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CONSTRUCTION OF THE SET OF 256-RUN DESIGNS OF RESOLUTION ≧ 5 AND THE SET OF EVEN 512-RUN DESIGNS OF RESOLUTION ≧ 6 WITH SPECIAL REFERENCE TO THE UNIQUE SATURATED DESIGNS[1]

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0. Summary. This investigation was originally motivated by the problem of determining the maximum number of variables which can be accommodated in a 2_V^{k-p} design in 256 runs and of constructing such a "saturated" design. This problem is solved through the application of an algorithm given by the authors in a previous paper (Draper and Mitchell (1967)) to the particular case $R = 5$, $q = k - p = 8$. To obtain the solution, the complete set of even 512-run designs of resolution $\geqq 6$ and the complete set of 256-run designs of resolution $\geqq 5$ are constructed. Tables are given which immediately provide generating relations for all of these designs, "optimally" blocked.

1. Introduction. In a previous paper (Draper and Mitchell (1967)), the authors presented a method for constructing saturated designs, i.e., designs which accommodate the maximum possible number of variables, of types 2_R^{k-p} and $2_{R+1}^{(k+1)-p}$, where R (an odd integer) and $q(= k - p)$ are specified. This method involves:

(i) the stage by stage construction of the set of distinct even $2^{(k+1)-p}$ designs of resolution $\geqq R + 1$, followed by

(ii) the erasure of variables from these designs (i.e., the removal of a specified variable from each word in the defining relation of each design) to obtain the set of distinct odd 2^{k-p} designs of resolution $\geqq R$.

(Note: An even 2^{k-p} design is one whose defining relation contains all even words; an odd 2^{k-p} design is one whose defining relation consists of 2^{p-1} even words and 2^{p-1} odd words, where the identity I is counted as an even word in each case. All 2^{k-p} designs are either even or odd.)

It was shown that, through the erasure of a variable from an even blocked $2_{R+1;S'}^{(k+1)-p-t}$ design, where S' is an even integer not less than $(R + 1)/2$, any of the 2_R^{k-p} designs thus constructed can be blocked in such a way that interactions involving less than $(R + 1)/2$ factors are not confounded with blocks.

(Note: The use of notation of the form $2_{R+1;S'}^{(k+1)-p-t}$ to describe a blocked design serves to indicate that t blocking generators have been added to the p generators

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of a $2_{R+1}^{(k+1)-p}$ design in such a way that the shortest word in the full defining relation generated by these $(p + t)$ generators has length S'.) A method was given for the construction of the $2_{R+1;S'}^{(k+1)-p-t}$ designs necessary for this procedure.

In both the construction and the blocking procedures, it is frequently necessary to test for the *equivalence* of two designs. (Two designs are defined to be equivalent if the defining relation of one can be mapped into the defining relation of the other through a relabeling of the variables). A "sequential conjecture" method was developed for this purpose.

The procedures were programmed for the computer and illustrated by the example $R = 5$, $q = 7$. Thus the unique saturated 2_{V}^{11-4} design, and the unique even saturated 2_{VI}^{12-4} design, were constructed and blocked.

In the case $R = 3$, saturated designs of resolutions III and IV can be constructed more easily by other methods (Box and Hunter (1961a)) for all values of q. This is also true of resolution V cases, when $q \leqq 7$. (Box and Hunter (1961b)).

A more difficult case is that in which $R = 5$ and $q = 8$, where the maximum number of variables which can be accommodated has not previously been determined. A specific 2_{V}^{17-9} design was given by Addelman (1965) who remarked that although his "systematic trial and error procedure does not guarantee that 17 is the maximum number of two-level factors that can be accommodated in a resolution V plan with 256 treatment combinations, · · · it is unlikely that more than 17 factors can be accommodated in such a plan." This remark was based on an examination of the ascending trend shown by the difference:

$$s = \text{slack} = \begin{pmatrix}\text{max. no. of factors} \\ \text{allowed by available d.f.}\end{pmatrix} - \begin{pmatrix}\text{max. no. of} \\ \text{factors possible}\end{pmatrix}$$

in the cases $R = 5$, $q \leqq 7$. The "maximum number of factors allowed by available d.f." can be obtained for a given q by recognizing that in a design of resolution V, all the k main effects and the $k(k - 1)/2$ two-factor interactions (as well as the overall mean) must be estimated clear of one another. Therefore, the total number ($2^q = 2^{k-p}$) of sets of confounded estimates available must exceed or equal $(1 + k + k(k - 1)/2)$.

The upper bound for the "maximum number of factors possible" obtained by this calculation gets progressively looser (as indicated by increasing slack values) as q increases; for $q = 4, 5, 6$, and 7, the slack is $s = 0, 1, 2$, and 4 respectively. For $q = 8$, the existence of a 2_{V}^{17-9} design can be shown to imply that $s \leqq 5$, with $s = 5$ if 17 is actually the maximum number of factors possible. Addelman therefore suggests that, unless there is a departure from the ascending trend of s, his 2_{V}^{17-9} design should, indeed, accommodate the largest number of variables possible. In this paper, we present the results obtained by our procedures in this same case ($R = 5$, $q = 8$).

We first constructed the set of all distinct even $2^{(k+1)-p}$ designs of resolution $\geqq 6$, where $q = k - p = 8$. These designs are listed in Table 1, together with their word length patterns. (Each word length pattern consists of the

TABLE 1

Even 512-Run Designs of Resolution ≧ 6

No.	v	Word Length Pattern							b	Ref.	Delete Variables
		6	8	10	12	14	16	18			
1.1	10	1	0	0	0	0	0	0	16	6.1	10, 11, 12, 13, 14
1.2	10	0	1	0	0	0	0	0	16	6.1	1, 10, 12, 13, 14
1.3	10	0	0	1	0	0	0	0	16	6.1	2, 5, 10, 11, 13
2.1	11	3	0	0	0	0	0	0	16	6.1	10, 11, 12, 13
2.2	11	2	1	0	0	0	0	0	16	6.1	10, 11, 12, 15
2.3	11	2	0	1	0	0	0	0	16	6.1	5, 10, 11, 13
2.4	11	1	2	0	0	0	0	0	16	6.1	6, 10, 13, 14
3.1	12	6	0	0	1	0	0	0	16	6.1	13, 14, 15
3.2	12	6	1	0	0	0	0	0	16	6.1	10, 11, 12
3.3	12	5	1	1	0	0	0	0	16	6.1	10, 11, 13
3.4	12	4	3	0	0	0	0	0	16	6.1	10, 13, 14
3.5	12	4	3	0	0	0	0	0	16	6.1	1, 10, 11
4.1	13	10	4	0	1	0	0	0	16	6.1	13, 14
4.2	13	10	3	2	0	0	0	0	16	6.1	10, 11
4.3	13	12	3	0	0	0	0	0	16	6.1	11, 14
4.4	13	9	5	1	0	0	0	0	16	6.1	10, 13
4.5	13	8	7	0	0	0	0	0	16	6.1	1, 10
5.1	14	17	10	3	1	0	0	0	16	6.1	13
5.2	14	15	14	1	1	0	0	0	16	6.3	10
5.3	14	16	11	4	0	0	0	0	16	6.1	10
5.4	14	18	7	6	0	0	0	0	16	6.1	11
5.5	14	15	15	0	0	1	0	0	8	7.1	4, 9
6.1	15	28	21	12	2	0	0	0	16	6.1	—
6.2	15	27	24	9	3	0	0	0	8	9.1	10, 11, 14
6.3	15	25	30	3	5	0	0	0	16	6.3	—
6.4	15	27	23	12	0	1	0	0	8	7.1	9
6.5	15	30	15	18	0	0	0	0	16	7.3	10
7.1	16	45	41	34	6	1	0	0	8	7.1	—
7.2	16	44	45	28	10	0	0	0	8	9.1	10, 11
7.3	16	48	30	48	0	0	1	0	16	7.3	—
8.1	17	68	85	68	34	0	0	0	8	9.1	10
9.1	18	102	153	153	102	0	0	1	8	9.1	—

numbers of words of lengths 6, 7, 8, and so on, present in the defining relation.) Every design in this table can be constructed and blocked by deleting variables (in the manner indicated in the table) from one of five blocked *reference designs*, whose generators are given in Table 2.

TABLE 2

Reference Designs

Design 6.1 $2^{15-6-4}_{VI;IV}$	Design 6.3 $2^{15-6-4}_{VI;IV}$	Design 7.1 $2^{16-7-3}_{VI;IV}$	Design 7.3 $2^{16-7-4}_{VI;IV}$	Design 9.1 $2^{18-9-3}_{VI;IV}$
W_1 = 12345(10)	W_1 = 12345(10)	W_1 = 12345(10)	W_1 = 12345(10)	W_1 = 12345(10)
W_2 = 12367(11)	W_2 = 12367(11)	W_2 = 12367(11)	W_2 = 12367(11)	W_2 = 12367(11)
W_3 = 12389(12)	W_3 = 12389(12)	W_3 = 12389(12)	W_3 = 12468(12)	W_3 = 12389(12)
W_4 = 12468(13)	W_4 = 12468(13)	W_4 = 12468(13)	W_4 = 12579(13)	W_4 = 12468(13)
W_5 = 12579(14)	W_5 = 13579(14)	W_5 = 12579(14)	W_5 = 14569(14)	W_5 = 12579(14)
W_6 = 14569(15)	W_6 = 1456789(15)	W_6 = 14569(15)	W_6 = 15678(15)	W_6 = 14569(15)
		W_7 = 14789(16)	W_7 = 1235689(16)	W_7 = 24789(16)
B_1 = 1247	B_1 = 1247			W_8 = 1345678(17)
B_2 = 1269	B_2 = 1259	B_1 = 1247	B_1 = 1238	W_9 = 2356789(18)
B_3 = 1348	B_3 = 1378	B_2 = 1258	B_2 = 1269	
B_4 = 1356	B_4 = 1456	B_3 = 1269	B_3 = 1356	B_1 = 1247
			B_4 = 1478	B_2 = 1258
				B_3 = 1269

The set of all distinct 256-run odd designs of resolution $\geqq 5$, which is directly related to the set of even 512-run $2^{(k+1)-p}$ designs of resolution $\geqq 6$, is listed in Table 3. Each of these designs can be obtained from one of the reference designs of Table 2 through the deletion of a set of variables together with the erasure of a single variable.

In Section 2 we shall discuss the construction and use of these tables, and shall emphasize those results which are of particular interest to us.

2. Discussion of the tables.

Table 1. This table lists the even 512-run $2^{(k+1)-p}$ designs of resolution $\geqq 6$. The number of each design is written in the form $(p \cdot a)$, where p is the number of generators and a is a number which orders those designs having the same value of p. (The order used here is that in which the designs were actually constructed by the computer at each stage.)

The number of designs at each stage is seen, in Table 1, to be as follows:

(2.1)

No. of generators:	1	2	3	4	5	6	7	8	9
No. of designs:	3	4	5	5	5	5	3	1	1

This pattern shows an increasing number of distinct designs in the lower stages, followed by a gradual "leveling off" in the upper-middle stages, then a rapid drop to a single saturated design.

The column headed "v" ($= k + 1$, in our previous notation) in Table 1 gives the number of variables which are accommodated in each design. We see that the single design (9.1) which was found at the last stage accommodates 18 variables, i.e., 18 is the maximum number of variables which can be incorporated into a

TABLE 3

256-Run Designs of Resolution $\geqq 5$

No.	k	Word Length Pattern													b	Ref.	Delete	Erase
		5	6	7	8	9	10	11	12	13	14	15	16	17				
1.1/0	9	1	0	0	0	0	0	0	0	0	0	0	0	0	16	6.1	10, 11, 12, 13, 14	15
1.2/0	9	0	0	1	0	0	0	0	0	0	0	0	0	0	16	6.1	1, 10, 12, 13, 14	15
1.3/0	9	0	0	0	0	1	0	0	0	0	0	0	0	0	16	6.1	2, 5, 10, 11, 13	15
2.1/1	10	2	1	0	0	0	0	0	0	0	0	0	0	0	16	6.1	10, 11, 12, 13	15
2.2/1	10	1	1	1	0	0	0	0	0	0	0	0	0	0	16	6.1	10, 11, 12, 15	14
2.2/2	10	2	0	0	1	0	0	0	0	0	0	0	0	0	16	6.1	10, 11, 12, 15	2
2.3/1	10	1	1	0	0	1	0	0	0	0	0	0	0	0	16	6.1	5, 10, 11, 13	15
2.3/3	10	2	0	0	0	0	1	0	0	0	0	0	0	0	16	6.1	5, 10, 11, 13	2
2.4/1	10	0	1	2	0	0	0	0	0	0	0	0	0	0	16	6.1	6, 10, 13, 14	15
2.4/2	10	1	0	1	1	0	0	0	0	0	0	0	0	0	16	6.1	6, 10, 13, 14	12
3.1/1	11	3	3	0	0	0	0	1	0	0	0	0	0	0	16	6.1	13, 14, 15	12
3.2/1	11	3	3	1	0	0	0	0	0	0	0	0	0	0	16	6.1	10, 11, 12	14
3.2/2	11	4	2	0	1	0	0	0	0	0	0	0	0	0	16	6.1	10, 11, 12	15
3.3/1	11	2	3	1	0	1	0	0	0	0	0	0	0	0	16	6.1	10, 11, 13	15
3.3/2	11	3	2	0	1	1	0	0	0	0	0	0	0	0	16	6.1	10, 11, 13	14
3.3/3	11	3	2	1	0	0	1	0	0	0	0	0	0	0	16	6.1	10, 11, 13	5
3.4/1	11	1	3	3	0	0	0	0	0	0	0	0	0	0	16	6.1	10, 13, 14	15
3.4/2	11	2	2	2	1	0	0	0	0	0	0	0	0	0	16	6.1	10, 13, 14	12
3.4/4	11	3	1	1	2	0	0	0	0	0	0	0	0	0	16	6.1	10, 13, 14	9
3.5/2	11	2	2	2	1	0	0	0	0	0	0	0	0	0	16	6.1	1, 10, 11	15
4.1/1	12	4	6	4	0	0	0	0	1	0	0	0	0	0	16	6.1	13, 14	15
4.1/2	12	4	6	3	1	0	0	1	0	0	0	0	0	0	16	6.1	13, 14	12
4.1/4	12	6	4	1	3	0	0	1	0	0	0	0	0	0	16	6.1	13, 14	10
4.2/2	12	4	6	2	1	2	0	0	0	0	0	0	0	0	16	6.1	10, 11	14
4.2/3	12	5	5	2	1	1	1	0	0	0	0	0	0	0	16	6.1	10, 11	15
4.2/5	12	6	4	0	3	2	0	0	0	0	0	0	0	0	16	6.1	10, 11	1
4.3/2	12	6	6	2	1	0	0	0	0	0	0	0	0	0	16	6.1	11, 14	15
4.4/2	12	3	6	4	1	1	0	0	0	0	0	0	0	0	16	6.1	10, 13	15
4.4/3	12	4	5	4	1	0	1	0	0	0	0	0	0	0	16	6.1	10, 13	11
4.4/4	12	5	4	2	3	1	0	0	0	0	0	0	0	0	16	6.1	10, 13	14
4.5/2	12	2	6	6	1	0	0	0	0	0	0	0	0	0	16	6.1	1, 10	15

4.5/4	12	4	4	4	3	0	0	0	0	0	0	0	0	0	16	6.1	1, 10	14
4.5/5	12	4	4	4	3	0	0	0	0	0	0	0	0	0	16	6.1	1, 10	11
5.1/1	13	7	10	6	4	3	0	0	1	0	0	0	0	0	16	6.1	13	14
5.1/2	13	7	10	7	3	1	2	1	0	0	0	0	0	0	16	6.1	13	11
5.1/3	13	5	12	7	3	3	0	1	0	0	0	0	0	0	16	6.1	13	12
5.1/4	13	8	9	5	5	2	1	1	0	0	0	0	0	0	16	6.1	13	10
5.1/5	13	9	8	3	7	3	0	1	0	0	0	0	0	0	16	6.1	13	9
5.2/1	13	5	10	10	4	1	0	0	1	0	0	0	0	0	16	6.3	10	5
5.2/4	13	6	9	9	5	0	1	1	0	0	0	0	0	0	16	6.3	10	15
5.2/5	13	7	8	7	7	1	0	1	0	0	0	0	0	0	16	6.3	10	14
5.3/2	13	6	10	8	3	2	2	0	0	0	0	0	0	0	16	6.1	10	11
5.3/4	13	7	9	6	5	3	1	0	0	0	0	0	0	0	16	6.1	10	13
5.3/5	13	8	8	4	7	4	0	0	0	0	0	0	0	0	16	6.1	10	1
5.4/2	13	8	10	4	3	4	2	0	0	0	0	0	0	0	16	6.1	11	10
5.4/3	13	6	12	4	3	6	0	0	0	0	0	0	0	0	16	6.1	11	14
5.5/3	13	3	12	12	3	0	0	0	0	1	0	0	0	0	8	7.1	4, 9	11
5.5/5	13	7	8	8	7	0	0	0	0	1	0	0	0	0	8	7.1	4, 9	16
6.1/1	14	11	17	11	10	9	3	1	1	0	0	0	0	0	16	6.1	—	13
6.1/3	14	12	16	10	11	8	4	2	0	0	0	0	0	0	16	6.1	—	10
6.1/4	14	10	18	14	7	6	6	2	0	0	0	0	0	0	16	6.1	—	11
6.2/1	14	10	17	14	10	6	3	2	1	0	0	0	0	0	8	9.1	10, 11, 14	17
6.2/2	14	12	15	10	14	8	1	2	1	0	0	0	0	0	8	9.1	10, 11, 14	18
6.2/3	14	11	16	13	11	5	4	3	0	0	0	0	0	0	8	9.1	10, 11, 14	16
6.3/2	14	10	15	16	14	2	1	4	1	0	0	0	0	0	16	6.3	—	10
6.4/3	14	11	16	12	11	8	4	0	0	1	0	0	0	0	8	7.1	9	16
6.4/4	14	9	18	16	7	6	6	0	0	1	0	0	0	0	8	7.1	9	11
6.4/5	14	12	15	8	15	12	0	0	0	0	1	0	0	0	8	7.1	9	4
6.5/4	14	12	18	8	7	12	6	0	0	0	0	0	0	0	16	7.3	10	16
7.1/1	15	17	28	20	21	22	12	4	2	1	0	0	0	0	8	7.1	—	16
7.1/4	15	18	27	18	23	22	12	6	0	0	1	0	0	0	8	7.1	—	9
7.1/5	15	15	30	26	15	16	18	6	0	1	0	0	0	0	8	7.1	—	11
7.2/1	15	16	28	24	21	16	12	8	2	0	0	0	0	0	8	9.1	10, 11	16
7.2/2	15	17	27	21	24	19	9	7	3	0	0	0	0	0	8	9.1	10, 11	18
7.3/5	15	18	30	15	15	30	18	0	0	0	0	1	0	0	16	7.3	—	10
8.1/2	16	24	44	40	45	40	28	24	10	0	0	0	0	0	8	9.1	10	11
9.1/1	17	34	68	68	85	85	68	68	34	0	0	0	0	1	8	9.1	—	10

512-run resolution VI design. This implies at once that the maximum number of variables which can be accommodated by a 256-run resolution V design is 17.

If we examine the word length patterns of the designs of Table 1, we see that there are only two designs, 3.4 and 3.5, which have identical word length patterns. We therefore conclude that, with a single exception, all distinct 512-run resolution VI designs of even words have distinct word length patterns. In general, we shall say that a set S of designs has the *distinct word pattern property* if and only if the word length patterns of every pair of distinct designs in S are distinct.

It has been shown by Draper and Mitchell (1967) that the set of 256-run even resolution VI designs has this property, as well as the set of 128-run resolution V designs. The same property holds for the set of designs which remains if we omit either of the designs 3.4 or 3.5 from the set of all even 512-run resolution VI designs. Similarly, in the 256-run resolution V case, we shall see that the distinct word pattern property holds if we omit only two designs from the entire set.

In the cases we have studied, therefore, sets of designs having the distinct word pattern property are very large, in that they include (or very nearly include) every design in the entire class of designs under examination. Although the same degree of "comprehensiveness" cannot be guaranteed in general, the distinct word pattern property does suggest a possible alternative to the *permutation subroutine* (which is the computer program corresponding to our "sequential conjecture" procedure) for testing the equivalence of two designs. Since it is much faster for the computer to compare word length patterns than to go through the entire permutation subroutine, this alternative is especially useful when repeated application of the subroutine would require a prohibitive amount of computer time. The disadvantage of testing equivalence by comparing word length patterns is that, in our stage by stage procedure, a design which had the same word length pattern as one previously found would automatically be discarded, even if the two designs were not equivalent. The set of designs constructed using the word length comparison to test equivalence is therefore not necessarily the *complete* set of designs of the specified type. However, encouraged by the fact that this (possibly incomplete) set of designs has, in all the cases we have studied, included the saturated design, we can use the word length comparison test to attack some problems for which the application of the permutation subroutine is impractical.

One such problem is the *blocking* of the designs listed in Table 1. Using the word length comparison test for equivalence, we have constructed blocking arrangements for each of these designs, adding as many blocking generators as we could to each design. The column headed "b" in Table 1 lists this maximum number of blocks in each case. The resulting blocked designs are of type $2_{R';S'}^{(k+1)-p-t}$, where $R' \geqq 6$ and $S' \geqq 4$. We remark that in no case can the number of blocks in such a design be greater than 16, if $q = k - p = 8$. This can be seen as follows: The blocking generators $B_1, B_2, \cdots, B_t$ are composed of the variables $(1, 2, \cdots, q + 1 = 9)$ and must themselves be the generators of a resolution S' design, where $S' \geqq 4$. But the maximum number of generators which can be incorporated into a nine-variable design of resolution $\geqq 4$ is four. Hence $t \leqq 4$ and $b = 2^t \leqq 16$. Table 1 shows that the maximum possible number of blocks (16) can be attained for v as high as 16, thanks to the remarkable receptivity of the design 7.3 to the addition of blocking generators.

Similarly, a $2_{R';S'}^{(k+1)-p-t}$ design (where $R' \geqq 6$ and $S' \geqq 4$) must be such that if $k = p + q \geqq 16$, then $(k + 1) - p - t = q - t + 1 \geqq 6$, since each block (viewed as a separate design) must contain at least 64 runs to be of resolution $\geqq 4$ and accommodate more than 16 variables. Hence if $q = 8$, $t \leqq 3$ for $2_{R';S'}^{(k+1)-p-t}$ designs in which the number of variables $v = k + 1 \geqq 17$. In the cases $v = 17$ and $v = 18$, we have found designs and blocking arrangements which achieve the maximum possible number of blocks (8). Corresponding to each value of v, therefore, there exists at least one design in Table 1 which achieves the maximum possible number of blocks.

Each of the designs in Table 1, blocked into the number of blocks indicated in column "b", can be obtained (through deletion of variables) from one of five reference designs, which are given in Table 2. Table 1 gives the appropriate reference design in each case, together with the variables which are to be deleted. An example which illustrates how this is done is given in the discussion of Table 2 which follows.

Table 2. This table gives the generators of the five blocked reference designs 6.1, 6.3, 7.1, 7.3, and 9.1. Of these designs, the base designs of all but 6.1 are "dead-end" designs, where we define a "dead-end" design to be one with which all candidates in the stage by stage construction are incompatible (Draper and Mitchell (1967)). In other words, no further generators can be added to any of the base designs of 6.3, 7.1, 7.3, or 9.1 without violating the resolution conditions or increasing the number of runs. All even 512-run designs of resolution $\geqq 6$ can be derived from these four "dead-end" designs through the deletion of variables.

The design 6.1, which is not a "dead-end" design, was included as a reference design for reasons of blocking. It is the only design of Table 1 (other than the "dead-end" designs) which cannot be obtained *together with* its best blocking arrangement through the deletion of variables from a "best-blocked" design at a higher stage. The inclusion of 6.1, therefore, completes our set of reference designs, which can now be used to obtain all the designs of Table 1 in "best-blocked" form. (We use the term "best-blocked" to mean the blocking arrangement which allows the maximum number of blocks possible in the set of arrangements considered.)

We shall now illustrate the use of Tables 1 and 2 with the following example. Suppose we wish to obtain the design 4.5 in a "best-blocked" form. Table 4.1 indicates that this design may be derived from the reference design 6.1 by deleting the variables 1 and 10. In order to delete these variables, we must first isolate them as indicator variables (i.e., so that each is associated with only one generator) in the base design of 6.1. We observe in Table 2 that the variable 10 is already isolated, but the variable 1 is not. In other words, we must clear the variable 1 from all the blocking generators and from all but one of the generators in the base design, without, of course, interfering with the isolation of variable 10. We shall do this by replacing each generator which contains 1 by its product with $W_2 = 12367(11)$. (Note that we could have chosen, instead of W_2, *any* of the other generators (W_3, W_4, W_5, W_6) which contain 1 but not 10.) The new set of generators thus obtained is as follows:

$$
\begin{aligned}
&W_1' = 4567(10)(11), &\quad B_1' &= 346(11),\\
&W_2' = 12367(11), &\quad B_2' &= 379(11),\\
&W_3' = 6789(11)(12), &\quad B_3' &= 24678(11),\\
&W_4' = 3478(11)(13), &\quad B_4' &= 257(11).\\
&W_5' = 3569(11)(14), & &\\
&W_6' = 234579(11)(15), & &
\end{aligned}
\tag{2.2}
$$

Note that the variables 1 and 10 are isolated in generators W_2' and W_1', respectively.

We now remove W_1' and W_2' from (2.2) to obtain the generators of the $2_{\mathrm{VI;IV}}^{13-4-4}$ design 4.5:

$$
\begin{aligned}
&W_1'' = 6789(11)(12), &\quad B_1'' &= 346(11),\\
&W_2'' = 3478(11)(13), &\quad B_2'' &= 379(11),\\
&W_3'' = 3569(11)(14), &\quad B_3'' &= 24678(11),\\
&W_4'' = 234579(11)(15), &\quad B_4'' &= 257(11).
\end{aligned}
\tag{2.3}
$$

For convenience, we could, at this point, relabel the variables (2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15) as (1, 2, ⋯ , 13).

In a similar fashion, all the designs of Table 1 can be obtained from the reference designs of Table 2. From the designs of Table 1, we can in turn derive, through the erasure of variables, the complete set of 256-run resolution V designs, each in "best-blocked" form.

Table 3. This table lists the complete set of distinct 256-run odd designs of resolution $\geqq 5$ together with their word length patterns. Each design is identified by means of a number written in the form $(p\cdot a/b)$. The meaning of this notation is that $(p\cdot a)$ is the design of Table 1 from which the design $(p\cdot a/b)$ is derived (through the erasure of a variable), and $((p - 1)\cdot b)$ is the even design of Table 1 which corresponds to the even words of $(p\cdot a/b)$.

We note that design 9.1/1 is the unique saturated resolution V design in 256 runs. Addelman's (1965) speculation that 17 is the maximum number of variables which can be accommodated in a 256-run resolution V design is thus confirmed, his 2_{V}^{17-9} design being equivalent to design 9.1/1.

We should also draw attention to the $2_{\mathrm{V;III}}^{15-7-4}$ design 7.3/5 because of its blocking properties. Comparing columns "k" and "b" in Table 3, we see that this design is optimum in the sense that it accommodates more variables than any other 16-block design in the table. In fact, each block of this design is, in itself, a saturated resolution III design in 16 runs.

As an illustration of the derivation of the designs of Table 3 from the reference designs of Table 2, we shall consider the following example: Suppose we wish to obtain the $2_{\mathrm{V;III}}^{12-4-4}$ design 4.5/5. Table 3 indicates that this can be done by deleting, from the reference design 6.1, the variables 1 and 10, and then erasing the variable 11. We have already illustrated the deletion of the variables 1 and 10 from design 6.1; the resulting generators are given in (2.3). We now "erase" the variable 11 simply by removing it from each of these generators to give the following set of generators of 4.5/5.

$$
\begin{aligned}
W_1 &= 6789(12), & B_1 &= 346, \\
W_2 &= 3478(13), & B_2 &= 379, \\
W_3 &= 3569(14), & B_3 &= 24678, \\
W_4 &= 234579(15), & B_4 &= 257.
\end{aligned}
\tag{2.4}
$$

The variables (2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15) can, if we wish, be relabeled in any convenient manner, e.g., as (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12).

In a practical situation, we would probably not have, initially, a specific design in mind, as in the above example. We would first use the tables to pick out those designs which satisfied our requirements, and then choose a design from this set. Once a particular design has been chosen, it can be constructed and blocked as illustrated in the above example.

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