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RECURRENCE RELATIONS BETWEEN MOMENTS OF ORDER STATISTICS FOR EXCHANGEABLE VARIATES

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Let $X_{i:n}$ $(i = 1, 2, \cdots, n)$ be the order statistics obtained by re-arranging in non-decreasing order of magnitude the variates X_i having common marginal cdf $P(x)$. Denote by $F_{i:n}(x)$ and $\mu_{i:n}$ the cdf and expected value of $X_{i:n}$. Recurrence relations for moments and other functions of the $X_{i:n}$ have been derived by many authors, usually on the assumption that the X_i are independent continuous variates. The most basic of these relations states that for $r = 1, 2, \cdots, n - 1$,

$$n\mu_{r:n-1} = r\mu_{r+1:n} + (n - r)\mu_{r:n}. \tag{1}$$

In a paper which has appeared since a longer version of this note was submitted for publication, Young [6] shows (in effect) that (1) and hence results deducible from (1) continue to hold if the X_i are exchangeable, continuous or discrete variates, i.e., if $\Pr\{X_1 \leqq x_1, X_2 \leqq x_2, \cdots, X_n \leqq x_n\}$ is symmetric in $x_1, x_2, \cdots, x_n$. In this note we point out a simple argument which establishes (1) and multivariate generalizations thereof for exchangeable variates. Also, we give an application of the result

$$F_{n-1:n}(x) = nF_{n-1:n-1}(x) - (n - 1)F_{n:n}(x), \tag{2}$$

which is just the special case $r = n - 1$ of the counterpart of (1) for cdf's.

We illustrate our argument on the bivariate case. Let $F_{r,s:n}(x, y)$ denote the joint cdf of $X_{r:n}$ and $X_{s:n}$ $(1 \leqq r < s \leqq n;\ x \leqq y)$ and let $\mu_{r,s:n} = \mathcal{E}(X_{r:n}X_{s:n})$. Now of the n variates X_i drop one at random and let $Y_{i:n-1}$ $(i = 1, 2, \cdots, n - 1)$ denote the ith order statistic in the reduced set of $n - 1$ exchangeable variates. Then according as the variate dropped is one of the (a) first r, (b) next $s - r$, (c) last $n - s$, of the $X_{i:n}$ $(1 \leqq r < s \leqq n - 1)$, we see that $Y_{r:n-1}$, $Y_{s:n-1}$ are distributed jointly as (a) $X_{r+1:n}$, $X_{s+1:n}$ or (b) $X_{r:n}$, $X_{s+1:n}$ or (c) $X_{r:n}$, $X_{s:n}$. Since the events (a), (b), (c), have respective probabilities r/n, $(s - r)/n$, $(n - s)/n$, it follows that for any x, y $(x \leqq y)$

$$nF_{r,s:n-1}(x, y) = rF_{r+1,s+1:n}(x, y) + (s - r)F_{r,s+1:n}(x, y) + (n - s)F_{r,s:n}(x, y). \tag{3}$$

Differentiating or differencing, multiplying by $e^{itx+iuy}$ and integrating or summing, we obtain the same relation between pdf's, characteristic functions, and

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TABLE 1

Upper 5 and 1% points of $X_{n-1:n}$, the second largest among n equi-correlated standard normal variates with correlation coefficient

$\rho = 0.5$

n	2	3	4	5	6	7
5%	1.100	1.400	1.569	1.685	1.773	1.843
1%	1.713	1.981	2.134	2.242	2.324	2.390
n	8	9	10	11	12	
5%	1.901	1.950	1.993	2.031	2.065	
1%	2.443	2.490	2.532	2.569	2.597	

hence raw moments of any order (provided these moments exist). In particular, this gives the result

$$(4) \qquad n\mu_{r,s:n-1} = r\mu_{r+1,s+1:n} + (s-r)\mu_{r,s+1:n} + (n-s)\mu_{r,s:n},$$

established by Govindarajulu [3] for independent identically distributed continuous variates. For the equi-correlated multivariate normal case (with common marginal cdf). (4) may also be proved with the help of expressions for the moments of order statistics given by Owen and Steck [5].

As an application of (2) consider the problem of testing n "treatment" means against a control "mean" (Dunnett [1]). Let Z_{ij} and Z_{0h} ($i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, k$; $h = 1, 2, \cdots, l$) be mutually independent normal variates, Z_{ij} and Z_{0h} being respectively $N(\mu_i, \sigma^2)$ and $N(\mu_0, \sigma^2)$, with σ^2 assumed known. In order to test simultaneously whether any of the treatment means $\bar{Z}_i$ differ from the control mean $\bar{Z}_0$ we may use the statistic

$$(5) \qquad X_{n:n} \equiv \max X_i = \max_{i=1,2,\cdots,n}(\bar{Z}_i - \bar{Z}_0)/\sigma(1/k + 1/l)^{\frac{1}{2}}.$$

Here the X_i are equi-correlated standard normal variates with $\rho = k/(k + l)$. The cdf of $X_{n:n}$ for various ρ has been tabulated by Gupta [4] whose tables may therefore be used to obtain $F_{n-1:n}(x)$ and hence percentage points of $X_{n-1:n}$. For the case $k = l$, i.e., $\rho = \frac{1}{2}$, upper 5 and 1 % points are given in Table 1. Young [6] also tabulates upper percentage points of $X_{n-1:n}$ for $n \leqq 8$ but we disagree with some of his values. The more general statistic, in which σ is replaced by an estimator S such that $\nu S^2/\sigma^2$ is distributed as χ^2 with ν d.f., independently of the numerator in (5), can now be handled by studentization. As pointed out by Fisher [2] in connection with harmonic analysis a test of the second largest variate becomes of special interest when the test on the largest is inconclusive, that is, close to the chosen level of significance.

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