0) = a[u(P_0, \cdots, P_0, R, P_0, \cdots, P_0) - U(\mathbf{P}^0)]$ from (3), which by (4) converts to $V_i(R) - V_i(P_0) = a[u_i(R)\, u_i(P_0)]$ or $v_i(R) = au_i(R) + b_i$.□

This brings us to Step 3, for which the following axioms and definitions are used.

AXIOM 4 (Nontriviality). *$\mathbf{P} \prec \mathbf{Q}$ for some $\mathbf{P}, \mathbf{Q} \varepsilon \mathcal{H}$.*

DEFINITION 1 (Order for $\mathcal{R}$). *If $R, R' \varepsilon \mathcal{R}$, then $R \precsim R'$ if and only if $(R, \cdots, R) \precsim (R', \cdots, R')$, where $(R, \cdots, R), (R', \cdots, R') \varepsilon \mathcal{H}$.*

DEFINITION 2. (Null States). *$s_i \varepsilon S$ is* null *if and only if $P \sim Q$ for every $\mathbf{P}, \mathbf{Q} \varepsilon \mathcal{H}$ for which $P_j = Q_j$ for each $j \neq i$.*

AXIOM 5 (Monotonicity). *If $\mathbf{P} = (P_1, \cdots, P_{i-1}, R, P_{i+1}, \cdots, P_n)$ and $\mathbf{P}' = (P_1, \cdots, P_{i-1}, R', P_{i+1}, \cdots, P_n)$ are in $\mathcal{H}$, then $\mathbf{P} \precsim \mathbf{P}'$ if $R \precsim R'$, and $\mathbf{P} \prec \mathbf{P}'$ if $R \prec R'$ and $s_i$ is nonnull.*

Axiom 4 helps to insure a unique probability distribution on the states. With the other axioms at hand, Axiom 5 provides each $u_i$ for non-null $s_i$ with the same ordering: if $s_i$ and $s_j$ are non-null, then $u_i(R) \leqq u_i(R')$ if and only if $u_j(R) \leqq u_j(R')$ when $R, R' \varepsilon \mathcal{R}$.

THEOREM 3. *If $u$ on $\mathcal{A} = \mathcal{H}$ is any real-valued function satisfying (1) and (2), and if Axioms 4 and 5 hold, then, letting $u(R) = u(R, \cdots, R)$ for each $R \varepsilon \mathcal{R}$, there is a unique real-valued function $\pi$ on $S$ such that*

$$u(P_1, \cdots, P_n) = \sum_{i=1}^n \pi(s_i)u(P_i) \quad \text{for all} \quad (P_1, \cdots, P_n) \varepsilon \mathcal{H}, \tag{7}$$

$$\pi(s) \geqq 0 \quad \text{for each} \quad s \varepsilon S, \text{ and } \sum_{i=1}^n \pi(s_i) = 1, \tag{8}$$

*and, under these conditions, $\pi(s) = 0$ if and only if $s$ is null.*

PROOF. Given $u$ on $\mathcal{H}$ satisfying (1) and (2), and Axioms 4 and 5, let $\mathbf{P}^0 = (P_0, \cdots, P_0) \varepsilon \mathcal{H}$ be fixed. Since (1) and (2) imply Axioms 1, 2, and 3 [Theorem

1], there are $u_i$ functions on $\mathcal{R}$ satisfying (4) [Theorem 2]. Let $u$ and the $u_i$ be transformed according to (3) and (6) so that $u(\mathbf{P}^0) = 0$ and $u_i(P_0) = 0$ for each $i$. If $s_i$ is null, set $\pi(s_i) = 0$, so that in (7) $\pi(s_i)u(R) = 0$ for all $R \varepsilon \mathcal{R}$, corresponding to $u_i \equiv 0$ which follows from $u_i(P_0) = 0$, Definition 2, and (4).

Next, let $I = \{i \mid s_i \text{ is not null}\}$. Axiom 4 insures that $I$ is not empty. If $i \varepsilon I$, then Definition 1, Axiom 5, (1) and (4) yield $u_i(R) \leqq u_i(R')$ if and only if $R \preceq R'$, for all $R, R' \varepsilon \mathcal{R}$. Hence each $u_i$ for $i \varepsilon I$ satisfies a property similar to (1) with the same ordering for each $i$, and, by (5) of Theorem 2, each $u_i$ satisfies the same expectation property. With $k \varepsilon I$ fixed it then follows from the last part of Theorem 1 that, for each $i \varepsilon I$, there are numbers $a_i > 0$ and $c_i$ such that $u_i(R) = a_i\varphi(R) + c_i$ for all $R \varepsilon \mathcal{R}$ where $\varphi \equiv u_k$ and $a_k = 1, c_k = 0$. Since $u_i(P_0) = u_k(P_0) = 0$ by construction, $c_i = 0$ for each $i \varepsilon I$ and therefore $u_i(R) = a_i\varphi(R)$ for all $i \varepsilon I$. Hence by (4), $u(\mathbf{P}) = \sum_{i \varepsilon I} a_i\varphi(P_i)$ for all $\mathbf{P} \varepsilon \mathcal{H}$. Letting $\pi(s_i) = a_i/\sum_I a_i$ for each $i \varepsilon I$ and $u(R) = (\sum_I a_i)\varphi(R)$ for all $R \varepsilon \mathcal{R}$, we then obtain $u(\mathbf{P}) = \sum_i \pi(s_i)u(P_i)$ for all $\mathbf{P} \varepsilon \mathcal{H}$ along with $\pi(s) \geqq 0$, $\sum_i \pi(s_i) = 1$, and hence $u(R, \cdots, R) = u(R)$ for each $R \varepsilon \mathcal{R}$. If $v$ satisfies (1) and (2), then by Theorem 1, $v = au + b$, $a > 0$, so that if $u(\mathbf{P}) = \sum \pi(s_i)u(P_i)$, $\sum \pi(s_i) = 1$, then $v(\mathbf{P}) = \sum \pi(s_i)v(P_i)$.

Let $u$ and $\pi$ satisfy (1), (2), (7), and (8) and let Axioms 4 and 5 hold. Since $I$ is not empty, $u(R) < u(R')$ and hence $R \prec R'$ for some $R, R' \varepsilon \mathcal{R}$. If $v$ and $\pi'$ also satisfy (1), (2), (7) and (8), then, by Theorem 1, $v = au + b$, $a > 0$ and by (6) of Theorem 2 there are numbers $b_1, \cdots, b_n, b_1 + \cdots + b_n = b$ such that $\pi'(s_i)v(R) = a\pi(s_i)u(R) + b_i$ for all $R \varepsilon \mathcal{R}$, $i = 1, \cdots, n$. These equations then give $au(R)[\pi'(s_i) - \pi(s_i)] = b_i - b\pi'(s_i)$, which, since $u(R) < u(R')$ for some $R, R' \varepsilon \mathcal{R}$, requires that $\pi'(s_i) - \pi(s_i) = b_i - b\pi'(s_i) = 0$. This holds for any $s_i \varepsilon S$ and therefore $\pi' \equiv \pi$. Finally, if $s_i$ is null and $\mathbf{P} = (P_1, \cdots, P_{i-1}, R, P_{i+1}, \cdots, P_n)$ and $\mathbf{P}' = (P_1, \cdots, P_{i-1}, R', P_{i+1}, \cdots, P_n)$, then, by Definition 2, $\mathbf{P} \sim \mathbf{P}'$, which by (1) and (4) converts to $\pi(s_i)u(R) = \pi(s_i)u(R')$, which requires $\pi(s_i) = 0$ since $u(R) < u(R')$ for some $R, R' \varepsilon \mathcal{R}$. Conversely, if $\pi(s_i) = 0$, then (4) and (1) imply that $s_i$ is null. □

The $\pi(s_i)$ are the (subjective) state probabilities. In the foregoing $X$ is arbitrary with at least two elements (Axiom 4) and $u$ need not be bounded if $X$ contains an infinite number of elements. Our second axiomatization, in Section 5, requires $X$ to be uncountable.

The basic subjective expected utility model for the $n$-state case involves a set $F$ of acts (functions on $S$ to $X$), with $f(s)$ the consequence assigned by $f \varepsilon F$ to $s \varepsilon S$. The model is

$$(9) \quad f \preceq g \quad \textit{if and only if} \quad \textstyle\sum_{i=1}^n \pi(s_i)u(f(s_i)) \leqq \sum_{i=1}^n \pi(s_i)u(g(s_i)),$$

for all $f, g \varepsilon F$. (9) results from (1) and (7) by identifying $f$ with $\mathbf{P} = (P_1, \cdots, P_n)$ in which $P_i(f(s_i)) = 1$ for each $i$, and by identifying $g$ with $\mathbf{Q} = (Q_1, \cdots, Q_n)$ where $Q_i(g(s_i)) = 1$ for each $i$.

**4. Discussion.** Our definition of $\preceq$ on $\mathcal{R}$ on the basis of $\preceq$ on $\mathcal{H}$ is typical. It corresponds to Savage's D2 and our definition of null states corresponds to

Savage's D3: Axiom 4 is like Savage's P5 and Axiom 5 is like his sure-thing principle, P2 and P3. There is nothing in the theory of Section 3 that corresponds directly to Savage's axiom for comparability of events (subsets of $S$), P4. This is due to the different approaches used and not to any specific desire to avoid an axiom like P4. Savage applies $\lesssim$ to $F$ throughout his axioms [including P4, where $\lesssim$ between events is defined in terms of $\lesssim$ between acts], where $F$ is the set of (all?) functions on $S$ to $X$, similar to the definition given above except that Savage's axioms (notably P5 and P6) require $S$ to be infinite. As he notes (pp. 38–39) his axioms imply that $S$ can be partitioned into arbitrarily many equally-likely events. Because of this he is able to derive his subjective expected utility model

$$(10) \qquad f \lesssim g \quad \textit{if and only if} \quad \int u\,(f(s))\,dP(s) \leqq \int u\,(g(s))\,dP(s)$$

for all $f, g \in F$ (with $P$ the individual's finitely-additive probability measure on the set of all events), without resorting to some extrasituational device such as a well-balanced roulette wheel or canonical experiment. [Savage proves that (10) holds when $f$ and $g$ are bounded acts, where $f$ is bounded if $P(a \leqq u(f(s)) \leqq b) = 1$ for some numbers $a$ and $b$. He and I have discovered that his postulates imply that consequence utilities are bounded: consequently all acts are bounded and (10) holds for all $f, g \in F$. This result will be proved in a forthcoming edition of his book and relates to results proved in Fishburn [8].]

Our $\mathcal{R}$ and $\mathcal{H}$ in Section 3 correspond respectively to $\mathcal{R}$ and $\mathcal{H}$ as used by Anscombe and Aumann: for them, $\mathcal{R}$ is the set of all "roulette lotteries" with prizes in $X$, and $\mathcal{H}$ is the set of all "horse lotteries." In our terms, they apply the Luce-Raiffa [17] version of the von Neumann-Morgenstern theory to $\mathcal{R}$ and to $\mathcal{R}^*$, where $\mathcal{R}^*$ is the set of all simple probability distributions on $\mathcal{H}$. Then they apply axioms similar to Axiom 4 to $\mathcal{R}$ and to $\mathcal{R}^*$ and interconnect $\mathcal{R}$ and $\mathcal{R}^*$ with their Assumption 1, which is essentially the first part of Axiom 5. (They do not define null states.) With these axioms, plus another applying to $\mathcal{R}^*$, they derive the state probabilities for finite $S$ from the von Neumann-Morgenstern utility functions on $\mathcal{R}$ and $\mathcal{R}^*$.

As the authors note, the novelty in the Anscombe-Aumann theory lies in the double application of the von Neumann-Morgenstern theory. Whatever novelty there is in our first axiomatization is due to the application of the von Neumann-Morgenstern theory to $n$-tuples of probability distributions.

**5. An even-chance axiomatization.** Our second axiomatization begins with Debreu's even-chance theory of utility [5] which applies $\lesssim$ to $F \times F$, then adds a non-triviality axiom and another axiom to complete Steps 2 and 3. In this section we view $F$ as the set $X^n$ of all $n$-tuples $(x_1, \cdots, x_n)$, $x_i \in X$ for each $i$, with $f(s_i) = x_i$ when $f = (x_1, \cdots, x_n)$. With $F = X^n$ we interpret $(f, f) \in F^2$ as an alternative resulting in $f$ with certainty, and interpret $(f, g') \in F^2, f \neq g$, as an alter native resulting in act $f$ or $g$ (not both) with equal probability: $(f, g) \sim (g, f)$ is implied by Axiom $2^*$.

Debreu's even-chance theory for the present context is summarized by the following four axioms and Theorem $1^*$.

AXIOM $0^*$ (Structure). *$(X, \mathfrak{I})$ is a connected and separable topological space.*

AXIOM $1^*$ (Order). *$\lesssim$ on $F^2$ is a weak order.*

AXIOM $2^*$ (Cancellation, Even-Chance). *If $(f, g) \lesssim (f', g')$ and $(f', g^*) \lesssim (f^*, g)$ then $(g^*, f) \lesssim (g', f^*)$.*

AXIOM $3^*$ (Archimedean-continuity). *$\{(f, g) \mid (f, g) \varepsilon F^2, (f, g) < (f', g')\}$ and $\{(f, g) \mid (f, g) \varepsilon F^2, (f', g') < (f, g)\}$ are open sets (in the product topology $\mathfrak{I}^{2n}$) for each $(f', g') \varepsilon F^2$.*

THEOREM $1^*$ (Debreu [5]). *Axioms $0^*$, $1^*$, $2^*$, and $3^*$ imply that there is a continuous, real-valued function $u$ on $F^2$ such that*

(11) $(f, g) \lesssim (h, k)$ *if and only if* $u(f, g) \leqq u(h, k)$, *for all* $f, g, h, k \varepsilon F$,

(12) $$u(f, g) = \tfrac{1}{2}u(f, f) + \tfrac{1}{2}u(g, g) \quad \text{for all} \quad f, g \varepsilon F,$$

*and if $v$ also is a real-valued function on $F^2$ satisfying (11) and (12) then $v$ is continuous and there are numbers $a > 0$ and $b$ such that*

(13) $$v(f, g) = au(f, g) + b \quad \text{for all} \quad f, g \varepsilon F.$$

Theorem $1^*$ corresponds to Theorem 1 and (11), (12), and (13) correspond respectively to (1), (2), and (3).

To complete the derivation we use the following axioms and definitions. In Definition $1^*$ and later, $\bar{x}$ is the *constant act* in $F$ assigning $x \varepsilon X$ to each $s \varepsilon S$.

AXIOM $4^*$ (Non-triviality). *$(f, f) < (g, g)$ for some $f, g \varepsilon F$.*

DEFINITION $1^*$ (Order for $X^2$). If $x, y, z, w \varepsilon X$ then $(x, y) \lesssim (z, w)$ if and only if $(\bar{x}, \bar{y}) \lesssim (\bar{z}, \bar{w})$.

DEFINITION $2^*$ (Null States). $s_i \varepsilon S$ is *null* if and only if $(f, g) \sim (h, k)$ whenever $\{f(s), g(s)\} = \{h(s), k(s)\}$ for all $s \neq s_i$.

AXIOM $5^*$ (Monotonicity). *If $\{f(s), g(s)\} = \{h(s), k(s)\}$ for all $s \neq s_i$ then $(f, g) \lesssim (h, k)$ if $(f(s_i), g(s_i)) \lesssim (h(s_i), k(s_i))$, and $(f, g) < (h, k)$ if $(f(s_i), g(s_i)) < (h(s_i), k(s_i))$ and $s_i$ is nonnull.*

For Step 2, the next theorem corresponds to Theorem 2 with (14), (15), and (16) the obvious counterparts to (4), (5), and (6).

THEOREM $2^*$. *Axioms $0^*$, $1^*$, $2^*$, $3^*$, and $5^*$ imply that, with $u$ on $F^2$ continuous and satisfying (11) and (12), there are real-valued functions $u_1, \cdots, u_n$ on $X^2$ such that*

(14) $$u(f, g) = \textstyle\sum_{i=1}^{n} u_i(f(s_i), g(s_i)) \quad \text{for all} \quad (f, g) \varepsilon F^2,$$

*and, under these conditions, each $u_i$ is continuous and*

(15) $$u_i(x, y) = \tfrac{1}{2}u_i(x, x) + \tfrac{1}{2}u_i(y, y) \quad \text{for all} \quad x, y \varepsilon X; \qquad i = 1, \cdots, n.$$

*If $u$ (continuous) and the $u_i$ satisfy (11), (12), and (14), and real-valued functions $v$ (continuous) on $F^2$ and $v_i$ on $X^2$ for $i = 1, \cdots, n$ satisfy (11), (12), and (14) then there are numbers $a > 0, b_1, \cdots, b_n$, such that (13) holds with $b = b_1 + \cdots + b_n$ and*

(16) $$v_i(x, y) = au_i(x, y) + b_i \quad \text{for all} \quad (x, y) \varepsilon X^2; \; i = 1, \cdots, n.$$

PROOF. Let the cited axioms hold with $u$ (continuous) satisfying (11) and (12). If $x, y, z, w \varepsilon X$ and $\{x, y\} = \{z, w\}$, then, by Axiom $1^*$ $[(f, g) \sim (f, g)]$ or Axiom $2^*$ $[(f, g) \sim (g, f)]$, $(\bar{x}, \bar{y}) \sim (\bar{z}, \bar{w})$: hence, by Definition $1^*$, $(x, y) \sim (z, w)$. This result and the first part of Axiom $5^*$ imply that, if $\{f(s), g(s)\} = \{h(s), k(s)\}$ for all $s \varepsilon S$, then $(f, g) \sim (h, k)$. Letting $u(f) \equiv u(f, f)$, it then follows from Fishburn [7], pp. 42–43, that there are real-valued functions $w_i$ on $X$ for $i = 1, \cdots, n$ such that $u(f) = \sum_i w_i(f(s_i))$ for all $f \varepsilon F$. Let $u_i(x, y) = \frac{1}{2}w_i(x) + \frac{1}{2}w_i(y)$ for all $x, y \varepsilon X$. Using (12) we then have $u(f, g) = \frac{1}{2}\sum_i w_i(f(s_i)) + \frac{1}{2}\sum_i w_i(g(s_i)) = \sum_i u_i(f(s_i), g(s_i))$, which is (14).

Let $u_i'$ on $X^2$ be defined as follows. Let $F_i^2 = \{(f, g) \mid f(s_j) = g(s_j) = x_0$ for all $j \neq i\}$, where $x_0 \varepsilon X$ is fixed, and let $u_i'(f(s_i), g(s_i)) = u(f, g)$ for $(f, g) \varepsilon F_i^2$. Then (see, e.g., Kelley [12], pp. 102–103) $u_i'$ is a continuous function on $X^2$ since $u$ is continuous on the product space $F^2$. If (14) holds, then $u_i(f(s_i), g(s_i)) = u_i'(f(s_i), g(s_i)) - \sum_{j \neq i} u_i(x_0, x_0)$, so that $u_i$ is also continuous on $X^2$.

Verifications of (15) and (16) are similar to those for (5) and (6) respectively. □

THEOREM $3^*$. *If $u$ is any continuous, real-valued function on $F^2$ satisfying* (11) *and* (12), *and if Axioms* $0^*$, $4^*$, *and* $5^*$ *hold, then letting $u(x, y) = u(\bar{x}, \bar{y})$ for all $(x, y) \varepsilon X^2$, there is a unique real-valued function $\pi$ on $S$ such that*

$$(17) \qquad u(f, g) = \sum_{i=1}^{n} \pi(s_i) u(f(s_i), g(s_i)) \quad \textit{for all} \quad (f, g) \varepsilon F^2$$

$$\pi(s) \geqq 0 \quad \textit{for each} \quad s \varepsilon S, \quad \textit{and} \quad \sum_{i=1}^{n} \pi(s_i) = 1,$$

*and, under these conditions, $\pi(s) = 0$ if and only if $s$ is null.*

We note that the model (9) results from (11) and (17) on letting $u(x) = u(x, x)$.

PROOF. Since the conditions of the first two lines of the theorem imply Axioms $1^*$, $2^*$, and $3^*$, these axioms were not listed in its statement. With Theorems $1^*$ and $2^*$ at our disposal, let $u$ on $F^2$ and the $u_i$ on $X^2$, continuous and satisfying (11), (12), and (14), be transformed according to (13) and (16) so that $u(\bar{x}_0, \bar{x}_0) = u_i(x_0, x_0) = 0$, $i = 1, \cdots, n$. If $s_i$ is null, then $u_i \equiv 0$, so that we set $\pi(s_i) = 0$ when $s_i$ is null.

Next, let $I = \{i \mid s_i$ is not null$\}$, which is not empty by Axiom $4^*$. If $i \varepsilon I$, then Definition $1^*$, Axiom $5^*$, (11) and (14) yield $u_i(x, y) \leqq u_i(z, w)$ if and only if $(x, y) \lesssim (z, w)$ for all $x, y, z, w \varepsilon X$. Hence each $u_i$ for $i \varepsilon I$ satisfies the same order-preserving property for the same weak order on $X^2$, and, by (15), each $u_i$ for $i \varepsilon I$ satisfies the same expectation property, analogous to (12) for $u$ on $F^2$. In addition, analogues of Axiom $2^*$ and $3^*$ also apply to $X^2$ for each $i \varepsilon I$: if $(x, y) \lesssim (x', y')$ and $(x', y^*) \lesssim (x^*, y)$, then $(y^*, x) \lesssim (y', x^*)$ since $(x, y) \lesssim (z, w)$ if and only if $u_i(x, x) + u_i(y, y) \leqq u_i(z, z) + u_i(w, w)$; $\{(x, y) \mid (x, y) \prec (x', y')\}$ and $\{(x, y) \mid (x', y') \prec (x, y)\}$ are open subsets of $X^2$ [in the product topology $\mathfrak{I}^2$] since $u_i$ on $X^2$ is continuous and $u_i(X, X)$ is an interval of reals by continuity and the connectedness of $(X^2, \mathfrak{I}^2)$, the latter from Axiom $0^*$. It then follows by analogy to Theorem $1^*$ that, for each $i \varepsilon I$ (with $t \varepsilon I$), there are numbers $a_i > 0$ and $c_i$ such that $u_i(x, y) = a_i\varphi(x, y) + c_i$ for all $(x, y) \varepsilon X^2$, where $\varphi \equiv u_t$ and $a_t = 1, c_t = 0$. Since $u_i(x_0, x_0) = 0$ by construction, $c_i = 0$ for each $i \varepsilon I$ and therefore $u_i(x, y) = a_i\varphi(x, y)$ for all $i \varepsilon I$. Substituting into (14), this gives $u(f, g) = \sum_I a_i\varphi(f(s_i), g(s_i))$ for all $(f, g) \varepsilon F^2$, which converts to (17) and (8) on letting $\pi(s_i) = a_i/\sum_I a_i$ for each $i \varepsilon I$ and $u(x, y) = (\sum_I a_i)$ $\varphi(x, y)$ for all $(x, y) \varepsilon X^2$. Since $\sum \pi(s_i) = 1, u(f, f) = u(x, x)$ when $f(s) = x$ for all $s \varepsilon S$. If $v(f, g) = au(f, g) + b$ for all $(f, g) \varepsilon F^2$, (17) then converts to $v(f, g) = \sum \pi(s_i)v(f(s_i), g(s_i))$.

The uniqueness proof for $\pi$ is like that used for Theorem 3. □

**6. Further Discussion.** In the first axiomatization, the structural properties which (along with the preference axioms) led to the fact that the $u_i$ were related by increasing linear transformations were the closure under convex combinations and related properties of mixture sets (Axiom 0) and the fact that the $u_i$ were defined on the same set $\mathcal{R}$. The even-chance axiomatization used $F = X^n$ and, instead of convex closure, relied on topological properties (Axiom 0*, continuity) to connect the $u_i$(on $X^2$) by increasing linear transformations. In both cases, this connection between the $u_i$ for non-null $s_i$ was our key to defining the unique state probabilities for the finite set $S$.

There is a close relation between Ramsey's original ideas [20] and our even-chance theory, which may be viewed as one reasonable transliteration of his basic ideas, supplemented by Debreu's insights. Suppes' theory [23] is also strongly motivated by Ramsey and is in many ways closer to Ramsey's theory than is ours. Both Ramsey and Suppes obtain a theorem similar to Theorem 1* (minus continuity) on a purely algebraic rather than topological basis. Ramsey does not explicitly spell out all the details leading to the model $u(f) = \sum_i \pi(s_i)u(f(s_i))$. Suppes is more complete in this respect and, in fact, goes beyond the finite-states model to the consideration of an arbitrary number of states, obtaining an expected-utility model under a finitely additive probability measure on $S$, as in (10). (Suppes' axiomatization is offered as an alternative to Savage's, which Suppes feels to contain several weak points not present in his theory. Suppes is also critical of some of the features of his own theory, and many of his comments apply to our axiomatizations. The reader may consult Suppes [23], [24] for further discussion.)

For *finite* $S$, our second axiomatization is stronger (less general) than is Suppes'. That is, our assumptions imply all eleven of Suppes' axioms (A.1 through A.11), and then some. His first five axioms are essentially equivalent to our Axioms 1* and 2*, with one notable difference: we apply $\lesssim$ to $X^n \times X^n$ whereas Suppes applies $\lesssim$ to $D \times D$ with $D \subseteq X^n$. His Axiom A.8 requires that all constant acts be in $D$, and A.6, A.7, A.10, and A.11, all involving $\lesssim$, specify other restrictions on $D$, A.11 being his midpoint axiom [$f, g \varepsilon D$ imply that there is an $h \varepsilon D$ such that, for every $s \varepsilon S$, $(f(s), g(s)) \sim (h(s), h(s))$]. Our system contains no direct analogues of these latter four axioms although each is a consequence of our theory: in terms of structure, we use $D = X^n$ and Axioms 0* and 3* compared to Suppes' A.6, A.7, A.8, A.10, and A.11. Suppes has no explicit counterpart to Axiom 4* (non-triviality) and does not use Savage's notion of a null state (or event). His resulting subjective probabilities are not necessarily unique but he notes that, for finite $S$, "various conditions which guarantee uniqueness are easy to give." [23], p. 68.

Suppes' other axiom, A.9, compares with the first part of Axiom $5^*$. In our terms, A.9 reads: if $(f(s), g(s)) \preceq (h(s), k(s))$ for each $s \in S$, then $(f, g) \preceq (h, k)$. The first part of Axiom $5^*$ reads: if $\{f(s), g(s)\} = \{h(s), k(s)\}$ for all $s \neq s_i$ and $(f(s_i), g(s_i)) \preceq (h(s_i), k(s_i))$, then $(f, g) \preceq (h, k)$. Granting $(f, g) \sim (f, g) \sim (g, f)$ in all cases, we showed in the proof of Theorem $2^*$ that $\{f(s), g(s)\} = \{h(s), k(s)\}$ implies $(f(s), g(s)) \sim (h(s), k(s))$. The first part of Axiom $5^*$ is an immediate consequence of this and A.9. Similarly, Axiom $5^*$ and the transitivity of $\preceq$ imply A.9, with one reservation. To show this, suppose $(f(s), g(s)) \preceq (h(s), k(s))$ for each $s \in S$. Then, by repeated applications of Axiom $5^*$ and transitivity we obtain, with acts written as $n$-tuples of consequences, $([f(s_1), \cdots, f(s_{n-1}), f(s_n)], [g(s_1), \cdots, g(s_{n-1}), g(s_n)]) \preceq ([f(s_1), \cdots, f(s_{n-1}), h(s_n)], [g(s_1), \cdots, g(s_{n-1}), k(s_n)]) \preceq ([f(s_1), \cdots, f(s_{n-2}), h(s_{n-1}), h(s_n)], [g(s_1), \cdots, g(s_{n-2}), k(s_{n-1}), k(s_n)]) \preceq \cdots \preceq ([f(s_1), h(s_2), \cdots, h(s_n)], [g(s_1), k(s_2), \cdots, k(s_n)]) \preceq ([h(s_1), \cdots, h(s_n)], [k(s_1), \cdots, k(s_n)])$. For this demonstration it is of course necessary to assume that all acts involved are contained in $D$, which is not necessarily so in Suppes' scheme.

**7. On open axiomatizations.** Following Luce and Suppes [18], p. 269, we define an axiomatization for the subjective expected-utility model (9) or (10) as

a. *finitely open (open) with respect to* $S$ if the axioms apply to any finite (finite or infinite) number $n$ of states, $n \geqq 2$;

b. *open with respect to* $X$ if $X$ can be of any size ($\geqq 2$);

c. *open with respect to* $F$ if, with any allowable $S$ and $X$, $F$ can be any subset of the set of all functions on $S$ to $X$, with at least two elements;

d. *open in form* if, for any $(S, X, F)$ admissible under the axioms, the axioms hold for every nontrivial numerical realization of (9) or (10) as the case may be when $\preceq$ is defined on the basis of (9) or (10);

e. *pure* if each preference axiom can be stated using $\preceq$ between pairs of elements in $F$ only.

Savage's theory is open with respect to $X$ and is the only pure theory cited in this paper. Because it requires an infinite $S$ and arbitrarily large (finite) partitions of $S$ into equally-likely events it is neither open with respect to $S$ nor open in form.

No theory cited here is open with respect to $F$: most require all constant acts to be in $F$.

The theories of Suppes and Davidson-Suppes and our second axiomatization are similar in that they use the special probability $\frac{1}{2}$ for scaling utilities, which renders them not pure. They are finitely open in $S$ but not $X$ (Davidson-Suppes take $X$ finite: the others require $X$ infinite) and are not open in form because of special conditions on $u$. Suppes' theory is the only one cited in this paper (excluding Ramsey) that is open with respect to $S$.

The other three theories (our first, Anscombe-Aumann, Pratt-Raiffa-Schlaifer) are similar in that they use a continuum of "known" probabilities as a measurement device, which renders them not pure. They are finitely open in $S$, open in $X$ ($X$ finite is sometimes used but is not essential), and open in form.

No theory cited here is open in form *and* pure, which raises a series of questions. For example, are there axiomatizations for (9) that are finitely open in $S$, open in form, and pure when $F = X^n$? If so, is any open in $X$? What about $F \subseteq X^n$? $\cdots$

Having raised these questions I can say little more about the first than to note that it has a positive answer, although the axiomatization I have in mind assumes that $X$ has two elements only. This axiomatization amounts to a modified version of the Kraft-Pratt-Seidenberg [15] or Scott [22] theory for the existence of a probability measure on a finite set (using $F = \{x, y\}^n$ in place of the set of events), or to an application of the theory of additive conjoint measurement for finite sets, developed by Adams [1], Scott [22], and Tversky [25]. With $n \geqq 2$ states the axioms are

A0′. *$X$ has two elements and $F = X^n$.*

A1′. *$f \precsim g$ or $g \precsim f$ when $f, g \in F$.*

A2′. *If $m > 1$; $f_1, \cdots, f_m, g_1, \cdots, g_m \in F$; $f_1(s_i), \cdots, f_m(s_i)$ is a permutation of $g_1(s_i), \cdots, g_m(s_i)$ for $i = 1, \cdots, n$; $f_j \precsim g_j$ for $j = 1, \cdots, m - 1$: then $g_m \precsim f_m$.*

A3′. *For some $x, y \in X$, $\bar{x} \prec \bar{y}$ or $\bar{y} \prec \bar{x}$.*

A4′. *If $x, y \in X$ and $\bar{x} \prec \bar{y}$ then $\bar{x} \precsim f$ for all $f \in \{x, y\}^n$.*

These axioms are a slight modification of Scott's axioms (p. 246), and, using Scott's result, they imply (9) where with $X = \{x, y\}$, $u(x) < u(y)$ if $\bar{x} \prec \bar{y}$, or $u(y) < u(x)$ if $\bar{y} \prec \bar{x}$. In most numerical realizations of (9) for this case $\pi$ will not be unique. The axiomatization given here is pure since the preference axioms apply $\precsim$ to $F$ throughout.

It is not at all clear to me at this time if and how the above axiomatization can be modified so as to remain pure, finitely open in $S$, open in form, and open (or finitely open) with respect to $X$ when $F = X^n$. In the next section we comment on a pure axiomatization for (9) when $F$ is a particular type of subset of $X^n$.

**8. Constant acts.** In considering constant acts I will confine the discussion to consequences that are possible under some available act and shall let $X_i$ be the set of such consequences when (under the hypothesis that) $s_i$ is the true state. With $n$ states, $F$ is therefore a subset of the product set $\prod_{i=1}^n X_i$. Alternatively, $F$ is a subset of $X^n$, with $X = \bigcup_{i=1}^n X_i$. To define consequence utilities and state probabilities in a meaningful way it is convenient, if not necessary, to assume that all constant acts are in $F$. This implicitly assumes that constant acts are logically and psychologically relevant (if not in fact available) and explicitly implies that $X_i = X_j$ for all $i, j$ in $\{1, \cdots, n\}$.

The difficulty with constant acts arises from their logical intractibility in many decision situations, especially those in which specific aspects of the states enter into the formulation of consequences. Imagine, if you will, a court trial with two states: "Mr. Accused committed the crime he is on trial for" and "Mr. Accused did not commit the crime he is on trial for." From the judge's viewpoint one consequence could be "Sentence Mr. Accused, who committed the crime he is on trial for, to ten years in prison." To suppose that there is a constant act with this consequence as the constant leads us to say that Mr. Accused, who committed the crime he is on trial for, did not commit the crime he is on trial for. If we try to avoid constant acts as such in this case and instead try to compare consequences directly we face the psychologically interesting possibility of asking the judge to compare preferentially two consequences such as "Sentence Mr. Accused, who commited the crime he is on trial for, to five years in prison" and "Acquit Mr. Accused, who did not commit the crime he is on trial for." The judge might very will find these two consequences incomparable and decline to make any direct preference-indifference judgment between them.

In this brief example $X_i \cap X_j = \varnothing$ when $i \neq j$. When this holds and $X = \cup X_i$ is finite, there is a pure axiomatization for (9) that is open in form, but the $\pi(s_i)$ for this case should not be viewed as subjective probabilities. To be specific, the theory of additive conjoint measurement noted in Section 7 says that the pure axioms A1′ and A2′ hold when $F \subseteq \prod_{i=1}^{n} X_i$ is finite if and only if there are real-valued functions $u_1, \cdots, u_n$ on $X_1, \cdots, X_n$ respectively such that

$$f \lesssim g \quad \text{if and only if} \quad \sum_{i=1}^{n} u_i(f(s_i)) \leqq \sum_{i=1}^{n} u_i(g(s_i)), \tag{18}$$

for all $f, g \in F$. Let $\pi(s_1), \cdots, \pi(s_n)$ be *any* sequence of positive numbers that sum to one. Then, with $X_i \cap X_j = \varnothing$ when $i \neq j$, define $u$ on $X$ by $u(x) = u_i(x)/\pi(s_i)$ when $x \in X_i$. (9) is thus obtained from (18) but adds nothing to the content of (18). Because the $\pi(s_i)$ are arbitrary we would not think of them as subjective probabilities.

This example and previous discussion indicate that, in order to coherently view the $\pi(s_i)$ in (9) as subjective probabilities, it is essential to have consequences that can occur under more than one state (consequences contained in more than one $X_i$). This does not require constant acts: it does require a sufficiently rich overlap of the $X_i$. For example, in our first axiomatization, which is based on the Anscombe-Aumann notion of horse lotteries, we do not require constant acts but do assume that $X_i = X_j$ for all $i, j$. Moreover, it appears that a modification of our first axiomatization brought about by assuming only a minimal overlap and linkage among the $X_i$ can result in (9) with unique state probabilities. I hope to comment further on this and related ideas at another time.

**9. Acknowledgments.** I am indebted to Professor L. J. Savage and the referee for valuable advice on the material in this paper.

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