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VARIATIONS ON A RENEWAL THEOREM OF SMITH

By C. C. Heyde

University of Sheffield

1. Introduction. Let X_i, $i = 1, 2, 3, \cdots$, be a sequence of independent random variables with finite expectations $\mu_n = EX_n$ such that

$$(\mu_1 + \mu_2 + \cdots + \mu_n)/n \to \mu \quad \text{as} \quad n \to \infty,$$

where μ is finite and strictly positive. Write $S_n = \sum_{i=1}^{n} X_i$, $n \geqq 1$, $M_n = \max_{1 \leqq k \leqq n} S_k$ and $N_n = \max_{1 \leqq k \leqq n} |S_k|$. In the paper [5], Smith obtains results on the asymptotic behaviour of certain sums of the form $\sum_{n=1}^{\infty} a_n \Pr(S_n \leqq x)$ as $x \to \infty$. He goes on to mention, but not to treat, another line of enquiry relating to sums of the type $\sum_{n=1}^{\infty} a_n \Pr(M_n \leqq x)$ which, as he points out, differs from the former in some important respects. It happens, however, that the results of Smith carry over quite readily, and with no change in the conditions, from the case of the S_n's to that of the M_n's or the N_n's and it is the object of this note to establish this for the key Theorem 1 of [5].

A fairly detailed discussion of the relation between $\sum_{n=1}^{\infty} a_n \Pr(S_n \leqq x)$ and $\sum_{n=1}^{\infty} a_n \Pr(M_n \leqq x)$ has been given in [2] and [3] for the particular case in which the X_i are identically distributed and this note is directed towards pointing out the existence of similar parallels in the general case. The random variables N_n do not appear to have been previously introduced in a renewal theoretic context. The theorem which we shall establish is an important tool for the study of the passage time random variables $M(x) = \max[k \mid M_k \leqq x]$ and $N(x) = \max[k \mid N_k \leqq x]$. In particular, we note that

$$EM(x) = \sum_{n=1}^{\infty} \Pr(M_n \leqq x), \qquad EN(x) = \sum_{n=1}^{\infty} \Pr(N_n \leqq x).$$

2. Results. In the following work we shall take $G_n(x)$ to mean either $\Pr(M_n \leqq x)$ or $\Pr(N_n \leqq x)$ (so that if a property holds for both $\Pr(M_n \leqq x)$ and $\Pr(N_n \leqq x)$ it holds for $G_n(x)$ and conversely). We shall obtain the following theorem.

Theorem. *Suppose the following conditions hold.*

(T1) *$\{X_n\}$ is a sequence of independent random variables with distribution functions $\{F_n(x)\}$ and finite expectations $\mu_n = EX_n$ such that*

$$(\mu_1 + \mu_2 + \cdots + \mu_n)/n \to \mu \quad \textit{as} \quad n \to \infty$$

where μ is finite and strictly positive.

(T2) *For every $\epsilon > 0$,*

$$\int_{n\epsilon}^{\infty} n^{-1} \sum_{r=1}^{n} \{1 - F_r(x)\}\, dx \to 0 \quad \textit{as} \quad n \to \infty.$$

(T3) *For some $\alpha > 0$, $\gamma \geqq 0$, and some non-negative function of slow growth*

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$L(x)$, the sequence of non-negative constants $\{a_n\}$ satisfies the asymptotic relation

$$\sum_{n=1}^{\infty} a_n x^n \sim \alpha(1-x)^{-\gamma} L((1-x)^{-1}) \quad as \quad x \to 1-0.$$

(T4) *$\sum_{n=1}^{\infty} a_n$ diverges.*
Then, in order that

$$\sum_{n=1}^{\infty} a_n G_n(x) \sim \alpha L(x)(\Gamma(1+\gamma))^{-1}(x/\mu)^{\gamma} \quad as \quad x \to \infty,$$

it is sufficient that the following two conditions are satisfied.

(T5(a)) *If k is any number such that $a_n = O(n^k)$, then there is a distribution function $K(x)$ of a negative-valued random variable with a finite moment of order $(k+2)$ such that $K(x) \geqq F_n(x)$ for all n and all x.*

(T5(b)) *If $-\kappa$ is the first moment of $K(x)$, then for some $\nu > \kappa$ and every $\epsilon > 0$,*

$$\liminf_{n\to\infty} \int_{-\infty}^{\epsilon n/\log n} n^{-1} \sum_{r=1}^{n} \{U(x) - F_r(x)\}\, dx > (2(k+1)\nu\epsilon)^{\frac{1}{2}}$$

where

$$U(x) = 1, \quad x \geqq 0,$$
$$= 0, \quad x < 0.$$

For an explanation of the significance of the component parts of the theorem the reader is referred to [5]. In particular, we note that the divergence of $\sum_{n=1}^{\infty} a_n$ ensures that the k of condition (T5(a)) satisfies $k \geqq -1$. The theorem generalizes certain of the results of Chow and Robbins [1] and Heyde [2] and can be used to generate asymptotic results on the moments of the random variables $M(x)$ and $N(x)$.

As a preliminary to establishing the theorem we need the following two lemmas.

Lemma 1. *$\{X_n\}$ is a sequence of independent random variables with finite expectations $\mu_n = EX_n$ such that*

$$(\mu_1 + \mu_2 + \cdots + \mu_n)/n \to \mu \quad as \quad n \to \infty$$

where $\mu \geqq 0$ is finite. If $n^{-1}S_n \to_P \mu$ as $n \to \infty$, then $n^{-1}M_n \to_P \mu$ and $n^{-1}N_n \to_P \mu$. ("P" stands for convergence in probability.)

Proof. Firstly we shall establish the result for M_n. Using Lévy's inequality (Loève [4], 247) we may write for arbitrary $\epsilon > 0$,

$$\Pr[\max_{1\leqq k \leqq n} |S_k - k\mu - \text{med}\{S_k - S_n + (n-k)\mu\}| \geqq n\epsilon] \leqq 2\Pr(|S_n - n\mu| \geqq n\epsilon)$$

and therefore, since $n^{-1}S_n \to_P \mu$,

$$(1) \quad \Pr[\max_{1\leqq k \leqq n} |S_k - k\mu - \text{med}\{S_k - S_n + (n-k)\mu\}| \geqq n\epsilon] \to 0 \quad \text{as} \quad n \to \infty.$$

However, the condition $n^{-1}(S_n - n\mu) \to_P 0$ ensures that n^{-1} med $\{S_k - S_n + (n-k)\mu\} \to 0$ as $n \to \infty$, so we obtain from (1) that for arbitrary $\epsilon > 0$,

$$(2) \qquad \Pr[\max_{1\leqq k \leqq n} |S_k - k\mu| \geqq n\epsilon] \to 0 \quad \text{as} \quad n \to \infty.$$

Now,

$$M_n - n\mu = \max_{1 \leqq k \leqq n} S_k - n\mu \leqq \max_{1 \leqq k \leqq n} (S_k - k\mu) \leqq \max_{1 \leqq k \leqq n} |S_k - k\mu|,$$

so that

$$\Pr[M_n \geqq n(\mu + \epsilon)] \leqq \Pr(\max_{1 \leqq k \leqq n} |S_k - k\mu| \geqq n\epsilon) \to 0 \quad \text{as} \quad n \to \infty$$

by condition (2). The result $n^{-1}M_n \to_P \mu$ then follows immediately upon noting that for $\epsilon > 0$,

$$\Pr[M_n \leqq n(\mu - \epsilon)] \leqq \Pr[S_n \leqq n(\mu - \epsilon)] \to 0 \quad \text{as} \quad n \to \infty.$$

Now let us examine the case of N_n. We have for arbitrary $\epsilon > 0$,

$$\begin{aligned} &\Pr[N_n \geqq n(\mu + \epsilon)] \\ (3) \qquad &= \Pr(N_n \geqq n(\mu + \epsilon); N_n = M_n) + \Pr(N_n \geqq n(\mu + \epsilon); N_n > M_n) \\ &\leqq \Pr[M_n \geqq n(\mu + \epsilon)] + \Pr[\min_{1 \leqq k \leqq n} S_k \leqq -n(\mu + \epsilon)], \end{aligned}$$

and $\Pr[M_n \geqq n(\mu + \epsilon)] \to 0$ as $n \to \infty$ since $n^{-1}M_n \to_P \mu$. Furthermore, from condition (2) we see that for arbitrary $\eta > 0$,

$$(4) \qquad \Pr[\min_{1 \leqq k \leqq n} (S_k - k\mu) \leqq -n\eta] \to 0 \quad \text{as} \quad n \to \infty,$$

and since $\min_{1 \leqq k \leqq n} (S_k - k\mu) \leqq \min_{1 \leqq k \leqq n} S_k$ we have from (4) that

$$\Pr[\min_{1 \leqq k \leqq n} S_k \leqq -n(\mu + \epsilon)] \leqq \Pr[\min_{1 \leqq k \leqq n} (S_k - k\mu) \leqq -n(\mu + \epsilon)] \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, returning to (3), we see that $\Pr[N_n \geqq n(\mu + \epsilon)] \to 0$ as $n \to \infty$. The result $n^{-1}N_n \to_P \mu$ then follows as it is easily seen that for $\epsilon > 0$,

$$\Pr[N_n \leqq n(\mu - \epsilon)] \leqq \Pr[|S_n| \leqq n(\mu - \epsilon)] \to 0 \quad \text{as} \quad n \to \infty.$$

This completes the proof of the lemma. We remark that in Lemma 2 of [5] the result $n^{-1}S_n \to_P \mu$ is established when the conditions (T1), (T2) and (T5(a)) with $k = -1$ are satisfied. Hence, our Lemma 1 shows that $n^{-1}M_n \to_P \mu$ and $n^{-1}N_n \to_P \mu$ under these conditions.

LEMMA 2. *Under the conditions* (T1), (T2) *and* (T5(a)) *with* $k = -1$,

$$0 \leqq \int_\mu^\infty [1 - G_n(nx)]\, dx \to 0 \quad as \quad n \to \infty.$$

PROOF. The obvious inequality

$$0 \leqq \Pr(N_n \leqq y) \leqq \Pr(M_n \leqq y) \leqq 1$$

yields

$$0 \leqq \int_\mu^\infty [1 - \Pr(M_n \leqq nx)]\, dx \leqq \int_\mu^\infty [1 - \Pr(N_n \leqq nx)]\, dx,$$

so in order to obtain the required result it suffices to show that

$$\int_\mu^\infty [1 - \Pr(N_n \leqq nx)]\, dx \to 0 \quad \text{as} \quad n \to \infty.$$

Integrating by parts, we have

$$(5)\quad n^{-1}EN_n = \int_0^\infty [1 - \Pr(N_n \leqq nx)]\,dx$$
$$= \int_0^\mu [1 - \Pr(N_n \leqq nx)]\,dx + \int_\mu^\infty [1 - \Pr(N_n \leqq nx)]\,dx.$$

Now from Lemma 1 and the comments following it we see that $n^{-1}EN_n \to \mu$ as $n \to \infty$ and that $\Pr(N_n \leqq nx) \to 0$ as $n \to \infty$ for $0 < x < \mu$. Hence, $\int_0^\mu [1 - \Pr(N_n \leqq nx)]\,dx \to \mu$ as $n \to \infty$ and from (5) we deduce the required result that $\int_\mu^\infty [1 - \Pr(N_n \leqq nx)]\,dx \to 0$ as $n \to \infty$. This completes the proof of the lemma.

Proof of Theorem. Having obtained the above lemmas we may use exactly the analysis of Smith [5] in his proof of his Theorem 1 and obtain for $0 < \eta < \mu$,

$$(6)\qquad [\mu^\gamma \Gamma(1 + \gamma)/t^\gamma L(t)]H_\eta(t) \to \alpha \quad \text{as} \quad t \to \infty,$$

where

$$H_\eta(x) = \textstyle\sum_{n=1}^\infty a_n G_n(x) U(x - n\eta).$$

Furthermore,

$$(7)\qquad \textstyle\sum_{n=1}^\infty a_n G_n(x) = H_\eta(x) + \sum_{n=1}^\infty a_n G_n(x)\{1 - U(x - n\eta)\},$$

and

$$\textstyle\sum_{n=1}^\infty a_n G_n(x)\{1 - U(x - n\eta)\}$$
$$\leqq \textstyle\sum_{n=1}^\infty a_n \Pr(M_n \leqq n\eta) \leqq \sum_{n=1}^\infty a_n \Pr(S_n \leqq n\eta).$$

This last sum is shown to be finite under the conditions of our theorem by Smith in his Section 4. Thus, from (6) and (7),

$$\lim_{t\to\infty} [\mu^\gamma \Gamma(1 + \gamma)/t^\gamma L(t)] \textstyle\sum_{n=1}^\infty a_n G_n(x) = \alpha,$$

since the divergence of $\sum_{n=1}^\infty a_n$ implies the divergence of $t^\gamma L(t)$ as $t \to \infty$. This establishes the result of the theorem.

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