The Annals of Mathematical Statistics
1968, Vol. 39, No. 1, 294–312

ABSTRACTS OF PAPERS

(Abstracts of papers to be presented at the Annual meeting, Washington, D.C., December 27–30, 1967. Additional abstracts appeared in the June, August, October, and December issues.

32. Stochastic point processes on topological spaces. Robert A. Agnew, Northwestern University.

A stochastic point process can be considered roughly as a random collection of countably many points from some underlying space. Goldman [*Ann. Math. Statist.* **38** (1967) 771–779] has obtained a variety of interesting results for point processes on n-dimensional Euclidean space, his work being a significant departure from the highly structured real line, where a large body of theory already exists. Motivated by Goldman's work, we investigate point processes on a locally compact Hausdorff space. The operations of superposition and decomposition are characterized, and some general inversion theorems are obtained. Further, it is demonstrated that these results hold for more general stochastic processes with an abstract parameter space. Finally, point processes on a sigma-compact, locally compact Abelian topological group are investigated. Stationary and uniform point processes are characterized, and some translation theorems are obtained. The Bernoulli and Poisson processes are characterized on countable and uncountable groups respectively. (Received 16 October 1967.)

33. Multidimensional partially balanced designs for models containing interaction terms (preliminary report). Donald A. Anderson, University of Wyoming. (By title)

Consider an experiment involving m factors $F_1, F_2, \cdots, F_m$ where factor F_i has s_i levels, say $S_i = \{F_{i1}, F_{i2}, \cdots, F_{is_i}\}$, $i = 1, 2, \cdots, m$. Let $S_{ij} = S_i \times S_j$ and define the class of sets $\mathcal{C}$ so that $S_{ij} \varepsilon \mathcal{C}$ iff factors F_i and F_j interact and $S_k \varepsilon \mathcal{C}$ iff factor F_k does not interact with any other factor. The definition of the multidimensional partially balanced (MDPB) design for the additive model, [Bose and Srivastava, *Sankhyā* **26** (1964)] has been extended to include models containing interaction terms where a MDPB association scheme is defined on $\mathcal{C}$. It is shown that if the direct product association scheme is defined on $\mathcal{C}$ then MDPB designs must be simple direct products of connected PB designs, e.g., in a four-dimensional design with interactions F_1F_2 and F_3F_4 we have $T = T_{13} \times T_{24}$ where T_{13} and T_{24} are PB designs involving factors (F_1, F_3) and (F_2, F_4), respectively. Three- and four-dimensional designs for models containing one interaction term and four-dimensional designs for the model with two interaction terms discussed above are obtained by mapping the elements of the sets of $\mathcal{C}$ onto the elements of sets in a class $\mathfrak{D}$ on which a MDPB association scheme is defined. These designs are more economic than the direct product designs and are also MDPB under the model restricted by the hypothesis that one or more of the interaction terms are zero. (Received 6 November 1967.)

34. Estimation through preliminary test estimators. Jesse C. Arnold and S. K. Katti, Florida State University.

This paper investigates the problem of using a preliminary test to select an estimator of an unknown parameter, from a set of estimators of interest. For the case of two estimators, say $\hat{\theta}_1$ and $\hat{\theta}_2$, our procedure is basically as follows: We construct regions R and $\bar{R}$ such that the $\text{MSE}(\hat{\theta}_1 \mid \theta) \leqq \text{MSE}(\hat{\theta}_2 \mid \theta)$ if $\theta \varepsilon R$, and $\bar{R}$ is the complement of R. If $\hat{\theta}_1 \varepsilon R$, we use $\hat{\theta}_1$ as the estimator of θ, and if $\hat{\theta}_1 \varepsilon \bar{R}$, $\hat{\theta}_2$ is selected as the estimator. The partition $(R, \bar{R})$ was found to be admissible with respect to a number of partitions, and under certain conditions, the preliminary test estimator was found to be superior to the most efficient estimator used in the preliminary test. When one of the estimators used in the preliminary test is a sufficient statistic, an estimation procedure is given which uses the conditional expectation given the sufficient statistic as one of the estimators. These results are presented for the general case of estimating k parameters and are derived with the aid of a k-variate generalization of the well known Rao-Blackwell theorem. This procedure causes the generalized expected mean square to be uniformly smaller than for the usual estimator given above. (Received 12 October 1967.)

35. On a test for several linear relations. A. P. Basu, University of Wisconsin.

In this paper we have proposed a test for the parameters of k linear relations among the p variables $\xi_1, \xi_2, \cdots, \xi_p$ where the ξ's are not observable and $p > k$. This extends the work of Villegas [*Ann. Math. Statist.* **35** (1964)] who considered the above problem for the special case $k = 1$. The applicability of the above model in an econometric problem is also considered. (Received 20 October 1967.)

36. Some asymptotic results, based on censored data, for the maximum likelihood estimate and the posterior distribution of parameters subject to constraints (preliminary report). A. P. Basu and J. D. Borwanker, University of Wisconsin and University of Minnesota.

Let $X_1 < X_2 < \cdots < X_r$ be the first r $(= [np])$ ordered observations from a random sample of size n drawn from a population with distribution function $F(x; \boldsymbol{\theta})$ where $\boldsymbol{\theta}$, a K-dimensional parameter, belongs to a compact subset Θ of the K-dimensional Euclidean space E_K. Assume that the parameters are subject to s restraints $h_i(\boldsymbol{\theta}) = 0$ $(i = 1, 2, \cdots, s)$. Further, let the true parameter value $\boldsymbol{\theta}_0$ be an interior point of Θ. Then, under some regularity conditions, it can be shown that the maximum likelihood estimator $\bar{\theta}_n = \bar{\theta}_n(X_1, \cdots, X_r)$ is strongly consistent and asymptotically normally distributed. Further, if the constraints $h_i(\boldsymbol{\theta})$ are linear, then the posterior distribution converges in variation to a singular normal distribution almost surely. These results may be considered as generalizations of some of the results of Halperin [*Ann. Math. Statist.* **23** (1952), 226–238], Aitchison and Silvey [*Ann. Math. Statist.* **29** (1958), 813–828] and LeCam [*Pub. Inst. Univ. Paris* (1958), 17–35]. (Received 30 October 1967.)

37. The power of some tests for uniformity of a circular distribution. Rudolph J. W. Beran, The Johns Hopkins University.

Let f be a probability density in $L_2[0, 1]$ and let $(x_1, x_2, \cdots, x_n)$ be independent observations on a random variable with the density $g \in L_2[0, 1]$. This paper studies the asymptotic distribution under g of the test statistic $T_n = n^{-1} \int_0^1 [\sum_{i=1}^n f(x + x_i) - n]^2 \, dx$. T_n may be used to test whether observations from an unknown distribution on a circle are, in fact, uniformly distributed. For any $f \not\equiv 1$, the test is consistent. As special cases, T_n includes statistics previously proposed for the problem by Ajne, Watson, and the author. To find the asymptotic characteristic function of T_n requires the eigenfunctions and eigenvalues of a non-homogeneous covariance kernel. A technique for approximating these to any degree of accuracy is presented. Inverting the characteristic function is awkward; however, a worked-out example suggests that a good approximation to the power of T_n can be obtained from its first two moments. The Bahadur slope and the expected significance level of T_n are calculated approximately from the asymptotic distributions. Unfortunately, the resulting criteria have serious drawbacks as measures of the performance of T_n. (Received 19 October 1967.)

38. Ergodic theory with recurrent weights. KENNETH N. BERK, Northwestern University.

Let w be the distribution function of a nonnegative random variable Y and let v be the solution of the renewal equation $v(x) = 1 + \int_{[0,x]} v(x - \alpha)\, dw(\alpha)$, $x \geqq 0$; $v(x) = 0$, $x < 0$. The measures generated by w and the nondecreasing function v are assumed to have support in T, an additive subsemigroup of $[0, \infty)$. Let $\{X_t, t \varepsilon T\}$ be a measurable stationary process on the probability space (Ω, Σ, P) for which EX_t exists. Then $\lim_{c\to\infty}(1/v(c)) \int_{[0,c]} X_t\, dv(t)$ exists with probability one. This result has the advantage of including both discrete and continuous ergodic theorems as special cases: (1) $Y = 1$ with probability one, and $T = \{0, 1, 2, \cdots\}$, and $v(x) = [x] + 1$; (2) Y is negative exponential, $w(\alpha) = 1 - e^{-\alpha}$, $T = [0, \infty)$, and $v(x) = 1 + x$. The proof is carried out in a more general operator-theoretic setting, in which the limit function is also discussed. Baxter proved the discrete version of the result in *J. Math. Mech.* **14** (1965) 277–288. (Received 6 November 1967.)

39. Single and sequential two-phase sampling plans for a dependent variable having auxiliary correlated variables. R. P. BHARGAVA, University of Toronto. (By title).

In this paper at first a single two-phase sampling plan for a dependent variable y having auxiliary variables $x_1, \cdots, x_k$ is considered for the case when an upper specification limit is prescribed for y. The assumptions made are that the vector random variable $(y, x_1, \cdots, x_k)$ has a multivariate (nonsingular) normal distribution with the covariance matrix known but with mean vector unknown. In particular, ρ, the multiple correlation coefficient between y and $x_1, \cdots, x_k$, is assumed to be known, as also σ_y^2, which is the variance of y. Assuming that it costs $s(> 1)$ dollars to measure y for an item and 1 dollar to measure $(x_1, \cdots, x_k)'$ and t dollars to measure $(y, x_1, \cdots, x_k)'$, optimum 1st and 2nd phase sample sizes (denoted respectively by $\hat{n}$ and $\hat{N}$) are determined for a given ρ which while giving the desired quality assurances, minimize the cost of a single two-phase sampling plan. By a 1st phase sample of size $\hat{n}$ is meant that for $\hat{n}$ items $(y, x_1, \cdots, x_k)'$ is measured and by a 2nd phase sample of size $\hat{N}$ is meant that for $\hat{N}$ items $(x_1, \cdots, x_k)'$ is measured. The optimum single two-phase sampling plan $(\hat{n}, \hat{N})$ is then compared with the comparable single sampling plan based on y alone and it is indicated when it (the two-phase one) should be preferred. For this case also two sequential two-phase sampling plans are given. The question of selection of the best subset of the auxiliary variables is discussed. (Received 20 November 1967.)

40. Robust estimation of multivariate linear trend. G. K. BHATTACHARYYA, University of Wisconsin.

Consider n independent random vectors $\mathbf{Y}_i = (Y_{1i}, Y_{2i}, \cdots, Y_{pi})$ representing observations of a p-component time series at times $i = 1, 2, \cdots, n$ and let $\mathbf{Y}_i$ have continuous cdf $\Psi(\mathbf{y} - \boldsymbol{\mu} - \boldsymbol{\theta}_i)$ where $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_p)$ is the linear trend parameter and $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_p)$ is the common location of $\mathbf{Y}_i$'s in the absence of trend. It is assumed that Ψ has symmetric marginal densities and that it is not degenerate. The following median and weighted median estimates of $\boldsymbol{\theta}$ are considered: $\hat{\boldsymbol{\theta}}_M = \text{median}\{(\mathbf{Y}_i - \mathbf{Y}_j)/(j - i); 1 \leqq i < j \leqq n\}$ and $\hat{\boldsymbol{\theta}}_L = \text{median}\{(\mathbf{Y}_i - \mathbf{Y}_j)/(j - i)$ with frequency $(j - i); 1 \leqq i < j \leqq n\}$. These are shown to be unbiased and asymptotically multivariate normal. Their relative efficiency with respect to the maximum likelihood estimate for normal models is obtained in terms of the asymptotic generalized variance. Bounds on the efficiency are discussed in some specific situations. A median estimate is proposed also for $\mathbf{\mu}$ and compared asymptotically with the classical estimate. (Received 23 October 1967.)

41. A new family of life distributions. Z. W. Birnbaum and S. C. Saunders, University of Washington and Boeing Scientific Laboratory.

Consideration of fatigue-failure due to crack-extension under certain assumptions leads to the two-parametric family $\mathfrak{F}$ of probability distributions: $P\{T \leqq t\} = \phi\{[(t/\beta)^{\frac{1}{2}} - (\beta/t)^{\frac{1}{2}}]/\alpha\}$, for $t \geqq 0$, where $\phi(u)$ is the standardized normal distribution function, as an appropriate family of fatigue-life distributions. This family has, among others, the property that if T has a distribution in $\mathfrak{F}$ then T^{-1} has a distribution in $\mathfrak{F}$. These properties make $\mathfrak{F}$ particularly useful for interpreting and using "Miner's rule" in the study of fatigue-life lengths under variable dynamic loading. In view of these applications, numerical techniques for computing maximum likelihood estimates for α and β have been developed in some detail. In particular, the expression $(rs)^{\frac{1}{2}}$, where r is the arithmetic mean and s the geometric mean of a sample of life-lengths, is already a practical estimate for β. (Received 8 November 1967.)

42. Estimation of the larger of two normal means. Saul Blumenthal and Arthur Cohen, New York University and Rutgers, The State University.

Let $X_{i1}, X_{i2}, \cdots, X_{in}$, $i = 1, 2$, be a pair of random samples from populations which are normally distributed with means θ_i, and common known variance τ^2. The problem is to estimate the function $\varphi(\theta_1, \theta_2) = \text{maximum}(\theta_1, \theta_2)$. In this paper we consider five different estimators (or sets of estimators) for $\varphi(\theta_1, \theta_2)$ and evaluate their biases and mean square errors. The estimators are (i) $\varphi(X_1, X_2)$, where X_i is the sample mean of the ith sample; (ii) the analogue of the Pitman estimator, i.e. the a posteriori expected value of $\varphi(\theta_1, \theta_2)$ when the prior distribution is the uniform distribution on two dimensional space; (iii) a class of estimators which are Bayes with respect to priors which are products of uniform and normal priors; (iv) hybrid estimators, i.e. those which estimate by $(X_1 + X_2)/2$ when $|X_1 - X_2|$ is small, and estimate by $\varphi(X_1, X_2)$ when $|X_1 - X_2|$ is large; (v) maximum likelihood estimator. The bias and mean square errors for these estimators are tabled, graphed, and compared. Also the invariance properties of these estimators are discussed with a rationale for restricting to invariant estimators. (Received 30 October 1967.)

43. Higher moments of a stopping rule. Bruce M. Brown, Purdue University.

Let $\{X_n\}$ be independent, $EX_n = 0$, $EX_n^2 = 1$, $S_n = \sum_1^n X_i$ for $n = 1, 2, \cdots$, with the stopping rule $t = t_m(c) = \inf\{n: |S_n| > cn^{\frac{1}{2}}\}$. A Lindeberg condition of order $v \geqq 2$, defined by the property $\sum_1^n E|X_j|^v I[|X_j| \geqq \epsilon n^{\frac{1}{2}}] = O(n^{v/2})$, $n \to \infty$, all $\epsilon > 0$, is shown to be sufficient for the convergence of $E(S_n \cdot n^{\frac{1}{2}})^v$ to the vth moment of a $N(0, 1)$ distribution as $n \to \infty$, and necessary when $v = 2k$, $k = 1, 2, \cdots$. Then it is shown that if the $\{X_n\}$ obey a Lindeberg condition of order $2k$, $Et^k < \infty$ for $c < c_k$ and all m, while $Et^k = \infty$ for m sufficiently large when $c \geqq c_k$, where c_k is the smallest positive root of the Hermite polynomial of order $2k$. Results of Chow-Teicher [*Ann. Math. Statist.* **37** (1966) 388–392] and Gundy-Siegmund [On a stopping rule and the central limit theorem, to appear in *Ann. Math. Statist.*] are thus extended to higher moments, and a conjecture of L. A. Shepp [A first passage problem for the Wiener process, to appear in *Ann Math. Statist.*], is verified. (Received 30 October 1967.)

44. Asymptotic Bayes sequential tests of the null hypothesis (preliminary report). THURMAN J. BROWN, JR., Michigan State University.

Let $\{X_t: t > 0\}$ be a Gaussian process with drift μ. This paper is concerned with approximating the optimal sequential procedure for testing $H_0: \mu = 0$ vs $H_1: \mu \neq 0$ for the large sample case when the prior distribution for the alternative is approximately Lebesgue and the loss is approximately proportional to $\|\mu\|^k$. The fixed sample size problem was treated by Rubin and Sethuraman [*Sankhyā Ser. A* **27** (1965) 347–356]. The solution is similar to that of Chernoff [Sequential tests for the mean of a normal distribution. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* 1 79–91. Univ. of California Press]. It consists of two strictly positive functions $a_0(t)$ defined on (τ, ∞), $\tau > 0$, and $a_1(t)$ defined on $(0, \infty)$ with $a_0(t) < a_1(t)$ which determine the following sequential procedure. Observe $\{X_t\}$ until $\|X_t\| = a_i(t)$. If $i = 0$ accept, if $i = 1$ reject. The asymptotic nature of the solution is derived, and standard numerical procedures are used to approximate the regions and the risk. Rubin and Sethuraman's work has shown that the general asymptotic testing problem may be reduced to the above case. (Received 2 November 1967.)

45. Some contributions to the theory of homogeneous random fields on locally compact Abelian groups (preliminary report). LAWRENCE A. BRUCKNER, The Catholic University of America.

Let $X(g)$ be a homogeneous random field on a locally compact abelian group G. Let I be a family of non-empty Borel sets of G. Regularity and singularity are defined relative to the family I. When I is closed under translations for all g in G, $X(g)$ may be written as a sum of an I-regular and I-singular component. These components are mutually orthogonal, homogeneous random fields which are subordinate to $X(g)$. Kolmogorov's theorems on linear transformations and subordinate stationary sequences are extended to the corresponding results on G. When G is discrete, necessary and sufficient conditions for I-regularity and I-singularity are given for two special I families, the family of singletons of G and the G-restriction of the neighborhood system at infinity in the one-point compactification of G. Interpolation formulas are also obtained for these cases. (Received 27 October 1967.)

46. Some properties of the incomplete beta function. T. CACOULLOS, New York University.

Several recursive relations and series expansions for the incomplete beta function $I_x(p, q)$ are obtained by using the difference-differential equations $x(1 - x)\, D\, I_x(p, q) = -p\, \Delta\, I_x(p, q) = q\, \Delta^*\, I_x(p, q)$ where D denotes differentiation with respect to x, and Δ and Δ^* denote the usual advancing finite differences with respect to p and q, respectively. Thus, well-known and other recursive formulae for $I_x(p, q)$, which follow from properties of the hypergeometric function $F(a, b, c; x)$, can be obtained directly. (Received 7 November 1967.)

47. On combinatorial composition and extension methods in the construction of designs. I. M. CHAKRAVARTI, University of North Carolina at Chapel Hill.

By *combinatorial composition* is meant a rule for combining two or more given designs or configurations. For instance, given two latin squares of order U and V respectively, it is possible to give a rule for combining these two latin squares into one of order UV. An example of *combinatorial extension* is the familiar problem of adjoining $n - r$ rows and $n - s$

columns to an $r \times s$ latin rectangle in n symbols, so that the resulting configuration is a Latin square or order n. This paper discusses many more examples where combinatorial composition and extension have been successfully used to construct incidence matrices of BIB designs, orthogonal Latin squares, response surface designs and other designs and configurations. (Received 7 November 1967.)

48. A Bayesian analysis of pilot surveys. Samprit Chatterjee, New York University.

In stratified sampling the optimum allocation is determined by using estimates of strata variances. These estimates are usually obtained from the results of a previous survey. Use of the estimated values leads to a non optimal allocation which results in a loss. Sometimes it may be possible however to conduct a pilot survey to obtain more reliable estimates of strata variances. In this paper we study the gain due to additional sampling. The analysis is carried out in the Bayesian framework. The strata variances are assumed to have a gamma prior distribution. A posterior and preposterior analysis is carried out. From this an "optimum" design for a pilot survey is derived. The design is "optimum" in the sense that it maximizes the net gain due to sampling or equivalently minimizes the posterior loss. (Received 23 October 1967.)

49. "Renewal" limit theorems for excessive functions of semi-Markov chains on arbitrary spaces. Erhan Çinlar, Northwestern University.

Let $(E, \mathfrak{F})$ be an arbitrary measurable space, and let $R = (-\infty, +\infty)$, $R^+ = [0, \infty)$, $\mathfrak{B}$ the Borel sets of R^+. A semi-Markovian kernel is a mapping Q: $E \times \mathfrak{F} \times \mathfrak{B} \rightarrow [0, 1]$ such that, (i) $Q(., F, B)$ is $\mathfrak{F}$-measurable for each fixed pair $F \varepsilon \mathfrak{F}$ and $B \varepsilon \mathfrak{B}$, (ii) $Q(x, \cdot, \cdot)$ is a measure on $\mathfrak{F} \times \mathfrak{B}$ for each fixed $x \varepsilon E$. Let L be the Banach space of $\mathfrak{F} \times \mathfrak{B}$-measurable functions f from $E \times R$ into R such that $f(\cdot, t)$ vanishes whenever $t < 0$, and with norm defined by $\|f\| = \sup_{x\varepsilon E} \int_{R^+} e^{-\lambda t} |f(x, t)| \, dt$, ($\lambda > 0$ is fixed). The paper is about the limiting behaviour as $t \uparrow \infty$ of functions $f \varepsilon L$ satisfying $f(x, t) = g(x, t) + \int_E \int_{R^+} f(y, t - u) Q(x, dy, du)$, $g \varepsilon L$. Limit theorems we give are generalizations of the renewal theorems of Blackwell and Smith (which become the special cases of ours where E consists of a single point). If E is countable, then results are well known through renewal theory. If E is arbitrary, then the problem needs special attention due to the lack of imbedded renewal processes. The kernel Q defines a semi-Markov chain with state space $(E, \mathfrak{F})$, (where, considering the movement of a particle in E, $Q(x, F, B)$ is the probability that the particle moves from x to a point $y \varepsilon F$ after staying at x some time $t \varepsilon B$). Most questions concerning the additive functionals of such a process, as well as some questions concerning the sample paths, reduce to questions concerning the existence, uniqueness, and the limiting behaviour of the solutions of the above equation. (Received 2 November 1967.)

50. Some probabilistic models of neuron firing (preliminary report). Roger D. Coleman and Joseph L. Gastwirth, The Johns Hopkins University.

Ten Hoopen and Reuver [*J. Appl. Prob.* **2** (1965) 286–292] have considered two renewal processes, one consisting of stimuli and the other of inhibitory impulses, which interact in the following way: whenever one or more inhibitory impulses occur, the next arriving stimulus is eliminated. Those stimuli which have not been eliminated form the response process. They analyzed the response process by obtaining the Laplace transform of the interresponse density function. When the inhibitory process is a Poisson process, we generalize the model by allowing the inhibitors to be effective only for a random time T. Moreover, when the stimuli also arrive according to a Poisson process, other models of interaction are discussed.

In all cases considered, the Laplace transform to the interresponse density is obtained. (Received 3 November 1967.)

51. Stochastically monotone Markov chains. D. J. DALEY, University of Washington.

The class of discrete time Markov chains $\{X_n\}$ with one-step transition functions satisfying Condition M: $pr\{X_{n+1} \leqq y \mid X_n = x\}$ *is for every fixed* y *non-increasing for increasing* x, arises in a natural way when it is sought to "bound" (in appropriate stochastic sense) the process $\{X_n\}$ by means of a "smaller" or "larger" process with the same transition probabilities. This class of Markov chains includes many simple models of applied probability theory, as for example random walks, queues, branching processes, genetics, and epidemics. A given process satisfying Condition M is readily bounded by another process with different transition probabilities, and this is of particular value when the latter process leads to simpler algebraic manipulations. Kalmykov [*Theor. Prob. Appl.* **7** (1962) 456–459] has a similar result. A stationary Markov chain $\{X_n\}$ satisfying Condition M has $\operatorname{cov}(f(X_0), f(X_n)) \geqq \operatorname{cov}(f(X_0), f(X_{n+1})) \geqq 0$ $(n = 1, 2, \cdots)$ for any monotonic function $f(\cdot)$. (Received 27 October 1967.)

52. Some non-regular functions of finite Markov chains. S. W. DHARMADHIKARI, Michigan State University.

We use the numbers $n(\epsilon)$ and the term regular function of a finite Markov chain introduced by Gilbert [*Ann. Math. Statist.* **30** (1959) 688–697]. Let $0 < \lambda \leqq \frac{1}{2}$, $0 < \theta < 2\pi$ and $\theta \neq \pi$. Let $c_j = \lambda^j \sin^2(j\theta/2)$, $j = 1, 2, \cdots$, and $c = \sum_{j=1}^{\infty} c_j$. Suppose $\{X_n\}$ is a stationary Markov chain with state-space $\{0, 1, 2, \cdots\}$ and transition matrix $\|m_{jk}\|$, where $m_{00} = 1 - c$ and, for $j \geqq 1$, $m_{0j} = c_j$ and $m_{j,j-1} = 1$. Let $Y_n = f(X_n)$ where $f(0) = \delta$ and $f(j) = \epsilon$ for $j \geqq 1$. Then $\{Y_n\}$ is a stationary process with $n(\delta) = 1$ and $n(\epsilon) = 3$. The following theorem is proved. THEOREM: (a) *If* θ *is an irrational multiple of* 2π *then* $\{Y_n\}$ *is not a function of a finite Markov chain.* (b) *If* $\theta = 2\pi\, \nu/N$, *where* ν *and* N *are relatively prime integers, then* $\{Y_n\}$ *is a function of a Markov chain with* $(N + 1)$ *states. Moreover, no representation as a function of a Markov chain with less than* $(N + 1)$ *states is possible. Thus, for* $N \geqq 4$, $\{Y_n\}$ *is a non-regular function of a finite Markov chain.* The main tool used in the proof is the Frobenius theorem on non-negative matrices. The example given by Fox and Rubin [Michigan State University, Research Memorandum 193, 1967] corresponds to $\lambda = \frac{1}{2}$ and $\theta = 2$. (Received 3 November 1967.)

53. On batch service with balking in the single server queueing process. RONALD S. DICK, C. W. Post College.

This paper extends the previous results of the writer for a single server queue (1) [R. Dick, On the queue length distribution with balking in a single server queueing process. *Ann. Math. Statist.* **37** (1966) abstract 762–3] to the more general case where the service is batched. r_1 is the least number of customers before service will begin, and r_2 is the maximal number of customers that can be served in one service. As in Dick (1966) after the waiting line exceeds a given length m, customers refuse to enter with probability $q = 1 - p$. Virtual waiting times and busy period results are also given. The input distribution of customers is exponential and the service distribution is general. (Received 1 November(1967.)

54. Starshaped transformations and the power of rank tests. KJELL A. DOKSUM, University of California.

h is starshaped if $h(0) = 0$ and $h(t)/t$ is nondecreasing for $t > 0$. On the class $\mathfrak{F}$ of continuous distributions F with $F(0) = 0$, define $<_*$ by: $F_1 <_* F_2$ iff $F_2^{-1}F_1$ is starshaped. Thus if $F_1 <_* F_2$, then F_2 is "less skew" (to the right) than F_1. On the class $\mathcal{G}$ of continuous distributions G with $G(0) = \frac{1}{2}$ and densities g, define $<_t$ by: $G_1 <_t G_2$ iff $G_2^{-1}G_1(x)$ and $-G_2^{-1}G_1(-x)$ are starshaped, and $g_2(0) \leqq g_1(0)$. Thus if $G_1 <_t G_2$, then G_2 has a "heavier tail" than G_1. Let $\beta_s(\varphi; F, \Delta)$ denote the power of the test φ in the two-sample scale problem with scale parameter $\Delta(H_0: \Delta \leqq 1$ vs. $H_1: \Delta > 1)$. THEOREM 1. *If φ is a monotone rank test, $F_1, F_2 \varepsilon \mathfrak{F}$, and $F_1 <_* F_2$, then $\beta_s(\varphi; F_2, \Delta) \leqq (\geqq) \beta_s(\varphi; F_1', \Delta)$ for each $\Delta \leqq 1(\geqq 1)$.* Thus the error probabilities are increasing functions of "skewness." Let $\beta_t(\varphi; G, \theta)$ denote the power of the test φ in the two-sample shift problem with shift parameter $\theta(H_0$: $\theta \leqq 0$ vs. $H_1: \theta > 0)$. THEOREM 2. *If φ is a monotone rank test, $G_1, G_2 \varepsilon \mathcal{G}$, and $G_1 <_t G_2$, then $\beta_t(\varphi; G_2, \theta) \geqq (\leqq) \beta_t(\varphi; G_1, \theta)$ for $\theta \leqq (\geqq) 0$.* Thus the error probabilities are increasing functions of "heavy tails." These results are used to derive minimax tests and to give bounds on the error probabilities of monotone sequential rank tests for the two-sample IFRA scale problem. (Received 25 October 1967.)

55. An approximation to the sample size in a selection procedure of Bechhofer and its extension to selection from a multivariate normal population (preliminary report). EDWARD J. DUDEWICZ, University of Rochester.

For populations $\pi_1, \cdots, \pi_k$ $(k \geqq 2)$ the observations from which are normally distributed with unknown means $\mu_1, \cdots, \mu_k$ (respectively) and common known variance $\sigma_1^2 = \cdots = \sigma_k^2 = \sigma^2$, Bechhofer [*Ann. Math. Statist.* **25** (1954) 16–39] studied a formulation and procedure for selecting the population associated with $\mu_{[k]} = \max(\mu_1, \cdots, \mu_k)$. (His procedure, which selects the correct population wp at least p^* if $\mu_{[k]} - \mu_{[k-1]} \geqq \lambda^* \sigma$, takes N independent observations independently from each population and selects the population yelding the largest sample mean.) We show that $N \sim -4 (\lambda^*)^{-2} \log (1 - P^*)$, the ratio tending to 1 as $N \to \infty$ (due to having $P^* \to 1$). The approximation for finite N is studied numerically. The nature of our proof shows that this approximation holds even if the components within each vector of observations $(X_{i1}, \cdots, X_{ik})$ $(i = 1, \cdots, N)$ are correlated, i.e. if we wish to select the component which has the highest mean from a multivariate normal population (with $\sigma_1^2 = \cdots = \sigma_k^2 = \sigma^2$). Since the configuration of $\mu_1, \cdots, \mu_k$ used to set N for this multivariate problem is as for the univariate problem, we have a large-sample solution for this multivariate problem. (Received 23 October 1967.)

56. The estimation of the probability that $Y < X$. PETER ENIS and SEYMOUR GEISSER, State University of New York at Buffalo.

Bayesian procedures are developed for estimating the probability that $Y < X$ when the functional form of the joint density $f(x, y \mid \alpha)$ is known but where either a subset or the entire set of parameters α are unknown, given that independent observations on $f(x, y \mid \alpha)$ are available. Basically there are two questions that one can ask here. The first is the estimation of the parameter $\theta = \theta(\alpha) = \Pr[Y < X \mid \alpha]$; i.e., the probability conditioned on α. This is essentially analogous to the classical theory. The other question is: Given a future pair of observations, say (X, Y), what is the predictive probability that $Y < X$. From the Bayesian point of view, the estimation of θ in the first problem is answered by finding the posterior density of θ given some prior density for θ. In particular, if one is required to give a point estimate he may choose it to be $\bar{\theta}$ the posterior mean of θ. The solution for the second question is to find the posterior probability (unconditionally), $\Pr[Y < X]$. It is shown that the point estimator, $\bar{\theta}$, and the unconditional probability, $\Pr[Y < X]$, are in fact equal. Certain special cases involving normal and exponential distributions are considered. The Bayesian estimates are then compared, when possible, with their confidence counterparts which are also here derived. (Received 20 October 1967.)

57. Stochastic approximation for smooth functions (preliminary report). VÁCLAV FABIAN, Michigan State University and Czechoslovak Academy of Sciences.

Let C be a class of functions defined on R^k, $f \varepsilon C$ iff there exist positive numbers r, K_1, K_2, a point $\theta \varepsilon R^k$ and a neighborhood N of θ such that: for any $s = 1, 2, \cdots, f$ has partial derivatives of order s in N, bounded in N by $(s + 1)!r^sK_1$. (i) The Hessian of f exists and is bounded in R^k. (ii) The vector $D(x)$ of the first partial derivatives satisfies $D(\theta) = 0$, $(x - \theta)'D(x) \geqq K_2\|x - \theta\|^2$. An approximation procedure is described which generates, for each $f \varepsilon C$, an approximating sequence $X_1, X_2, \cdots$, satisfying $E(X_n - \theta)^2 = o(t_n^{-1} \log^3 t_n)$ where θ is the stationary point of f and t_n is the number of observations necessary to construct $X_1, X_2, \cdots, X_n$. The procedure is obtained from that considered in Fabian (*Ann. Math. Statist.* **38** 191–200) by choosing suitably a_n, c_n, the design $[u_1, u_2, \cdots, u_m]$ and by letting $m = \log n \log_2^{-1} n$. (Received 7 November 1967.)

58. On the distribution of the log likelihood ratio test statistic when the true parameter is "near" the boundary of the hypothesis regions (preliminary report). PAUL I. FEDER, Yale University.

Let $P_\theta(x)$ be a family of probability measures indexed by a k-dimensional parameter $\theta = (\theta_1, \cdots, \theta_k)$, $\theta \varepsilon \Theta$. Assume that densities $f(x, \theta)$ exist with respect to a σ-finite measure μ. It is desired to test the hypotheses H_1: $\theta \varepsilon \omega_1$ vs. H_2: $\theta \varepsilon \omega_2$ where ω_1 and ω_2 are disjoint subsets of Θ. Let $\lambda^* = \sup_{\theta\varepsilon\omega_1} \prod_{j=1}^n f(X_j ; \theta)/\sup_{\theta\varepsilon\omega_2} \prod_{j=1}^n f(X, \theta)$ be the likelihood ratio test statistic. The asymptotic behavior of $-2 \log \lambda^*$ is considered when the true state of nature is *near* the boundaries of ω_1 and ω_2. That is, the true state of nature is regarded as a *sequence* of points θ_{0n} such that $\theta_{0n} - \theta_0 = o(1)$, where θ_0 is a boundary point of both ω_1 and ω_2. Three cases are considered: (a) $\theta_{0n} - \theta_0 = O(n^{-\frac{1}{2}})$. (b) $n^{\frac{1}{2}}(\theta_{0n} - \theta_0) \rightarrow \infty$ but $d(\theta_{0n}, \omega_1)$ and $d(\theta_{0n}, \omega_2)$ are both $O(n^{-\frac{1}{2}})$ (where $d(\theta, \varphi)$ is the Euclidean distance from the point θ to the set φ. (c) $n^{\frac{1}{2}}(\theta_{0n} - \theta_0) \rightarrow \infty$ and $\max \{d(\theta_{0n}, \omega_1), d(\theta_{0n}, \omega_2)\}$ is large when compared with $n^{-\frac{1}{2}}$. Under regularity conditions essentially those needed to imply the asymptotic normality of the mle, cases (a) and (b) give rise to noncentral versions of the results obtained by H. Chernoff [*Ann. Math. Statist.* (1954)] and case (c) leads to a degenerate distribution. These results also constitute a generalization of results obtained by S. S. Wilks [*Ann. Math. Statist.* (1938)] and A. Wald [*Tran. Amer. Math. Soc.* (1943)]. (Received 6 November 1967.)

59. Some results concerning stopping rules (preliminary report). LEON J. GLESER and S. ZACKS, Johns Hopkins University and Kansas State University.

As part of our investigation (unpublished) of sequential fixed-width confidence interval procedures for estimating the common mean of two distributions having unequal variances, we have found it necessary to extend some of the results of Chow and Robbins [*Ann. Math. Statist.* **36** (1965) 457–462] to cover somewhat more general kinds of stopping rules. For example, although Chow and Robbins treat only one-sided stopping boundaries, the present investigation requires us to consider two-sided (or even many-sided) stopping boundaries. Further, some of the boundaries of interest to us have a slightly more general functional form than do the stopping boundaries discussed in the Chow-Robbins paper. The results given in the present paper provide first-order asymptotic results for the distribution and expected value of the various stopping variables N (the random number of observations taken) considered, the form of the results being similar to those announced in the Chow-Robbins paper referred to above. (Received 16 October 1967.)

60. Asymptotic normality of linear combinations of functions of order statistics I. Z. GOVINDARAJULU, Case Western Reserve University. (By title)

This paper deals with the asymptotic normality of a class of linear combination of functions of order statistics in a random sample drawn from a continuous population, which can be written as the sum, $\sum_1^N a(t_i)H(U_{i,N})$ where $t_i(i/N+1)$, $i = 1, 2, \cdots, N$, nd $U_{i,N}$, $(i = 1, \cdots, N)$ denote the uniform order statistics in a random sample of size N. The class, in particular, includes the systematic statistics. The function $a(t_i)$ is expanded as $a(t_i) = a(U_{i,N}) + (t_i - U_{i,N}\, a'(U_{i,N}) +$ remainder and sufficient conditions for the asymptotic normality of the sum, when suitably standardized, are obtained. In particular, the weights $a(t_i)$ could be expected values of some suitable order statistics. The case when the sample size is random is also considered. Some applications are given. (Received 8 November 1967.)

61. Asymptotic normality of linear combinations of functions of order statistics II. Z. GOVINDARAJULU, Case Western Reserve University.

Asymptotic normality of the sum $\sum_1^N a(t_i)H(U_{i,N})$, where $t_i = (1/N+1)$, $i = 1, \cdots, N$, and $U_{i,N}$ $(i = 1, \cdots, N)$ are uniform order statistics in a random sample of size N has been studied by Chernoff, Gastwirth and Johns, Jr. [*Ann. Math. Statist.* **38** (1967) 52–72]. Although their results are quite interesting, their sufficient conditions are somewhat too restrictive and the normalizing constants are not easy to evaluate (See Theorem 1 of Chernoff et al.). The main purpose of this paper is, to unify, simplify, strengthen and generalize Theorems 1–3 of Chernoff et al., in a single theorem. We expand $H(U_{i,N}) = H(t_i) + (U_{i,N} - t_i)H'(U_{i,N}) +$ remainder and obtain sufficient conditions for the asymptotic normality of the sum when suitably standardized. These results are extended to the situation where the weights $a(t_i)$ are expected values of some suitable order statistic, to vector valued statistics and to statistics based on random samples. (Received 8 November 1967.)

62. Optimal ranking and selection procedures. Z. GOVINDARAJULU and H. SMITH HALLER, JR., Case Western Reserve University.

Let $\pi_1, \pi_2, \cdots, \pi_c$ denote populations having parameters $\theta_1, \theta_2, \cdots, \theta_c$ associated with their respective distributions. For the problem of ranking $\theta_1, \theta_2, \cdots, \theta_c$ we consider the risk associated with invariant decision functions and loss functions introduced by Eaton [*Ann. Math. Statist.* **38** (1967) 124–137]. If $X_{i,j}$ $(j = 1, 2, \cdots, n)$ is a random sample drawn from π_i $(i = 1, 2, \cdots, c)$ and $T = (T_1, T_2, \cdots, T_c)$ denotes the vector of statistics defined on the combined sample, the "natural" decision procedure is to order the values $\theta_1, \theta_2, \cdots, \theta_c$ according to the ordering of $T_1, T_2, \cdots, T_c$. A direct proof of Eaton's Theorem 4.2 is obtained which gives sufficient conditions for the "natural" decision procedure to be optimal. Locally optimal and asymptotically optimal (i.e. minimum risk, Bayes, minimax, and admissible) nonparametric decision procedures are defined. Sufficient conditions for the "natural" decision procedure based on rank order statistics to be optimal are obtained. These results are applied to statistics of the type introduced by Savage (*Ann. Math. Statist.* **27** (1956) 590–615) and Deshpande [*J. Indian Statist. Assoc.* **3** (1965) 20–29]. (Received 8 November 1967.)

63. c-sample rank orders. Z. GOVINDARAJULU and H. SMITH HALLER, JR., Case Western Reserve University.

The extension of the theory of rank orders to the c-sample situation is considered. As in the two-sample case [see Savage, *Ann. Math. Statist.* **27** (1956) 590–615] the primary emphasis is on obtaining conditions for admissible rank order tests. If the rank order probabilities are well ordered it is shown that a most powerful rank order test for the c-sample hypothesis exists. For the general class of slippage alternatives it is shown that the rank order probabilities are not well ordered. A partial ordering of rank order probabilities is established for alternatives having the monotone likelihood ratio property and for generalized Lehmann alternatives. In the latter case the ordering is obtained by using rank order statistics which belong to the Chernoff-Savage class. For ordered alternatives of the Lehmann type, the rank orders are shown to be locally well ordered by these statistics. (Received 8 November 1967.)

64. On selection and ranking procedures and order statistics from the multinomial distribution. S. S. GUPTA and KLAUS NAGEL, Purdue University.

For the multinomial distribution, two procedures R and T are proposed for selecting a subset to contain the cells with the largest and the smallest probability, respectively. Two theorems are proved to obtain the worst configuration where the overall minimum of the probability of a correct selection takes place. The actual probability of a correct selection, the expected proportion in the selected subset and the probability of selecting any fixed non-correct cell for the slippage-type configurations, are computed. Order statistics, the largest and the smallest, from the multinomial are discussed and their mean and variance are computed. (Received 15 November 1967.)

65. Construction of β-content-confidence level γ tolerance regions when sampling from the k-variate normal and sample sizes are large (preliminary report). IRWIN GUTTMAN, University of Wisconsin. (By title)

Suppose $\mathbf{X}_1, \cdots, \mathbf{X}_n$ are independent observation on $\mathbf{Y}$, where $\mathbf{Y}$ is distributed as $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. If $S(\mathbf{X}_1, \cdots, \mathbf{X}_n)$ is the region given by $S(\mathbf{X}_1, \cdots, \mathbf{X}_n) = \{\mathbf{Y} \mid (\mathbf{Y} - \bar{\mathbf{X}})'V^{-1}(\mathbf{Y} - \bar{\mathbf{X}}) \leqq K^{(k)}\}$ and if C is the coverage of this region, that is $C = (\int_s dN \mid \boldsymbol{\mu}, \boldsymbol{\Sigma})$, then the following result obtains: THEOREM. *If sampling from the k-variate normal distribution, the tolerance region S given above has coverage C where mean μ_c and variance σ_c^2 are, to terms of order $1/n$,*

$$\mu_c = \Psi_k(K^{(k)}) - [K^{(k)}]^{h/2}e^{-k^{(k)}/2}[2^{k/2+1}\Gamma(k/2)n]^{-1}$$

and

$$\sigma_c^2 = [K^{(k)}]^k e^{-K^{(k)}}[k2^{k-1}\Gamma^2(k/2)n]^{-1}.$$

An approximation to the distribution of C may now be made by fitting to a $B(p, q)$ distribution, and on doing this, approximations to $K^{(k)}$ may be made, where $K^{(k)}$ are chosen to make $\Pr(C \geqq \beta) = \gamma$. Comparison with a known approximation for $k = 1$ of Wald and Wolfowitz (*Amer. Math. Soc.* (1946)] are given, and tables for $k = 2$, 3 and 4 also given. (Received 27 October 1967.)

66. Investigation of rules for dealing with outliers in small samples from the normal distribution I: Estimation of the mean. IRWIN GUTTMAN and DENNIS E. SMITH, University of Wisconsin and HRB-Singer, Inc.

Suppose an experimenter takes a sample of n observations, hopefully all from $N(\mu, \sigma^2)$, but where a spurious observation from $N(\mu + a\sigma, \sigma^2)$ or $N(\mu, (1 + b)\sigma^2)$ may be present in the sample. In such a situation, the experimenter may be interested in a rule for dealing with outliers in his data. Three rules, each providing an unbiased estimate in the null case, are investigated via the "premium-protection" approach of Anscombe [*Technometrics* **2** (1960) 123–147] when the goal of the experiment is to estimate μ. Each of these rules is composed of a statistic t and a rejection region R. If $t \not\in R$, each rule uses estimator $\bar{y}$. If $t \in R$, these rules differ, and may be described as follows: (1) Anscombe's rule: If $t \in R$, the suspect observation is discarded, and estimation proceeds using the remaining $(n - 1)$ observations as a "new" sample. (2) The Winsorization rule: If $t \in R$, the suspect observation is given the value of its nearest neighbor, and estimation proceeds using this "new" sample of n observations. (3) The Semiwinsorization rule: If $t \in R$, the statistic t is given the value of the nearest rejection boundary. Estimation proceeds, subject to the constraint imposed by this procedure. In general, computation of premiums and protections for a sample of size n involves $(n - 1)$-fold integrals. Results are given for $n = 3, 4, 6, 8, 10$ in the case when σ^2 is known and for $n = 3$ when σ^2 is unknown. (Received 26 October 1967.)

67. Investigation of rules for dealing with outliers in small samples from the normal distribution II: Estimation of the variance. IRWIN GUTTMAN and DENNIS E. SMITH, University of Wisconsin and HRB-Singer, Inc. (By title)

Suppose an experimenter takes a sample of n observations, hopefully all from $N(\mu, \sigma^2)$, but where a spurious observation from $N(\mu, + a\sigma, \sigma^2)$ or $N(\mu, (1 + b)\sigma^2)$ may be present in the sample. In such a situation, the experimenter may be interested in a rule for dealing with outliers in his data. Three rules, each providing an unbiased estimate in the null case, are investigated via the "premium-protection" approach of Anscombe [*Technometrics* **2** (1960) 123–147] when the goal of the experiment is to estimate σ^2. Each of these rules is composed of a statistic t and a rejection region R. If $t \not\in R$, each rule uses estimator Ds^2 where D is a constant required by the constraint of unbiasedness. If $t \in R$, these rules differ, and may be described as follows: (1) Anscombe's rule: If $t \in R$, the suspect observation is discarded, and estimation proceeds using the remaining $(n - 1)$ observations as a "new" sample. (2) The Winsorization rule: If $t \in R$, the suspect observation is given the value of its nearest neighbor, and estimation proceeds using this "new" sample of n observations. (3) The Semiwinsorization rule: If $t \in R$, the statistic t is given the value of the nearest rejection boundary. Estimation proceeds, subject to the constraint imposed by this procedure. In general, computation of premiums and protections for a sample of size n involves $(n - 1)$-fold integrals. Results are given for $n = 3, 5, 7, 9, 11$ in the case when μ is known and for $n = 3$ when μ is unknown. (Received 26 October 1967.)

68. Metric spaces of distribution functions (preliminary report). R. P. HACKLEMAN, Carnegie-Mellon University.

A fundamental notion in many areas of probability and statistics is that of the distance between two probability distributions. The research reported here is concerned with the study of a particular metric space of probability distribution functions (df's). The set of all df's on the real line is metrized by defining the distance between two df's to be the area between their graphs. It is shown that the resulting metric space is complete and separable, and a compactness criterion for subsets is derived. Other properties of this metric space are investigated, including the relationships among convergence in this metric and other types of convergence commonly studied in probability theory. (Received 1 November 1967.)

69. Testing the validity of a basic model occurring in optimum πps sampling. T. V. HANURAV, Texas A & M University.

The problem of finding optimum sampling strategies to estimate the mean of a finite population under various optimality criteria was discussed earlier by the author [*Sankhyā* (1966), *J. Roy. Statist. Soc.* (1967)]. If y_i is the unknown value of the main variable and x_i

is the known value of an auxiliary variable, then under any super population model θ satisfying (i) $E_\theta(y_i \mid x_i) = ax_i$, (ii) $V_\theta(y_i \mid x_i) = \sigma^2 x_i^2$ and (iii) $\text{Cov}_\theta(y_i, y_j \mid x_i, x_j) = 0$, it is known that the optimum strategy for a fixed cost (measured by expected effective sample size) n would be (1) to select a πps without replacement sample of size n and (2) to use the Horvitz-Thompson estimator. The author gave methods of achieving (1) [*J. Roy. Statist. Soc.* (1967); *Ann Math Statist.* (1966) Abstract] which satisfy a number of other optimum properties. It is of interest to test the validity of the above model on the basis of the sample drawn. Two methods for this are presented assuming that θ is a multi-variate normal and that the distribution of the optimum estimator is nearly normal. In method 1 we stratify the population, assume different a's and σ^2's for the different strata and test for their equality. In method 2, we assume a more general model like (i) $E_\delta(y_i \mid x_i) = ax_i + b$, (ii) $V_\delta(y_i \mid x_i) = \sigma^2 x_i^g$ and (iii) $\text{Cov}_\delta(y_i, y_j \mid x_i, x_j) = 0$ and test for $b = 0, g = 2$. (Received 6 November 1967.)

70. A new table of percentage points of the Pearson Type III distribution. H. LEON HARTER, Aerospace Research Laboratories. (By title)

Recently the U. S. Water Resources Council has proposed standardization of the analysis of peak flood discharges by fitting a Pearson Type III distribution to the logarithms of the data. This action has served to draw attention to the inadequacy of available tables of percentage points of the Pearson Type III distribution and the need for better tables. Many tables of percentage points of the related chi-square distribution are available in the literature, perhaps the most comprehensive being those published by the author in 1964. These could be used to obtain percentage points of the Pearson Type III distribution, but it would be much more convenient to have a table from which percentage points of the latter distribution could be read directly for uniformly spaced values of the skewness coefficient. The author has therefore, by a modification of the programs used to compute his 1964 tables of percentage points of the chi-square distribution, obtained percentage points, corresponding to cumulative probability P = .0001, .0005, .001, .005, .01, .02, .025, .04, .05, .1(.1).9.95, .96, .975, .98, .99, .995, .999, .9995, .9999, of the standardized Pearson Type III distribution with skewness $\alpha_3 \equiv g_1 = 0.0(0.1)9.0$. This paper includes the five-decimal-place table, accurate to within a unit in the last place, together with a description of the method of computation and a discussion of possible applications, including the estimation of the return periods of floods. (Received 6 November 1967.)

71. A new estimation theory for sample surveys. H. O. HARTLEY and J. N. K. RAO, Texas A & M University.

We develop a new estimation theory for sample surveys based on the principle of maximum likelihood. Assuming (with virtually no loss of generality) that only a finite number of known discrete real values are feasible for the character of interest, we give a new definition of the likelihood function. For some simple sampling procedures, we show that most of the familiar estimators, introduced from heuristic principles, are either identical with, or closely related to, the maximum likelihood estimators. For simple random sampling without replacement, we establish asymptotic optimality properties of the maximum likelihood estimator as the population size or as both population and sample sizes tend to infinity in a manner realistically approximated by actual survey situations. (Received 3 November 1967.)

72. Iteration to the mean preference of a multi-normal population based on successive election results. MELVIN J. HINICH, Carnegie-Mellon University.

Given n issues, let $X' = (X_1, \cdots, X_n)$ be the vector of preferred positions on these issues for a random individual in the multi-normal population $N(\mu, \Sigma)$. Davis and Hinich

(*Mathematical Applications in Political Science*, II, edited by Joseph Bernd, SMU Press, 1966) have shown that given two platform positions θ and φ and assuming a uniform loss metric for the space, a majority of the population prefers θ over φ if θ is *closer* to the population mean preference μ (in the metric space). Using this result it is possible to construct an iteration scheme for two platforms such that they will converge to μ as a function of a number of successive elections. The scheme basically involves the selection of an orthogonal basis for the space, centered at the initial position of one of the politicians. This procedure does *not* require sampling of the vector preferences X of the population, but depends only on the binary outcome of each selection. (Received 13 October 1967.)

73. On the sampling distribution of the multiple correlation coefficient. VINCENT HODGSON, Florida State University.

Let R be the sample multiple correlation coefficient between one variable and the $p - 1$ other variables in a random sample of size n from a p-variate normal population in which the corresponding population multiple correlation coefficient is P. We define $\tilde{R} = R/(1-R^2)^{\frac{1}{2}}$ and $\tilde{P} = P/(1 - P^2)^{\frac{1}{2}}$, taking here and throughout the paper positive square roots. Then $\tilde{R}^2$ is distributed exactly like $[\chi^2_{p-2} + (\xi + \tilde{P}X_{n-1})^2]/\chi^2_{n-p}$ where ξ is a standardized normal variate, χ^2_{p-2}, χ^2_{n-1}, and χ^2_{n-p} are chi-square variates with degrees of freedom equal to their subscripts, and where ξ, χ^2_{p-2}, χ^2_{n-1}, and χ^2_{n-p} are independent. It follows that $[\tilde{R}(n - p - \frac{1}{2})^{\frac{1}{2}} - (p - 2 + \{n - \frac{3}{2}\}\tilde{P}^2)^{\frac{1}{2}}]/[1 + \frac{1}{2}\tilde{R}^2 + \frac{1}{2}\tilde{P}^2]^{\frac{1}{2}}$ is approximately a standardized normal variate. These results are generalizations of work by Ruben [*J. Roy. Statist. Soc. B* **28** (1966) 513–525] on the case $p = 2$. (Received 6 November 1967.)

74. Weak approachability in a two-person game. TIEN-FANG HOU, University of California, Berkeley.

In this paper we consider a two-person game with a 2×2 random vector payoff matrix $M = \|m_{ij}\|$ in a closed bounded two-dimensional convex set and we seek to answer the following questions: Given a set S in 2-space, can one of the players guarantee that the center of gravity of the payoffs ($\sum_1^N Y_n/N$) is in or arbitrarily near S, with probability approaching 1 as N tends to infinite? Is the other player able to keep $\sum_1^N Y_n/N$ from S at least a distance $\Delta > 0$, with probability approaching 1 as N tends to infinite? Our main result is that there is a complete solution for each matrix M with a convex Ω, where $\Omega = \{pq\bar{m}_{11} + p(1 - q)\bar{m}_{12} + (1 - p)q\bar{m}_{21} + (1 - p)(1 - q)\bar{m}_{22} : \text{for all } 0 \leqq p \leqq 1 \text{ and } 0 \leqq q \leqq 1\}$ and $\bar{m}_{ij}$ is the expectation of the random variable m_{ij}. For matrices with nonconvex Ω, we have sufficient conditions. (Received 13 September 1967.)

75. Analysis of one-way classification variance components model when the design is unbalanced. TOKE JAYACHANDRAN, Naval Postgraduate School.

Consider the one-way classification variance components model $y_{ik} = \mu + a_i + e_{ik}$, $k = 1, 2, \cdots, n_i$; $i = 1, 2, \cdots, 1$, with a_i I.I.D. $N(0, \sigma_a^2)$ and e_{ik} I.I.D. $N(0, \sigma^2)$. An exact test procedure for the hypothesis H: σ_a^2 is derived. The method extends quite easily to 2-way classification variance components models and 2-way classification mixed models. A new set of estimators for the variance components σ_a^2 and σ^2 are proposed. The variances of these estimators are smaller than the variances of the estimators based on the analysis of variance table, for certain values of the parameters. (Received 6 November 1967.)

76. A monotone binomial analogue of Ramanujan's equation. KUMAR JOGDEO, and S. M. SAMUELS, Courant Institute, New York University and University of California, Santa Cruz.

Among the multitude of interesting problems Ramanujan wrote about, while corresponding with Hardy, the following one was concerned regarding the expansion of e^n, where n is a non-negative integer. Translated in terms of probabilities, it amounts to studying the number y_n in the equation $\frac{1}{2} = P[Y < n - 1] + y_n P[Y = n]$ where Y is a Poisson random variable with the parameter n. Ramanujan (1911) first conjectured that $\frac{1}{3} < y_n \leqq \frac{1}{2}$ and gave a more precise estimate later. A natural question arises, whether the above equation related to Poisson probabilities is a limiting form of a binomial analogue in the following sense. If one writes $\frac{1}{2} = P[X < m] + z_{m,n}P[X = m]$, where X is the sum of n Bernoulli trials with the common success probability m/n, then one may ask whether $z_{m,n}$ has similar bounds as y_n. In the present paper this question has been answered in the affirmative by showing that $\frac{1}{3} < z_{m,n} \leqq \frac{1}{2}$ for $2m \leqq n$ and $\frac{1}{2} \leqq z_{m,n} < \frac{2}{3}$ for $n \geqq 2m$ which in fact, are the sharpest possible bounds. Further, it has been shown that the binomial random variables with integer valued expectations play a special role in the operation of taking limits: the converging sequences are strictly monotone. In particular, it follows that for such random variables the median and the expected value coincide. (Received 1 November 1967.)

77. The median significance level and other small sample measures of test efficacy. BRIAN L. JOINER, National Bureau of Standards.

The concepts of the "median significance level" and the "significance level of the average" are introduced and some relationships among these measures and the recently introduced "expected significance level" and "average critical value" are considered. The latter have been introduced by Hogben, Pinkham and Wilk (1962), Hogben (1963) and Dempster and Schatzoff (1965); and Geary (1966), respectively. The median significance level is defined as the median of the distribution of the observed significance level for a given alternative, and for one-sided tests is shown to be the inverse function of Geary's "median critical value." The "significance level of the average" is analogously defined to be the inverse function of the average critical value. Some simple examples are given illustrating the relationships among the several criterion. (Received 30 October 1967.)

78. Quiz show problems. JOSEPH B. KADANE, Yale University.

A quiz show contestant may choose the category of his next question. Associated with each category is a probability p_i of knowing the right answer to the question. If he answers the question correctly, the contestant will be given a reward t_i and be required to choose a category not previously chosen. If he answers incorrectly, he will receive the consolation prize y_i, and will leave the game with y_i plus his previous earnings. Suppose also that category i will require time t_i to get the question, answer it, and be ready to choose another question. Knowing a discount rate β, $0 \leqq \beta < 1$, and the parameters p_i, x_i, y_i and t_i, how should the contestant maximize his expected discounted winnings? This question divides into two connected problems: Given r categories it has been decided to attempt, what is the optimal order in which to attempt them? Second, if there are n possible categories $(n \leqq \infty)$ of which the contestant may choose r, which are the optimal categories to choose? At least two special cases of this model have been discussed previously: the gold-mining problem (Bellman) and the obstacle course problem (Goodman). A third problem, discrete search, is closely related but is not a quiz show problem. (Received 8 November 1967.)

79. The asymptotic behavior of the Smirnov test compared to standard "optimal procedures". GEORGE KALISH and PIOTR W. MIKULSKI, University of Maryland. (Introduced by PIOTR W. MIKULSKI)

Let $X_1X_2 \cdots X_m$, $Y_1Y_2 \cdots Y_n$ be independent random samples from absolutely continuous distribution functions F and G, respectively. Consider the hypothesis $H: F = G$ against the sided shift alternative $A: G(u) = F(u - \theta)$, $\theta > 0$. Standard two-sample test procedures belonging to three types were studied: (a) locally asymptotically most powerful test, (b) likelihood ratio tests, (c) locally most powerful rank tests. Each of these tests was compared to Smirnov test using asymptotic relative efficiency in the sense of Pitman. It has been shown, that given any test belonging to one of three mentioned types, one can construct a distribution under which Smirnov test is "better" in the sense of Pitman, its relative asymptotic efficiency exceeding any given value. As special cases Fisher-Yates, t- and Wilcoxon tests are considered. Direct extension applies to rank tests maximizing the efficacy like Van der Verden test. The method used involves an estimate for the lower bound of the power of Smirnov tests due to Birnbaum and Ramachandramurty. Once a bound for Pitman's efficiency is obtained, the construction involves for rank tests a density which possesses a small proportion highly concentrated around the mode. Mixtures of two densities with the same mode and up to scale parameters belonging to the same type, produce similar effect for parametric tests, provided that scale parameters differ sufficiently. (Received 2 October 1967.)

80. Estimators of components of variance with small mean square error. JEROME H. KLOTZ and ROY C. MILTON, University of Wisconsin.

For the components of variance one way layout ($X_{ij} = \mu + a_i + e_{ij}$ where $a_i:N(0, \sigma_a^2)$; $e_{ij}:N(0, \sigma_e^2)$; $i = 1, \cdots, I$; $j = 1, \cdots, J$) let $S_a^2 = J \sum_i (X_{i\cdot} - X_{\cdot\cdot})^2$ and $S_e^2 = \sum_i \sum_j (X_{ij} - X_{i\cdot})^2$. For estimating σ_a^2, we show analytically and numerically that the mean square error (mse) of the unbiased estimator is larger than the mse of the maximum likelihood estimator (mle) max $[((S_a^2/I - S_e^2/I(J - 1))/J, 0]$ which is in turn larger than the mse for the estimator max $[((S_a^2/(I + 1) - S_e^2/I(J - 1))/J, 0]$ of S. Zacks. All three are inadmissible using the approach of Stein. The Tiao-Tan and Stone-Springer [*Biometrika* (1965)] posterior means are both shown numerically to have larger mse than the unbiased estimator by up to an order of magnitude. However, the Tiao-Tan and Stone-Springer posterior modes and the above estimator of Zacks have unordered mse. For estimating σ_e^2, the mse for the following estimators are in decreasing order: unbiased, Hodges-Lehmann $S_e^2/(I(J - 1) + 2)$, min $[S_e^2/(I(J - 1) + 2), (S_e^2 + S_a^2)/(IJ + 1)]$. The mle min $[S_e^2/I(J - 1), (S_e^2 + S_a^2)/IJ]$ also has smaller mse than the unbiased. Numerical values indicate that the means of the Tiao-Tan and Stone-Springer posterior distributions have larger mse than the mle. Mean square errors are unordered for the mle, the modes of the Tiao-Tan and Stone-Springer posterior distributions, and min $[S_e^2/(I(J - 1) + 2), (S_e^2 + S_a^2)/(IJ + 1)]$. (Received 6 November 1967.)

81. Multivariate orthogonal polynomials. K. H. KRAMER, U. S. Steel Corporation.

The use of orthogonal polynomials for fitting a polynomial in a single variable is well known. Its basic feature is that it provides a method of doing regression analysis with uncorrelated estimators of regression coefficients. Although the advantage of such a procedure would appear to be even greater in the case of a polynomial in several variables, the problem of how to do this seems to have been neglected. For that reason, the writer has developed a method of computing multivariate orthogonal polynomials. The method was applied successfully to a polynomial regression problem which (because of multicolinearities in the regressors) could not be handled successfully by the usual regression technique. (Received 26 October 1967.)

82. Tables for the probability integral of the maximum of two correlated t variates (preliminary report). P. R. Krishnaiah, J. V. Armitage and M. C. Breiter, Aerospace Research Laboratories. (By title)

Let $x_1, \cdots, x_p$ be jointly distributed as a p-variate normal with zero means, common unknown variance σ^2, and known correlation matrix $\Omega = (\rho_{ij})$. Also, let s^2/σ^2 be a chi-square variate with n degrees of freedom distributed independently of $x_1, \cdots, x_p$. Then the joint distribution of $t_1, \cdots, t_p$ where $t_i = x_i/(s^{-1}n^{-\frac{1}{2}})$ is known [see Dunnett and Sobel, *Biometrika* **41** 153–169] to be a (central) p-variate t distribution. Now, let $\int_{-\infty}^{c} f(z)\, dz = (1 - \alpha)$ where $f(z)$ is the density function of z and $z = \max(t_1, \cdots, t_p)$. Extensive tables of the percentage points of z are given by Krishnaiah and Armitage (*Sankhyā Ser. B* **28** 31–56) when $\rho_{ij} = \rho$ for $i \neq j = 1, 2, \cdots, p$. In this paper, the authors tabulated the values of α for $p = 2, c = 1.0(0.1)5.5$ and for different values of n and ρ_{12}. These tables are useful in obtaining upper bounds on $\int_{-\infty}^{c} f(z)\, dz$ when $p > 2$. The use of these tables in the application of the multiple comparison tests proposed by Krishnaiah [*Sankhyā Ser. A* **27** 65–72] against one-sided alternatives are discussed. The applications of these tables in ranking and selection procedures are also indicated. (Received 23 October 1967.)

83. Some applications of the indicator random variable. Eustratios G. Kounias, University of Connecticut. (Introduced by Robert H. Riffenburgh)

It is known that if $A_1, A_2, \cdots, A_n$ are events of a probability space then $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n} \sum_{j>i} P(A_i A_j) + \cdots + (-1)^{n-1} P(A_1 A_2 \cdots A_n)$. Assuming that we only know $P(A_i)$ and $P(A_i A_j)$ for all $i, j = 1, 2, \cdots, n$, then although we cannot find the $P(\bigcup_{i=1}^{n} A_i)$ we can find some lower and upper bounds for it. One way of finding bounds for the $P(\bigcup_{i=1}^{n} A_i)$ is to consider the indicator random variable of the event $\bigcup_{i=1}^{n} A_i$ and then use Tchebychev's type inequalities. Under this procedure the best lower and upper bounds are found and given explicitly. Another lower bound is also given which is not as sharp but is more elegant than the above. It is found that the quadratic form $P'Q^+P$ is a lower bound for the $P(\bigcup_{i=1}^{n} A_i)$, where P' is the row vector $\{P(A_1), \cdots, P(A_n)\}$, Q is the $n \times n$ matrix $\{P(A_i A_j)\}$, and Q^+ is its generalized inverse. An interesting case arises when we have an infinite number of events $A_i, i = 1, 2, \cdots$, and we know only $P(A_i)$, and $P(A_i A_j) i, j = 1, 2, \cdots$, then different cases are investigated, a generalization of the Borel-Cantelli lemma is presented and a criterion for the strong convergence of a sequence of random variables is given. (Received 31 October 1967.)

84. Consistency conditions on, and bounds on critical homomorphically aggregated probabilities of, general dependent probability systems of a finite number of events macrospecified only by the triple $\{m, \sum_{i=1}^{m} P_i, \sum_{i=j}^{m} P_{ij}\}$. Seymour M. Kwerel, Baruch School of Business, City University of New York.

Let m = the number of events. $\sum_{i=1}^{m} P_i$ = the sum of the probabilities of occurrence of the m individual events. $\sum_{i<j}^{m} P_{ij}$ = the sum of the probabilities of simultaneous occur-

rence of each distinct pair of events. A general representation of the family of dependent probability systems associated with the macro-specified triple $\{m, \sum_{i=1}^{m} P_i, \sum_{i<j}^{m} P_{ij}\}$ is developed in terms of the quantities of the triple and an additional set of associated quantities. Using this general representation, necessary and sufficient conditions for consistency of the triple $\{m, \sum_{i=1}^{m} P_i, \sum_{i<j}^{m} P_{ij}\}$ and its associated family of dependent probability distributions are then developed. Macro-specified bounds on certain critical homomorphically-aggregated probabilities of the family of dependent probability systems associated with the consistent macrospecified triple $\{m, \sum_{i=1}^{m} P_i, \sum_{i<j}^{m} P_{ij}\}$ are developed. A broad but insightful typology of behavioral classes of dependent probability systems is developed on the basis of the quantities of the triple $\{m, \sum_{i=1}^{m} P_i, \sum_{i<j}^{m} P_{ij}\}$; and the behavioral properties, as well as the stringency of the macro-specified bounds on the critical homorphically-aggregated probabilities, of the associated family of dependent probability systems in each behavioral class is developed and discussed. (Received 2 November 1967.)

85. Sets of three studentized observations. ANDRE G. LAURENT, Wayne State University.

Let X_1, X_2, X_3, be random variables with the same probability distribution defined on the whole real line; let $\bar{X} = \sum X_i/3$ and $s^2 = \sum (X_i - \bar{X})^2/3$; let $t_i = (X_i - \bar{X})/s$, $i = 1$ to 3; let $f(t)$ be the pdf of t_i; let $g = t_1t_2t_3$; let $t_{(i)}$ be the ith order statistic, t', t'', the closest t_i's. Then the pdf of $t_{(i)}$ is $3f(t)$ in $(-2^{\frac{1}{2}}, -2^{-\frac{1}{2}})$, $(-2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})$, $(2^{-\frac{1}{2}}, 2^{\frac{1}{2}})$, $i = 1$ to 3, respectively. All $H(t_{(1)}, t_{(2)}, t_{(3)})$ are functions of $t_{(i)}$ alone, hence the distribution of H is easily obtained. If $X' = (X_1, X_2, X_3)$ has a radial distribution (special case: X_i's constitute a normal sample) $f(t)$ is Thompson's distribution. The paper gives the distribution of $t_{(i)}$, $t_{(i)} \pm t_{(j)}$, $t' \pm t''$, g and several other statistics. The covariance matrix of $t' = (t_{(1)}, t_{(2)}, t_{(3)})$ is given. The problem of testing normality and outliers is considered. The alternative hypothesis: $E[X_i] \neq E[X_j] = E[X_k]$ is studied. (Received 6 November 1967.)

86. Studentized triplets of observations for the exponential and rectangular models (preliminary report). ANDRE G. LAURENT, Wayne State University. (By title)

Let X_1, X_2, X_3 be a sample of three observations of a random variable X. The distributions of t_i, $t_{(i)}$, and several functions of the ordered statistics are derived in case X has an exponential and a rectangular distribution. The notations used are those of "sets of three Studentized observations". (Received 6 November 1967.)

87. An estimate of the incomplete Gamma distribution. ANDRE G. LAURENT, Wayne State University. (By title)

Let X follow the incomplete gamma distribution $P(X \leqq x) = G(x; \alpha, \lambda)$. Let $X_1, \cdots, X_n$ be a sample of n observations, $\bar{X} = \sum X_i/n$, $g = \prod X_i$. The minimum variance unbiased estimate $\hat{G}$ of $G(x; \alpha, \lambda)$ as a function of $\bar{X}$ and g has the same formal expression as the conditional probability $P(X \leqq x \mid \bar{X}, g)$. The latter can be obtained only if one is able to derive the joint distribution of $\bar{X}$ and g. This is done here in case $n = 3$. The conditional distribution (also marginal) of $g/\bar{X}^3$, given $\bar{X}$, is obtained, and $\hat{G}$, for $n = 3$, is $G_3 = K(g) \int_{t_1(g)}^{x} t^{-\frac{1}{2}}[(1-t)^2t - 4g/27\bar{X}^3]^{-\frac{1}{2}}\, dt$, $t_1 \leqq x \leqq t_2$, where $t_1 \leqq t_2 < t_3$ are the roots of the polynomial between brackets. When $n > 3$ an unbiased estimate can be obtained by averaging $\hat{G}_3$ over all subsamples of size three. (Received 6 November 1967.)

88. Distribution free procedures for testing a possible change in continuous distribution law at an unknown time point. SUNG WOOK LEE, Kansas State University.

Let $x_1, \cdots, x_n$ be a sequence of observations of independent random variables with continuous cdf's $F_1, \cdots, F_n$. We are concerned with the following testing problem. H_0: all F_i are identical vs. $H_1 : F_1 = \cdots = F_m ; F_{m+1} = \cdots = F_n, F_1(z) > F_n(z)$ for all z, $-\infty < z < \infty$, where the point of change m is unknown. The same problem was studied by Chernoff and Zacks [*Ann. Math. Statist.* **35** (1964) 990–1018] when F is assumed to be normal; and by Kander and Zacks [*Ann. Math. Statist.* **37** (1966) 1196–1210] when F is assumed to be of the exponential type. Two nonparametric test procedures are studied. The two test statistics are: $\sum_{i<j} \delta_{ij}$ and $\sum_i i\, r_i$, where $\delta_{ij} = 0$ if $X_i < X_j\, i, j = 1, \cdots, n, = 0$ otherwise; and r_i is the rank of x_i, i.e. $r_i = 1 + \sum_{j=1}^{n} \delta_{ji}$. Limiting distributions of the statistics are found to be normal in either case, under H_0 and H_1. The two precedures are found to be consistent, and unbiased under certain assumptions. When F is known to be normal and if m if known, an appropriate test is the Student t test under the assumption of $F_n(z) = F_1(z - \theta), \theta > 0$. Hence, the ARE of these tests compared to the t test are studied. When F is not known, the ARE compared to the Mann-Whitney U test is considered. Values of the power functions are computed for the exponential and uniform cases when the sample sizes are small. These are compared with values of the limiting power functions. (Received 3 November 1967.)