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A NOTE ON STOCHASTIC DIFFERENCE EQUATIONS[1]

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Let $\{y_t \mid -\infty < t < \infty\}$ be a mean zero stochastic process which satisfies the autoregressive equation

$$(1) \qquad y_t + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \cdots + \beta_p y_{t-p} = u_t , \qquad \beta_p \neq 0,$$

where the u_t's are uncorrelated random variables with means zero and common variance σ^2. The β_i may be functions of t. While the solution of (1) and the computation of the covariance function of $\{y_t\}$ may be accomplished in a variety of ways, a particularly efficient method is to use the Green's function technique of linear difference equations. The formulas we obtain are not only of theoretical interest; but also yield a convenient algorithm for calculating the covariance function $\mathcal{E} y_t y_s$.

Let $\{\psi_i(t) \mid 1 \leqq i \leqq p\}$ be a fundamental set of solutions of the homogeneous equation

$$(2) \qquad y_t + \beta_1 y_{t-1} + \cdots + \beta_p y_{t-p} = 0$$

and let $H(t, \xi)$ be the one-sided Green's function [1]. That is,

$$H(t, \xi) = (-1)^{p-1}(C(\xi))^{-1} \begin{vmatrix} \psi_1(t) & \psi_2(t) & \cdots & \psi_p(t) \\ \psi_1(\xi) & \psi_2(\xi) & \cdots & \psi_p(\xi) \\ \psi_1(\xi+1) & \psi_2(\xi+1) & \cdots & \psi_p(\xi+1) \\ \cdot & \cdot & & \cdot \\ \psi_1(\xi+p-2) & \psi_2(\xi+p-2) & \cdots & \psi_p(\xi+p-2) \end{vmatrix}$$

where $C(\xi) = \det \|\psi_j(\xi + i - 1)\|$ is the Casorati. Then

$$(3) \qquad y_t = \sum_{r=0}^{\infty} H(t, t-r) u_{t-r+p-1}$$

provided the series converges (in some sense). For example, (3) will converge in mean square if

$$(4) \qquad \sum_{r=0}^{\infty} |H(t, t-r)|^2 < \infty.$$

Since $H(t, t-r) = 0$ for $r = 0, 1, \cdots, p-2$, we may write (3) as

$$(5) \qquad y_t = \sum_{s=0}^{\infty} H(t, t-s-p+1) u_{t-s}.$$

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The covariance function of the $\{y_t\}$ process is

$$(6)\quad \sigma(t, t-r) = \mathcal{E} y_t y_{t-r} = \sigma^2 \sum_{s=0}^{\infty} H(t, t-r-s-p+1) H(t-r, t-r-s-p+1).$$

In the interesting case in which the β_i are constants, H depends only on the difference of its arguments:

$$(7)\quad y_t = \sum_{s=0}^{\infty} H(s+p-1) u_{t-s}$$

and the condition for (4) to be valid is that all the zeros of the indicial polynomial $P(z) = \sum_{k=0}^{p} \beta_k z^{p-k}$, $(\beta_0 = 1)$, be less than one in modulus. The covariance function (6) becomes

$$(8)\quad \sigma(r) = \sigma^2 \sum_{s=0}^{\infty} H(r+s+p-1) H(s+p-1) = \sigma^2 \sum_{q=p-1}^{\infty} H(q) H(r+q).$$

For example, suppose that the roots x_i, $1 \leqq i \leqq p$, of $P(z) = 0$ are all distinct and less than one in modulus. Then the Green's function is

$$(9)\quad H(r) = \sum_{\alpha=1}^{p} (P'(x_\alpha))^{-1} x_\alpha^{\ r}$$

and the covariance function is

$$(10)\quad \sigma(r) = \sigma^2 \sum_{\alpha,\gamma=1}^{p} (P'(x_\alpha) P'(x_\gamma))^{-1} x_\alpha^{\ r+p-1} x_\gamma^{\ p-1} (1 - x_\alpha x_\gamma)^{-1}$$

where $P'(x_\alpha)$ is the derivative of $P(z)$ evaluated at $z = x_\alpha$.

If x is a zero of multiplicity p of $P(z)$, and $|x| < 1$, then the Green's function is

$$(11)\quad H(r) = \binom{r}{p-1} x^r x^{-(p-1)}$$

and the covariance function is

$$(12)\quad \sigma(r) = \sigma^2 \binom{r+p-1}{r} x^r \, {}_2F_1(r+p, p, r+1; x^2)$$

where ${}_2F_1$ is the hypergeometric function. It is possible to write ${}_2F_1(r+p, p, r+1; x^2)$ in terms of elementary functions.

Properties of the Green's function may also be invoked to simplify (or at least systematize) certain calculations involved in constructing the covariance function. For example, if a linear difference operator may be factored, then the Green's function of the product is the convolution of the Green's function of each factor.

REFERENCE

[1] MILLER, K. S. (1966). *An Introduction to the Calculus of Finite Differences and Difference Equations*. Dover, New York.