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AN ELEMENTARY PROOF OF ASMPTOTIC NORMALITY OF LINEAR FUNCTIONS OF ORDER STATISTICS[1]

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1. Introduction and summary. Let $X_1, \cdots, X_n$ be independent random variables, identically distributed with continuous distribution function F. Let $X_{(1)}, \cdots, X_{(n)}$ denote the corresponding order statistics and F_n the empiric distribution function, which we assume to be right continuous. Jung [4] found the asymptotic mean and variance of linear functions of the form

$$T_n = n^{-1} \sum_{i=1}^{n} J(i/n)X_{(i)} = \int_{-\infty}^{\infty} xJ(F_n(x))\, dF_n(x)$$

when the function J has four bounded derivatives. More recently, it has been shown ([1], [2]) that under suitable restrictions

$$\mathcal{L}\{n^{\frac{1}{2}}[T_n - \int_{-\infty}^{\infty} xJ(F(x))\, dF(x)]\} \to N(0, \sigma^2) \tag{1.1}$$

where

$$\sigma^2 = 2 \iint_{s<t} J(F(s))J(F(t))F(s)[1 - F(t)]\, ds\, dt. \tag{1.2}$$

Here $N(0, \sigma^2)$ denotes the normal distribution with mean zero and variance σ^2, and (1.1) uses a standard notation for convergence in distribution.

The purpose of this note is to give a self-contained proof of (1.1) under the assumptions that the X_i have finite mean and that J' exists and is continuous and of bounded variation except at finitely many jumps of J. This theorem can be subsumed in a corrected version of Govindarajulu [2], where appeal is made to the results of [3]. The present proof is both more elementary and shorter. Chernoff, Gastwirth and Johns [1] do not require boundedness of J, but invoke compensating assumptions on F. Their methods of proof are quite different from those of [2] and of the present paper.

2. The result. Let G be any inverse of F. We remark here that for any choice of G all integrals below will exist wp 1.

Theorem. (1.1) *holds if* $\sigma^2 < \infty$ *and*

(A) $E|X_1| = \int_0^1 |G(u)|\, du < \infty$

(B) J *is continuous on* $[0, 1]$ *except for jump discontinuities at* $a_1, \cdots, a_M$, *and* J' *is continuous and of bounded variation on* $[0, 1] - \{a_1, \cdots, a_M\}$.

Proof. We will give the proof for the case in which J' is continuous and of bounded variation on $[0, 1]$. The treatment of the terms I_{1n} and I_{3n} below remains valid in the general case if all integrals are assumed to be over $[0, 1] -$

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$\{a_1, \cdots, a_M\}$. Although the argument given here for I_{2n} does not generalize to the discontinuous case, it is not difficult to give a separate argument treating the contribution to I_{2n} by a jump in J'. Details may be found in [5].

Write $U_n(x)$ for the empiric distribution function of the uniform random variables $R_i = F(X_i)$ after n observations. Then the left side of (1.1) is $\mathcal{L}\{I_n\}$, where

$$I_n = n^{\frac{1}{2}}[\int_0^1 G(u)J(U_n(u))\, dU_n(u) - \int_0^1 G(u)J(u)\, du].$$

By the mean value theorem, there exists a $0 < \theta < 1$ such that

$$J(U_n(u)) - J(u) = J'(V_n(u))(U_n(u) - u)$$

where $V_n(u) = \theta U_n(u) + (1 - \theta)u$. We may therefore write

$$I_n = I_{1n} + I_{2n} + I_{3n},$$
$$I_{1n} = \int_0^1 G(u)J'(u)W_n(u)\, du + \int_0^1 G(u)J(u)\, dW_n(u),$$
$$I_{2n} = \int_0^1 G(u)[J'(V_n(u)) - J'(u)]W_n(u)\, dU_n(u),$$
$$I_{3n} = n^{-\frac{1}{2}} \int_0^1 G(u)J'(u)W_n(u)\, dW_n(u),$$

where $W_n(u) = n^{\frac{1}{2}}[U_n(u) - u]$. Since (A) implies that

$$\lim_{u\to 0^+} uG(u) = \lim_{u\to 1^-} (1 - u)G(u) = 0,$$

we have after integration by parts that wp 1

$$-I_{1n} = \int_0^1 J(u)W_n(u)\, dG(u). \tag{2.1}$$

Now $U_n(u)$ may be expressed as the arithmetic mean of n independent Bernoulli rv's, each distributed as $U_1(u)$. The right side of (2.1) therefore becomes

$$n^{-\frac{1}{2}} \sum_{i=1}^n H(R_i),$$

where $H(R_1) = \int J(u)[U_1(u) - u]\, dG(u)$ has mean zero and variance σ^2. (Since the hypotheses of the theorem imply that

$$\int_0^1 \int_0^1 |J(u)|\, |J(t)| E[|U_1(u) - u|\cdot|U_1(t) - t|]\, dG(u)\, dG(t)$$

is finite, the interchange of expectation and integration required to obtain the variance σ^2 is valid.) Thus I_{1n} has the required asymptotic distribution by the central limit theorem.

Next note that

$$|I_{2n}| \leqq \sup_{0\leqq u\leqq 1} |J'(V_n(u)) - J'(u)|\cdot\sup_{0\leqq u\leqq 1} |W_n(u)|\cdot\int_0^1 |G(u)|\, dU_n(u).$$

By the Glivenko-Cantelli theorem and uniform continuity of J', the first factor on the right converges to zero wp 1. A well-known theorem of Kolmogorov asserts that sup $|W_n(u)|$ has a limit in distribution, and by the strong law of large numbers

$$\int |G(u)|\, dU_n(u) = n^{-1} \sum_{i=1}^n |G(R_i)| \text{ converges to } \int |G(u)|\, du < \infty \text{ wp } 1.$$

Combining these results gives $I_{2n} \to 0$ in probability.

By writing I_{3n} as a sum of integrals over the continuity set of W_n and its complement we may obtain that wp 1

$$I_{3n} = \tfrac{1}{2}n^{-\frac{1}{2}} \int_0^1 G(u)J'(u)\, d[W_n(u)]^2 + \tfrac{1}{2}n^{-\frac{3}{2}} \textstyle\sum_{i=1}^n G(R_i)J'(R_i).$$

The second term on the right clearly converges to zero wp 1. After integration by parts the first term becomes wp 1

$$-\tfrac{1}{2}n^{-\frac{1}{2}} \int_0^1 [W_n(u)]^2\, d[G(u)J'(u)].$$

For brevity we will treat only the integral over $[\frac{1}{2}, 1]$, assuming that $\frac{1}{2}$ is a continuity point of G; it will be clear how to treat the full integral. If $V(u)$ is the total variation of GJ' on $[\frac{1}{2}, u]$, we have that

$$\begin{aligned} E|n^{-\frac{1}{2}} \textstyle\int_{\frac{1}{2}}^1 [W_n(u)]^2\, d[G(u)J'(u)]| &\leqq E[n^{-\frac{1}{2}} \textstyle\int_{\frac{1}{2}}^1 [W_n(u)]^2\, dV(u)] \\ &= n^{-\frac{1}{2}} \textstyle\int_{\frac{1}{2}}^1 u(1-u)\, dV(u). \end{aligned}$$

To establish $I_{3n} \to 0$ in probability it is therefore sufficient to show

$$\int_{\frac{1}{2}}^1 u(1-u)\, dV(u) < \infty. \tag{2.2}$$

We need only consider the case $G(1) = \infty$, and may choose r so that $G(r) > |G(\frac{1}{2})|$. It is enough to treat the integral over $[r, 1]$. If $V_a^b(f)$ denotes the total variation of f on $[a, b]$, an obvious estimate is

$$\begin{aligned} V(u) &\leqq \sup_{[\frac{1}{2},u]} |G(t)| \cdot V_{\frac{1}{2}}^u\,(J') + \sup_{[\frac{1}{2},u]} |J'(t)| \cdot V_{\frac{1}{2}}^u\,(G) \\ &= G(u) \cdot V_{\frac{1}{2}}^u(J') + \sup_{[\frac{1}{2},u]} |J'(t)| \cdot [G(u) - G(\tfrac{1}{2})] \leqq KG(u) \end{aligned}$$

for $u \geqq r$ and some $K > 0$. Integrating $\int_r^1 u(1-u)\, dV(u)$ by parts and using this bound establishes (2.1) and concludes the proof.

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