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ON TESTS OF THE EQUALITY OF TWO COVARIANCE MATRICES

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0. Introduction. Let $X = (X_1, \cdots, X_p)'$, $Y = (Y_1, \cdots, Y_p)'$ be independently and normally distributed column vectors with unknown means $\xi = (\xi_1, \cdots, \xi_p)'$, $\eta = (\eta_1, \cdots, \eta_p)'$ and unknown positive definite covariance matrices Σ_1, Σ_2 respectively. We are interested here to test the null hypothesis $H_0 : \Sigma_1 = \Sigma_2$. This problem remains invariant under the group G of affine transformations (linear transformations together with translations) of the form $X \to AX + b_1$, $Y \to AY + b_2$ where A is a $p \times p$ non-singular matrix and b_1, b_2 are p-dimensional column vectors. Let $X_1, \cdots, X_{N_1}$ be the samples of sizes N_1 and N_2 from X, Y respectively. Writing

$$\bar{X} = \textstyle\sum_1^{N_1} X_i/N_1, \qquad \bar{Y} = \sum_1^{N_2} Y_i/N_2,$$

$$S_1 = \textstyle\sum_1^{N_1} (X_i - \bar{X})(X_i - \bar{X})' \quad \text{and} \quad S_2 = \sum_1^{N_2} (Y_i - \bar{Y})(Y_i - \bar{Y})';$$

a set of maximal invariants in the sample space with respect to G (with sufficiency and invariance reduction in either order, see Hall, Wijsman and Ghosh (1965)), is $R_1, \cdots R_p$, the characteristic roots of $S_1 S_2^{-1}$. The corresponding set of maximal invariants in the parametric space under G is $\theta_1, \cdots \theta_p$, the characteristic roots of $\Sigma_1 \Sigma_2^{-1}$. In terms of maximal invariants our testing problem can be reduced to that of testing the null hypothesis:

$$H_0 : \theta_1 = \cdots = \theta_p = 1 \tag{0.1}$$

We will consider here the following alternative.

$$H_1 : \textstyle\sum_1^p \theta_i > p. \tag{0.2}$$

The dual alternative $\sum_1^p \theta_i < p$ is reduced to (0.2) by interchanging the roles of the X's and Y's.

For this problem several invariant tests are known to us (i) a test based on $|S_2|/|S_1|$, (ii) a test based on tr $S_1 S_2^{-1}$, (iii) Roy's test based on the largest and the smallest characteristic roots of $S_1 S_2^{-1}$, (iv) Kiefer and Schwartz's test (1965) based on $|S_1 + S_2|/|S_2|$.

From Anderson and Das Gupta (1964) it follows that the power of each of the above tests for testing H_0 against H_1 is a monotonically increasing function of each θ_i. Kiefer and Schwartz's test is admissible for this problem. We will suggest here another test based on tr $S_2(S_1 + S_2)^{-1}$ which is locally best invariant.

1. Locally best invariant test. Let $\mathfrak{X}$ be the space of maximal invariant R in the sample space and Ω be the space of corresponding maximal invariant θ in

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the parametric space where R and θ are diagonal matrices with diagonal elements $R_1, \cdots, R_p$ and $\theta_1, \cdots, \theta_p$ respectively. For each point θ in Ω suppose that $p(\cdot; \theta)$ is a probability density function of R with respect to the Lebesgue measure μ. For fixed α (level of significance), $\theta < \alpha \leq 1$, we shall be interested in testing the hypothesis $H_0: \theta = \theta_0 = (\theta_1^0, \cdots, \theta_p^0)$ against the alternative $H_1: \theta = (\theta_1, \cdots, \theta_p) \neq \theta_0$ such that $\sum_1^p (\theta_i - \theta_i^0) > 0$.

For notational covenience we will write θ as vector.

ASSUMPTION. Suppose that

$$(1.1) \quad p(r; \theta)/p(r; \theta_0) = 1 + \sum_1^p (\theta_i - \theta_i^0)\{g(\theta, \theta_0) + K(\theta, \theta_0)U(r)\} + B(r; \theta, \theta_0)$$

where $g(\theta, \theta_0)$ and $K(\theta, \theta_0)$ are bounded for θ in the neighborhood of θ_0, $K(\theta, \theta_0) > 0$, $B(r; \theta, \theta_0) = o(\sum_1^p (\theta_i - \theta_i^0))$ and $U(r)$ is bounded and has continuous distribution function for each θ in Ω.

DEFINITION. If the assumption is satisfied we shall say that a test is locally best invariant for testing H_0 against H_1 if its critical region is given by $U(r) \geqq C$ (C depending on α and θ_0).

In our application we will take θ_0 to be an identity matrix. The probability density of the maximal invariant R is given in James ((1964), Equation 65) from which the probability ratio is given by (writing $N_1 - 1 = N_1$ and $N_2 - 1 = N_2$)

$$(1.2) \quad f(r; \theta)/f(r; I) = |\theta|^{-N_1/2} {}_1F_0((N_1 + N_2)/2; -\theta^{-1}, R)/{}_1F_0((N_1 + N_2)/2; -I, R)$$

where

$$(1.3) \quad {}_1F_0((N_1 + N_2)/2; -\theta^{-1}, R) = \int_{O(p)} |I + \theta^{-1}HRH'|^{-(N_1+N_2)/2} \, dH$$

and dH stands for the Haar measure on the orthogonal group $O(p)$. From (1.2) simple calculation yields.

$$\begin{aligned} &f(r; \theta)/f(r; I) \\ (1.4) \quad &= 1 + \tfrac{1}{2} \sum_1^p (\theta_i - 1)\{N_2 - K \sum_1^p (1 + R_i)^{-1}\} + B(\theta, R) \\ &= 1 + \tfrac{1}{2} \sum_1^p (\theta_i - 1)\{N_2 - K \operatorname{tr} S_2(S_1 + S_2)^{-1}\} + B(\theta, R) \end{aligned}$$

where K is a positive constant and $B(\theta, R) = o(\sum_1^p (\theta_i - 1))$. Hence we have the following theorem.

THEOREM. *For testing H_0 against H_1 the test, which rejects H_0 if* $\operatorname{tr} S_2(S_1 + S_2)^{-1}$ *is less than constant, is locally best invariant.*

It is easy to see that the acceptance region $\operatorname{tr} S_2(S_1 + S_2)^{-1} \geqq$ constant, satisfies the condition that if $(R_1, \cdots, R_p)$ is in the region so is $(\bar{R}_1, \cdots, \bar{R}_p)$ with $\bar{R}_i < R_i$ for all i. Hence from Anderson and Das Gupta (1964) it follows that the power of the test with the above acceptance region is monotonically increasing function of each θ_i $(i = 1, \cdots, p)$.

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