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ON INFINITELY DIVISIBLE LAWS AND A RENEWAL THEOREM FOR NON-NEGATIVE RANDOM VARIABLES[1]

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0. Summary. Let $\{X_n\}$ be an infinite sequence of independent non-negative random variables such that, for some regularly varying non-decreasing function $\lambda(n)$, with exponent $1/\beta$, $0 < \beta < \infty$, as $n \to \infty$,

$$P\{X_1 + \cdots + X_n/\lambda(n) \leqq x\} \to K(x)$$

at all continuity points of some d.f. $K(x)$. Let $\Lambda(x)$ be the function inverse to $\lambda(n)$, let $R(x)$ be any other regularly varying function of exponent $\alpha > 0$. Then, if $N(x)$ is the maximum k for which $X_1 + \cdots + X_k \leqq x$, it is proved that, as $x \to \infty$,

$$\mathcal{E}R(N(x)) \sim I(\alpha\beta)R(\Lambda(x))$$

where

$$I(\alpha\beta) = \int_0^\infty u^{-\alpha\beta}\, dK(u)$$

and this latter integral may diverge.

1. Introduction. Let $X_1, X_2, \cdots$ be an infinite sequence of independent, non-negative, random variables. Write $S_n = X_1 + \cdots + X_n$ and let $N(x)$ be the maximum suffix k such that $S_k \leqq x$. The simplest elementary renewal theorem states that, when the $\{X_n\}$ are identically distributed $\mathcal{E}\{N(x)/x\} \to 1/\mathcal{E}X_n$, as $x \to \infty$. Let $\Psi(x)$ be an unbounded, non-decreasing function of regular variation (Feller (1966), p. 269). A more general theorem of Smith (1964), (1966) allows the $\{X_n\}$ to be *non-identically distributed* and shows that if S_n/n tends in probability to a finite limit μ as $n \to \infty$ (that is, the *weak law of large numbers* holds) then $\mathcal{E}\psi(N(x)) \sim \psi(x/\mu)$ as $x \to \infty$. This result has been carried further by Williamson (1966) who showed that if there exists a proper distribution function $K(x)$ such that $P\{S_n/n \leqq x\} \to K(x)$, as $n \to \infty$, at every continuity point of $K(x)$, then $\mathcal{E}N(x) \sim x\int_0^\infty u^{-1}\, dK(u)$ as $x \to \infty$. However, Williamson found it necessary to assume that certain extra conditions were satisfied, and his theorem is not as strong as the one of Smith for the case when $K(x)$ is degenerate (and a weak law of large numbers holds). In the present paper we shall prove the following theorem.

Theorem 1. *Let $\{X_n\}$ be an infinite sequence of mutually independent, non-negative, random variables. Let $\lambda(x)$ be an unbounded, strictly increasing, and continuous function of regular variation with exponent $1/\beta$, $\beta > 0$. Let $\Lambda(x)$ be*

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the function inverse to $\lambda(n)$. *Suppose there is a proper distribution function* $K(x)$ *such that, as* $n \to \infty$.

$$(1.1) \quad P\{(X_1 + X_2 + \cdots X_n)/\lambda(n) \leqq x\} \to K(x), \qquad K(0+) < 1,$$

at every continuity point of $K(x)$.[2] *Then, if* $R(x)$ *is any regularly varying function of index* $\alpha > 0$, *as* $x \to \infty$,

$$\mathcal{E}R(N(x)) \sim I(\alpha\beta)R(\Lambda(x)),$$

where

$$I(\alpha\beta) = \int_0^\infty u^{-\alpha\beta}\, dK(u).$$

The integral $I(\alpha\beta)$ *may be divergent, in which case it is to be understood that*

$$\mathcal{E}R(N(x))/R(\Lambda(x)) \to +\infty \quad as \quad x \to +\infty.$$

Note that the inverse of a regularly varying function is a regularly varying function and that, consequently, $\Lambda(x)$ is regularly varying with index β. We also remark that it will be seen later that $I(\alpha\beta)$ is necessarily finite when a condition "$\Phi(0+) > 0$", to be explained later, is satisfied.

The special case of Theorem 1 when the $\{X_n\}$ are identically distributed, non-negative random variables is not uninteresting. Feller (1966), p. 424, Theorem 2, provides an appropriate convergence theorem. Unfortunately, as he gives it, it does not give results when his exponent has the value unity. From the material he provides, however, one can prove quite easily the following.

THEOREM A. *Let* $\{X_n\}$ *be an infinite sequence of mutually independent, identically distributed, non-negative random variables with distribution function* $F(x)$. *Let* $\Lambda(x)$ *be a continuous and non-decreasing regularly varying function with exponent* β, $0 < \beta \leqq 1$, *and suppose that, as* $x \to \infty$,

$$(1.2) \qquad x^{-1}\int_0^x \{1 - F(u)\}\, du \sim 1/\Gamma(2 - \beta)\Lambda(x).$$

Let $\lambda(n)$ *be the regularly varying function which is inverse to* $\Lambda(x)$. *Then, as* $n \to \infty$,

$$P\{(X_1 + \cdots + X_n)/\lambda(n) \leqq x\} \to G(x)$$

at all points of continuity, where $G(x)$ *is the stable distribution function with Laplace-Stieltjes transform* $G^*(s) = e^{-s\beta}$.

Incidentally, when $0 < \beta < 1$ (i.e. $\beta \neq 1$), the condition (1.2) is equivalent to the simpler

$$\{1 - F(x)\} \sim 1/\Gamma(1 - \beta)\Lambda(x).$$

From Theorem A and Theorem 1 we then have the following renewal theorem for non-negative and identically distributed random variables:

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[2] If $K(0+) = 1$ we get a trivial result.

THEOREM 2. *Under the conditions of Theorem A, for any regularly varying function $R(x)$, with exponent $\alpha > 0$, as $x \to \infty$,*

$$\mathcal{E}\{R(N(x))\} \sim [\Gamma(\alpha + 1)/\Gamma(\alpha\beta + 1)]R(\Lambda(x)).$$

Notice that Theorem 2 does *not* provide information if $\beta = 0$. However, the special case $\{\beta = 0; \alpha = 1\}$ has been discussed, to some extent, in an elementary fashion already (Smith, (1961)). All we need to deduce Theorem 2 from the earlier ones is the evaluation of $I(\alpha\beta)$ for the stable law $G(x)$. We have:

$$\int_0^\infty u^{-\alpha\beta}\, dG(u) = \int_0^\infty \{\int_0^\infty e^{-\lambda u}\lambda^{\alpha\beta-1}/\Gamma(\alpha\beta)\, d\lambda\}\, dG(u) = \int_0^\infty e^{-\lambda^\beta}\lambda^{\alpha\beta-1}/\Gamma(\alpha\beta)\, d\lambda$$
$$= \Gamma(\alpha + 1)/\Gamma(\alpha\beta + 1), \qquad \text{as claimed.}$$

2. Proof of Theorem 1. We begin by mentioning certain results from "standard" probability theory which we shall use. Let $\{F_n(x)\}$ be the respective distribution functions of the variables $\{X_n\}$. Then, denoting Laplace-Stieltjes transforms thus: $F_n^*(s)$, for s real and positive, we see that (1.1) implies

$$\prod_{j=1}^n F_j^*(s/\lambda(n)) \to K^*(s), \qquad \text{as } n \to \infty. \tag{2.1}$$

Let us write, for $x > 0$,

$$\Phi_n(x) = (\lambda(n))^{-1}\sum_{j=1}^n \int_0^{x\lambda(n)} \{1 - F_j(u)\}\, du.$$

Then, plainly, $\Phi_n(x)$ is a non-decreasing function of x and it is not a difficult piece of analysis to deduce from (2.1) that

$$\Phi_n^*(s) \to s^{-1}\log (K^*(s))^{-1}, \qquad n \to \infty.$$

Therefore by the continuity theorem for Laplace-Stieltjes transforms (Feller (1966), p. 410), there is a non-decreasing function $\Phi(x)$ (possibly with a jump at $x = 0$, which must be taken into account when calculating transforms) and, at continuity points, $\Phi_n(x) \to \Phi(x)$ as $n \to \infty$. Furthermore,

$$\Phi^*(s) = s^{-1}\log (K^*(s))^{-1}.$$

We can quickly determine whether $\Phi(x)$ has a jump at $x = 0$, since we must evidently have

$$\Phi(0+) = \lim_{n\to\infty} s^{-1}\log (K^*(s))^{-1}.$$

Incidentally, it is not hard to show that, when $\Phi(0+) > 0$, $I(\alpha\beta)$ is finite for all $\alpha > 0$. If we define, for $x > 0$, non-increasing functions

$$M_n(x) = \sum_{j=1}^n \{1 - F_j(x\lambda(n))\},$$

then $\Phi_n(x) = \int_0^x M_n(u)\, du$ and so there must be a non-increasing function $M(x)$ such that $M_n(x) \to M(x)$, at all continuity points, as $n \to \infty$. Furthermore, for any $a > b > 0$,

$$\Phi(a) - \Phi(b) = \int_b^a M(u)\, du,$$

and so

$$\Phi(x) = \Phi(0+) + \int_0^x M(u)\, du.$$

This implies

(2.2) $$K^*(s) = \exp\,[-s\Phi(0+) - s\int_0^\infty e^{-sx} M(x)\, dx].$$

Because it has been so easy to derive these "classical" results for *non-negative* random variables, we thought it worthwhile and convenient to do so and at the same time introduce our notations. There is one further result we shall need which does not lie quite so near the surface. It transpires that, if (1.1) holds, $K(x)$ cannot be *any* distribution function whose transform happens to have the form (2.2). In fact $K(x)$ must be of "Lévy class L" which is the case if and only if $M(e^x)$ is a convex function for $-\infty < x < \infty$. A minor consequence is that $M(x)$ is continuous in $0 < x < \infty$. We refer to Feller (1966) or Gnedenko and Kolmogorov (1954) for a treatment of this convexity property.

One further preliminary needs discussion. A minor irritation in our work is that in various places we would wish a variable to be an integer and it may not be. When we write S_n and n is not an integer we shall mean by S_n the partial sum $X_1 + \cdots + X_k$, where k is the least integer not less than n. If $a < b$ are not integers we shall denote sums such as $\sum_{a<j\leqq b} u_j$ more simply as $\sum_a^b u_j$. This will save some typographical problems.

Let us now consider the proof of Theorem 1. We begin by showing that if the theorem is once proved for $R(x) \equiv x^\alpha$, $\alpha > 0$, then it will follow for the general case by a rather amusing trick. Indeed, we shall only need the theorem proved for α arbitrarily small. Let us suppose, therefore, that the theorem has been established when $R(x) = x^\alpha$. Suppose $L(x)$ is an arbitrary function of slow growth and $\gamma > \alpha$. Let $b(x)$ be a non-decreasing continuous function of $x > 0$, taking integer values when x is an integer, and such that

$$b(x) \sim x^{\gamma/\alpha}[L(x)]^{1/\alpha}, \qquad x \to \infty.$$

Since $\gamma > \alpha$, $b(x)/x$ is increasing and unbounded and we note that $b(x+1) - b(x) \geqq b(x)/x$.

Suppose $\{X_n\}$ to be a sequence of non-negative, independent random variables, such that

$$P\{(X_1 + \cdots + X_n)/\lambda(n) \leqq \zeta\} \to K(\zeta), \qquad n \to \infty.$$

Define a new sequence of random variables $\{Y_n\}$ as follows: If the suffix $k = b(r) + 1$ for some r then $Y_k = X_{r+1}$ (we can take $b(0) = 1$ so $Y_1 = X_1$); otherwise let $P\{Y_k = 0\} = 1$. Let us define a continuous non-decreasing $\vartheta(x)$ such that $\vartheta(n) = r$ for all integers n satisfying $b(r) \leqq n < b(r+1)$. It is not difficult to see that $\vartheta(x)$ varies regularly with exponent (α/γ) and that

$$P\{\textstyle\sum_1^n Y_j/\lambda(\vartheta(n)) \leqq \zeta\} \to K(\zeta), \qquad n \to \infty.$$

Thus $\{Y_n\}$ is a sequence of random variables to which Theorem 1 applies, with $\lambda(\vartheta(n))$ in place of $\lambda(n)$; we note that $\lambda(\vartheta(x))$ varies regularly, with exponent $(\alpha/\beta\gamma)$.

Now let $M(x)$ be the maximum k such that $Y_1 + \cdots + Y_k \leqq x$. Then, noting that $b(\Lambda(x))$ asymptotically equals the inverse of $\lambda(\vartheta(n))$, we have

$$\mathcal{E}\{[M(x)]^\alpha\} \sim [\Lambda(x)]^\gamma L(\Lambda(x))I(\beta\gamma). \tag{2.3}$$

But

$$\mathcal{E}\{[M(x)]^\alpha\} = \sum_{n=1}^\infty (n^\alpha - \overline{n-1}^\alpha)P\{Y_1 + \cdots + Y_n \leqq x\}$$

and, since so many Y's are almost surely zero, we find this equation means

$$\begin{aligned}\mathcal{E}\{[M(x)]^\alpha\} &= \sum_{r=1}^\infty ([b(r)]^\alpha - [b(r-1)]^\alpha)P\{X_1 + \cdots + X_r \leqq x\} \\ &= \mathcal{E}\{[b(N(x))]^\alpha\} = \mathcal{E}R(N(x)),\end{aligned}$$

if $R(x) = x^\gamma L(x)$. Thus, from (2.3) we have

$$\mathcal{E}R(N(x)) \sim R(\Lambda(x))I(\beta\gamma),$$

as was to be proved.

Let us now consider the proof of Theorem 1 for the case $R(x) \equiv x^\alpha$, $\alpha > 0$.

For fixed $\zeta > 0$ and varying $x > 0$, set $n = \Lambda(x/\zeta)$. Then $n \to \infty$ as $x \to \infty$, and

$$P\{N(x) \geqq \Lambda(x/\zeta)\} = P\{S_n \leqq x\} = P\{S_n \leqq \zeta\lambda(n)\} \to K(\zeta), \quad \text{as } x \to \infty, \tag{2.4}$$

provided ζ is a continuity point of K.

Since $\Lambda(x)$ has exponent $\beta > 0$, we can write $\Lambda(x) = x^\beta L(x)$ where $L(x)$ is a function of slow growth.

Thus

$$\Lambda(x/\zeta)/\Lambda(x) \to \zeta^{-\beta}, \qquad \text{as } x \to \infty,$$

and so, from (2.3), we have

$$P\{N(x)/\Lambda(x) \geqq \zeta^{-\beta}\} \to K(\zeta), \qquad \text{as } x \to \infty,$$

or,

$$P\{N(x)/\Lambda(x) \geqq u\} \to K(u^{-1/\beta}), \qquad x \to \infty. \tag{2.5}$$

But

$$\mathcal{E}\{[N(x)/\Lambda(x)]^\alpha\} = \alpha \int_0^\infty u^{\alpha-1}P\{N(x)/\Lambda(x) \geqq u\}\, du.$$

Thus we can infer from (2.4) and Fatou's lemma that

$$\liminf_{x\to\infty} \mathcal{E}\{[N(x)/\Lambda(x)]^\alpha\} \geqq \alpha \int_0^\infty u^{\alpha-1}K(u^{-1/\beta})\, du = I(\alpha\beta).$$

In the case when $I(\alpha\beta) = \infty$ the theorem is proved. From now on we suppose $I(\alpha\beta) < \infty$. By dominated convergence we see that, for any $C > 0$,

$$\lim_{x\to\infty} \alpha \int_0^C u^{\alpha-1}P\{N(x)/\Lambda(x) \geqq u\}\, du = \alpha \int_0^C u^{\alpha-1}K(u^{-1/\beta})\, du.$$

The proof will be complete if we can show that, for any $\epsilon > 0$, we can choose $C = C(\epsilon)$ so that

$$\int_C^\infty u^{\alpha-1} P\{N(x)/\Lambda(x) \geqq u\}\, du < \epsilon \tag{2.6}$$

for all sufficiently large x.

For integer j, let us write $G_j(x) \equiv P\{S_j \leqq x\} = P\{N(x) \geqq j\}$. Then (2.6) may be rewritten

$$\sum_{j \geqq C\Lambda(x)} \int_{(j/\Lambda(x))}^{((j+1)/\Lambda(x))} u^{\alpha-1} P\{N(x) \geqq u\Lambda(x)\}\, du < \epsilon,$$

which is equivalent to

$$\sum_{j \geqq C\Lambda(x)} G_{j+1}(x) \int_{(j/\Lambda(x))}^{((j+1)/\Lambda(x))} u^{\alpha-1}\, du < \epsilon.$$

Thus we see that our theorem will be proved if we can show that, for all large x,

$$\sum_{j > C\Lambda(x)} j^{(\alpha-1)} G_j(x) < \epsilon[\Lambda(x)]^\alpha. \tag{2.7}$$

The proof that, when $I(\alpha\beta) < \infty$, (2.7) must be true, is quite involved and is given the following two sections.

3. Completion of proof when $\Phi(0+) = 0$.

LEMMA 3.1. *If $\Phi^*(s) = \int_0^\infty e^{-sx}\, d\Phi(x)$ then, for all $x > 0$, $\Phi^*(1/x) \leqq \Phi(x)$. Consequently,*

$$(\Gamma(\alpha\beta))^{-1} \int_0^\infty e^{-\Phi(x)/x} x^{-(1+\alpha\beta)}\, dx \leqq I(\alpha\beta).$$

PROOF. The argument depends on the convexity of $M(e^x)$. Thus, if $x > 0$ and $y > 0$,

$$2M(x) \leqq M(xy) + M(xy^{-1}).$$

Evidently,

$$\begin{aligned}\Phi^*(x^{-1}) - \Phi(x) &= x^{-1} \int_0^\infty e^{-u/x} \Phi(u)\, du - \Phi(x) = \int_0^\infty e^{-y}[\Phi(yx) - \Phi(x)]\, dy \\ &= \int_0^\infty e^{-y}\{\int_x^{xy} M(u)\, du\}\, dy = J_1 - J_2, \text{ say,}\end{aligned}$$

where

$$J_1 = \int_1^\infty e^{-y}\{\int_x^{xy} M(u)\, du\}\, dy, \qquad J_2 = \int_0^1 e^{-y}\{\int_{xy}^x M(u)\, du\}\, dy.$$

If, in J_2, we set $u = x/v$ we find

$$J_2 = x \int_0^1 e^{-y}\{\int_1^{1/y} M(x/v) v^{-2}\, dv\}\, dy$$

and so, by the convexity of $M(e^x)$,

$$-J_2 \leqq x \int_0^1 e^{-y}\{\int_1^{1/y} M(xv) v^{-2}\, dv\}\, dy - 2xM(x) \int_0^1 e^{-y}\{\int_1^{1/y} v^{-2}\, dv\}\, dy.$$

But $M(x)$ is non-increasing, and so

$$-J_2 \leqq -xM(x) \int_0^1 e^{-y}(1 - y)\, dy = -xM(x)/e.$$

Similarly, the monotonicity of $M(x)$ yields

$$J_1 \leqq xM(x) \int_1^\infty e^{-y}(y-1)\,dy = xM(x)/e.$$

Consequently $J_1 - J_2 \leqq 0$ and $\Phi^*(x^{-1}) \leqq \Phi(x)$ as claimed. Thus

$$\int_0^\infty e^{-\Phi(x)/x}\, x^{-(1+\alpha\beta)}\,dx \leqq \int_0^\infty \exp[-\Phi^*(1/x)/x] x^{-(1+\alpha\beta)}\,dx = \int_0^\infty K^*(1/x)/x^{1+\alpha\beta}\,dx$$
$$= \int_0^\infty \int_0^\infty e^{-u/x}/x^{1+\alpha\beta}\,dx\,dK(u) = \Gamma(\alpha\beta) \int_0^\infty u^{-\alpha\beta}\,dK(u).$$

This proves the lemma. Notice, by the way, that it has not used $\Phi(0+) = 0$.

LEMMA 3.2. *Suppose $\Phi(0+) = 0$. Then, given any $\delta > 0$, one can find $0 < a < b < \delta$, such that*

$$(d/dx)(\Phi(x)/x) \leqq -\alpha\beta x^{-1} - x^{-\frac{1}{2}},$$

for all $a \leqq x \leqq b$.

PROOF. Let us set $\psi(x) = x^{-1}\Phi(x) = x^{-1}\int_1^x M(u)\,du$, and note that $\psi(x)$ is consequently differentiable and non-increasing. Suppose that for some $\delta > 0$ and almost all $x \leqq \delta$,

$$-x\psi'(x) \leqq \alpha\beta + x^{\frac{1}{2}}.$$

By integration we have

$$\psi(x) \leqq \psi(\delta) + \alpha\beta \log(\delta/x) + 2\delta^{\frac{1}{2}} - 2x^{\frac{1}{2}}.$$

Thus

$$\int_0^\delta e^{-\psi(x)}\, x^{-(1+\alpha\beta)}dx \geqq \exp[\psi(\delta) + 2\delta^{\frac{1}{2}}]\delta^{-\alpha\beta} \int_0^\delta \exp(2x^{\frac{1}{2}})x^{-1}\,dx.$$

The integral on the right diverges and so we have a contradiction, in view of Lemma 3.1. Thus, if E denotes the set of x-points where

$$-x\psi'(x) > \alpha\beta + x^{\frac{1}{2}},$$

then the intersection $E \cap (0, \delta)$ must have strictly positive Lebesgue measure for every $\delta > 0$. But $-x\psi'(x) = \psi(x) - M(x)$; and both $M(x)$ and $\psi(x)$ are continuous. Thus $-x\psi'(x)$ is continuous and so in any interval $(0, \delta)$ there must be a sub-interval (a, b), $0 < a < b < \delta$, such that

$$\inf_{x\epsilon(a,b)} - x\psi'(x) \geqq \alpha\beta + x^{\frac{1}{2}}.$$

This proves the lemma.

LEMMA 3.3. *If $\Phi(0+) = 0$ then there exists a constant $A > \alpha\beta$, together with $\epsilon > 0$ and $\eta > 0$, such that for all large n*

$$(\lambda(n))^{-1} \sum_{\epsilon n}^{n} \lambda(j)\{1 - F_j(\eta\lambda(j))\} \geqq A.$$

PROOF. Choose $\rho > 0$, $0 < \gamma < 1$. For any integer p and all $n \geqq \gamma^{-p}$ define

$$T_{n,p}(\rho, \gamma) = (\lambda(n))^{-1} \sum_{n\gamma^p}^{n\gamma^{(p-1)}} \int_0^{\lambda(j)} \{1 - F_j(\rho u)\}\,du$$
$$\geqq (\lambda(n))^{-1} \sum_{j=1}^{n\gamma^{(p-1)}} \int_0^{\lambda(n\gamma^p)} \{1 - F_j(\rho u)\}\,du$$
$$- (\lambda(n))^{-1} \sum_{j=1}^{n\gamma^p} \int_0^{\lambda(n\gamma^p)} \{1 - F_j(\rho u)\}\,du.$$

We know that, for every $\zeta > 0$, as $n \to \infty$,

$$(\lambda(n))^{-1} \sum_{j=1}^{n} \int_0^{\zeta\lambda(n)} \{1 - F_j(u)\}\, du \to \Phi(\zeta).$$

We also know that $\lambda(n)$ varies regularly, with exponent $1/\beta$. Thus, as $n \to \infty$,

$$\lambda(n\gamma^p) \sim \gamma^{1/\beta}\lambda(n\gamma^{(p-1)}),$$
$$\lambda(n\gamma^{(p-1)}) \sim \gamma^{(p-1)/\beta}\lambda(n).$$

Therefore

$$\liminf_{n\to\infty} T_{n,p}(\rho, \gamma) \geqq \gamma^{(p-1)/\beta}\rho^{-1}\Phi(\gamma^{1/\beta}\rho) - \gamma^{p/\beta}\rho^{-1}\Phi(\rho),$$

and so, for any large integer k,

$$\liminf_{n\to\infty} \sum_{p=1}^{k} T_{n,p}(\rho, \lambda) \geqq d_k, \text{ say,}$$

where

$$d_k = \{\Phi(\gamma^{1/\beta}\rho)/\gamma^{1/\beta}\rho - \phi(\rho)\rho^{-1}\} \sum_{p=1}^{k} \gamma^{p/\beta}.$$

Evidently, as $k \to \infty$, $d_k \to d_\infty$ with

$$d_\infty = \gamma^{1/\beta}(1 - \gamma^{1/\beta})^{-1}\{\Phi(\gamma^{1/\beta}\rho)/\gamma^{1/\beta}\rho - \Phi(\rho)\rho^{-1}\}.$$

But, by Lemma 3.2, we can choose γ and ρ in such a way that

$$-x\psi'(x) \geqq \alpha\beta + x^{\frac{1}{2}}$$

for all x such that $\gamma^{1/\beta}\rho \leqq x \leqq \rho$. However, by the mean value theorem, there is a θ, $0 < \theta < 1$, such that

$$\psi(\rho) - \psi(\rho\gamma^{1/\beta}) = \rho(1 - \gamma^{1/\beta})\psi'(\rho - \theta\rho[1 - \gamma^{1/\beta}]).$$

Thus

$$\begin{aligned}
&\gamma^{1/\beta}(1 - \gamma^{1/\beta})^{-1}\{\psi(\rho\gamma^{1/\beta}) - \psi(\rho)\} \\
&\geqq \rho\gamma^{1/\beta}\{\alpha\beta/[\rho - \theta\rho(1 - \gamma^{1/\beta})] + [\rho - \theta\rho[1 - \gamma^{1/\beta}]]^{-\frac{1}{2}}\} \geqq \rho\gamma^{1/\beta}\{\alpha\beta/\rho + \rho^{-\frac{1}{2}}\} \\
&= \gamma^{1/\beta}\{\alpha\beta + \rho^{-\frac{1}{2}}\}.
\end{aligned}$$

We can choose $\gamma < 1$ sufficiently close to unity to make $\gamma^{1/\beta}\{\alpha\beta + \rho^{\frac{1}{2}}\} > \alpha\beta$ without spoiling our argument. Thus $d_\infty > \alpha\beta$, and so there is a large k such that $d_k > \alpha\beta$. This means that, if we set $\epsilon = \gamma^{k/\beta}$,

$$\liminf_{n\to\infty} (\lambda(n))^{-1} \sum_{\epsilon n}^{n} \int_0^{\lambda(j)} \{1 - F_j(\rho u)\}\, du > \alpha\beta.$$

However, for any small $\delta > 0$,

$$\begin{aligned}
(\lambda(n))^{-1} \sum_{\epsilon n}^{n} \int_0^{\lambda(j)\delta} \{1 - F_j(\rho u)\}\, du &\leqq (\lambda(n))^{-1} \sum_{1}^{n} \int_0^{\lambda(n)\delta} \{1 - F_j(\rho u)\}\, du \\
&= (\rho\lambda(n))^{-1} \sum_{1}^{n} \int_0^{\lambda(n)\rho\delta} \{1 - F_j(u)\}\, du.
\end{aligned}$$

Therefore

$$\limsup_{n\to\infty} (\lambda(n))^{-1} \sum_{\epsilon n}^{n} \int_0^{\lambda(j)\delta} \{1 - F_j(\rho u)\}\, du \leqq \Phi(\rho\delta)/\rho,$$

and the right member can be made arbitrarily small by choice of δ (since $\Phi(0+) = 0$). It now follows that, for a suitably small δ,

$$\liminf_{n\to\infty} (\lambda(n))^{-1} \sum_{\epsilon n}^{n} \int_{\lambda(j)\delta}^{\lambda(j)} \{1 - F_j(\rho u)\}\, du > \alpha\beta.$$

But, since $\{1 - F_j(x)\}$ is a non-increasing function of x, this in turn implies

$$\liminf_{n\to\infty} (\lambda(n))^{-1} \sum_{\epsilon n}^{n} \lambda(j)\{1 - F_j(\rho\delta\lambda(j))\} > \alpha\beta.$$

If we set $\eta = \rho\delta$ the lemma is proved.

We are now in a position to complete the task of this section. Let $\Omega(x)$ be a non-decreasing function, $\Omega(0) = 0$, and let $\Omega(x)$ increase only through jumps at the points $x = \lambda(j)$, $j = 1, 2, \cdots$; the saltus at $x = \lambda(j)$ is to be $\{1 - F_j(\eta\lambda(j))\}$. Lemma 3.3 then shows that for some $A > \alpha\beta$ and all $x \geqq x_0(A)$,

$$x^{-1} \int_{\epsilon x}^{x} u\, d\Omega(u) \geqq A. \tag{3.1}$$

On the other hand,

$$(\lambda(n))^{-1} \int_{\epsilon\lambda(n)}^{\lambda(n)} u\, d\Omega(u) = (\lambda(n))^{-1} \sum_{\epsilon n}^{n} \lambda(j)\{1 - F_j(\eta\lambda(j))\}$$
$$\leqq (\eta\lambda(n))^{-1} \sum_{1}^{n} \int_{0}^{\eta\lambda(j)} \{1 - F_j(u)\}\, du.$$

Thus,

$$\limsup_{x\to\infty} x^{-1} \int_{\epsilon x}^{x} u\, d\Omega(u) \leqq \Phi(\eta)/\eta < \infty,$$

and so there exists a finite B such that

$$x^{-1} \int_{\epsilon x}^{x} u\, d\Omega(u) \leqq B \tag{3.2}$$

for all $x \geqq x_0$.

From (3.1) we have, if $y > x \geqq x_0$,

$$\int_{x}^{y} u^{-2}\{\int_{\epsilon u}^{u} v\, d\Omega(v)\}\, dv \geqq A \log(y/x).$$

But, by Fubini's theorem (or integration by parts),

$$\int_{x}^{y} u^{-2}\{\int_{\epsilon u}^{u} v\, d\Omega(v)\, du = \int_{x}^{y} d\Omega(u) + x^{-1} \int_{\epsilon x}^{x} u\, d\Omega(u)$$
$$- \epsilon \int_{\epsilon x}^{x} d\Omega(u) - y^{-1} \int_{x}^{y} u\, d\Omega(u).$$

Thus, for $y > x \geqq x_0$,

$$\int_{x}^{y} d\Omega(u) \geqq A \log(y/x) - B,$$

and so, if $y \geqq y_0$, where

$$y_0 = \exp[2B/(A - \alpha\beta)],$$

we shall have

$$\int_{x}^{y} d\Omega(u) \geqq \tfrac{1}{2}(A + \alpha\beta) \log(y/x) = \nu \log(y/x), \text{ say,}$$

where $\nu > \alpha\beta$. Thus, if $\lambda(n) \geqq (x/\eta)y_0$ and $(x/\eta) \geqq x_0$,

$$\sum_{\Lambda(x/\eta)}^{n} \{1 - F_j(\eta\lambda(j))\} \geqq \nu \log(\lambda(n)/(x/\eta)),$$

whence,

$$\sum_{\Lambda(x/\eta)}^{n} \{1 - F_j(x)\} \geqq \nu \log (\lambda(n)/(x/\eta))$$

and, *a fortiori*,

$$\sum_1^n \{1 - F_j(x)\} \geqq \nu \log (\lambda(n)/(x/\eta)). \tag{3.3}$$

From rather obvious probabilistic considerations we have

$$G_n(x) \leqq \prod_{j=1}^n F_j(x) \leqq \exp [-\sum_1^n \{1 - F_j(x)\}]. \tag{3.4}$$

Thus, if $\lambda(n) \geqq xy_0/\eta$ and if $x \geqq x_0\eta$, from (3.3) and (3.4) we find

$$G_n(x) \leqq (x/\eta\lambda(n))^\nu.$$

The inequality on n is equivalent to one of the form $n \geqq C\Lambda(x)$, for C sufficiently large. Thus

$$\sum_{n>C\Lambda(x)} n^{\alpha-1} G_n(x) \leqq (x/\eta)^\nu \sum_{n>C\Lambda(x)} n^{\alpha-1}[\lambda(n)]^{-\nu}. \tag{3.5}$$

At this point we need the following result on regularly varying functions.

Lemma 3.4. *If $Z(n)$ varies regularly with exponent γ and if $\gamma + \alpha < 0$ then, as $t \to \infty$,*

$$\sum_{n=t}^{\infty} n^{\alpha-1} Z(n) \sim t^\alpha Z(t)/|\gamma + \alpha|.$$

This lemma is closely analogous to a familiar result involving an integral in place of a summation (see Theorem 1, p. 273 of Feller (1966) where, however, the proof of a vital lemma is incorrect, though salvable). For this reason no proof of Lemma 3.4 seems necessary.

If, in (3.5) we let $\nu = \alpha\beta + \delta\beta$, $\delta > 0$, then the function

$$Z(n) = \{\lambda(n)\}^{-\nu}$$

varies regularly with exponent $-(\alpha + \delta)$. Thus we can appeal to Lemma 3.4 to find that, as $x \to \infty$,

$$\sum_{n>C\Lambda(x)} n^{\alpha-1}[\lambda(n)]^{-\nu} \sim [C\Lambda(x)]^\alpha \delta^{-1}[\lambda(CA(x))]^{-\nu} \sim [\Lambda(x)]^\alpha (\delta C^\delta x^\nu)^{-1}.$$

We then, finally, deduce from (3.5) that

$$\limsup_{x\to\infty} [\Lambda(x)]^{-\alpha} \sum_{n>C\Lambda(x)} n^{\alpha-1} G_n(x) \leqq (\delta\eta^\nu C^\delta)^{-1}.$$

Since C can be chosen arbitrarily large, the theorem is proved for the case $\Phi(0+) = 0$.

4. Completions of proof when $\Phi(0+) > 0$. The following lemma, needed here, generalizes a result buried in Smith (1964).

Lemma 4.1. *Let $\{X_n\}$ be a sequence of independent, non-negative, random variables with distribution functions $\{F_n(x)\}$. Set, for $x > 0$,*

$$\Phi_n(x) = (\lambda(n))^{-1} \sum_{j=1}^n \int_0^{x\lambda(n)} \{1 - F_j(u)\}\, du.$$

If, for some $x > 0$, $\Phi_n(x)$ *is a bounded function of* n *and if there is a* $\delta > 0$ *such that, for every* $x > 0$,

$$\liminf_{n\to\infty} \Phi_n(x) \geqq \delta, \tag{4.1}$$

then, for any $\alpha > 0$, $\epsilon > 0$, *we can find* $C(\epsilon)$ *such that for all large* x, *using established notation,*

$$\sum_{j>C\Lambda(x)} j^{(\alpha-1)} G_j(x) < \epsilon[\Lambda(x)]^{\alpha}.$$

PROOF. Suppose that $\Phi_n(\zeta) \leqq A < \infty$ for all n and some fixed $\zeta > 0$. From the fact that (4.1) holds we must be able to find an *unbounded* non-decreasing function $w(\lambda(n))$ such that, for all large n,

$$(\lambda(n))^{-1} \sum_{j=1}^{n} \int_0^{\lambda(n)/w(\lambda)} \{1 - F_j(u)\}\, du \geqq \delta.$$

This inequality remains valid if we replace $w(\lambda)$ by a more slowly increasing $w_1(\lambda) \leqq w(\lambda)$. Thus, by Lemma 9 of Smith (1964) we may suppose that $w(\lambda)$ is continuous, non-decreasing, and unbounded, and that $\lambda/w(\lambda)$ is an increasing function of λ. Let us define $l(\lambda) = \log w(\lambda)$; evidently $\lambda/l(\lambda)$ is also, for all large λ, an increasing function. Let us now set

$$t(n) = \lambda(n)l(n)/w(n).$$

Then

$$t(n)/l(t(n)) = \lambda(n) \log w(n)/w(n) \log w(t(n)).$$

However, for all large n, $\lambda(n) \geqq t(n)$; therefore

$$t(n)/l(t(n)) \geqq \lambda(n)/w(n).$$

Thus, whenever $\lambda(r) \geqq t(n)$, or, equivalently, $r \geqq \Lambda(t(n))$, we shall have

$$\lambda(r)/l(\lambda(r)) \geqq \lambda(n)/w(n).$$

Therefore, for all large n,

$$(\lambda(n))^{-1} \sum_{j=1}^{\Lambda(t(n))-1} \int_0^{\lambda(n)/w(\lambda(n))} \{1 - F_j(u)\}\, du$$
$$+ (\lambda(n))^{-1} \sum_{\Lambda t(n))}^{n} \int_0^{\lambda(j)/l(\lambda(j))} \{1 - F_j(u)\}\, du \geqq \delta,$$

that is, say, $T_1(n) + T_2(n) \geqq \delta$.

Choose a small $\eta > 0$. Since $t(n)/\lambda(n) = l(n)/w(n) \to 0$, as $n \to \infty$, we shall have $t(n) < \eta\lambda(n)$ for all large n. Also, since $w(\lambda(n)) \to \infty$ as $n \to \infty$, we shall have $\lambda(n)/w(\lambda(n)) < \zeta\eta\lambda(n)$ for all large n. Thus

$$T_1(n) < (\lambda(n))^{-1} \sum_1^{\Lambda(\eta\lambda(n))} \int_0^{\zeta\eta\lambda(n)} \{1 - F_j(u)\}\, du.$$

But $\Lambda(\eta\lambda(n)) \sim \eta^{\beta} n$ as $n \to \infty$, and so $\Lambda(\eta\lambda(n)) < 2\eta^{\beta} n$ for all large n. Therefore

$$T_1(n) < (\lambda(n))^{-1} \sum_1^{2\eta^{\beta} n} \int_0^{\zeta\eta\lambda(n)} \{1 - F_j(u)\}\, du.$$

However $\lambda(2\eta^{\beta}n) \sim 2^{1/\beta}\eta\lambda(n)$ as $n \to \infty$ and so $\lambda(2\eta^{\beta}n) > \eta\lambda(n)$ for all large n. Therefore

$$T_1(n) < (\lambda(n))^{-1} \sum_1^{2\eta^{\beta}n} \int_0^{\zeta\lambda(2\eta^{\beta}n)} \{1 - F_j(u)\}\, du < \lambda(2\eta^{\beta}n)(\lambda(n))^{-1}A.$$

We may then infer that

$$\limsup_{n\to\infty} T_1(n) < 2^{1/\beta}\eta A.$$

Thus, since η is arbitrarily small, we conclude that

$$\liminf_{n\to\infty} T_2(n) \geqq \delta$$

and so there must exist some $\delta_1 > 0$ such that, for all large n,

$$(\lambda(n))^{-1} \sum_{j=1}^{n\lambda(j)/l(\lambda(j))} \{1 - F_j(u)\}\, du \geqq \delta_1 . \tag{4.2}$$

For a given large x define $r^* = r^*(x)$ as the greatest integer such that $\lambda(r^*)/l(\lambda(r^*)) \leqq x$. Let $s^* = s^*(x)$ be the greatest integer such that $\lambda(s^*)/l(\lambda(s^*)) \leqq (1 + e)x$. Choose a large $C > 0$ and consider the following three cases.

(i) $C\Lambda(x) < n < r^*(x)$. By a familiar inequality (see e.g. Smith (1964)),

$$G_n(x) \leqq \exp [W_n(t)], \qquad t > 0, \tag{4.3}$$

where

$$W_n(t) = tx - t \sum_{j=1}^{n} \int_0^{\infty} e^{-tu}\{1 - F_j(u)\}\, du. \tag{4.4}$$

Let $t = 1/x$ and truncate the integrals at x in (4.4) and we find

$$W_n(x^{-1}) \leqq 1 - (ex)^{-1} \sum_{j=1}^{n} \int_0^{x} \{1 - F_j(u)\}\, du. \tag{4.5}$$

Since $\lambda(j)/l(\lambda(j)) \leqq x$ for $j = 1, 2, \cdots, n \leqq r^*(x)$ we have from (4.2) that

$$\sum_{j=1}^{n} \int_0^{x} \{1 - F_j(u)\}\, du > \lambda(n)\delta_1 .$$

Thus, from (4.5),

$$W_n(x^{-1}) \leqq 1 - \delta_1\lambda(n)(ex)^{-1},$$

and so

$$G_n(x) \leqq \exp \{1 - \delta_1\lambda(n)(ex)^{-1}\}, \qquad n \leqq r^*. \tag{4.6}$$

(ii) $r^*(x) < n \leqq s^*(x)$. By combining inequalities (3.4) and (4.3) we have that

$$G_n(x) \leqq \exp [R_n(t)] \tag{4.7}$$

where

$$R_n(t) = \tfrac{1}{2}tx - \tfrac{1}{2}t \sum_{j=1}^{n} \int_0^{\infty} e^{-tu}\{1 - F_j(u)\}\, du - \tfrac{1}{2} \sum_{j=1}^{n} \{1 - F_j(x)\}.$$

Thus, truncating integrals,

$$R_n(x^{-1}) \leqq \tfrac{1}{2} - (2ex)^{-1} \sum_{j=1}^{n} \int_0^{x} \{1 - F_j(u)\}\, du - \tfrac{1}{2} \sum_{r^*+1}^{n} \{1 - F_j(x)\}. \tag{4.8}$$

Let us set

$$\gamma_j = \int_0^{\lambda(j)/l(\lambda(j))} \{1 - F_j(u)\}\, du, \qquad I_j(x) = \int_0^x \{1 - F_j(u)\}\, du.$$

Then (4.2) can be rewritten

$$\gamma_1 + \gamma_2 + \cdots + \gamma_n > \delta_1\lambda(n). \tag{4.9}$$

If $j > r^*$ (so $\lambda(j)/l(\lambda(j)) > x$), we have

$$\int_x^{\lambda(j)/l(\lambda(j))} \{1 - F_j(u)\}\, du = \gamma_j - I_j(x).$$

Therefore

$$(\lambda(j)/l(\lambda(j)) - x)(1 - F_j(x)) \geqq \gamma_j - I_j(x). \tag{4.10}$$

But, if $j \leqq s^*(x)$, then $\lambda(j)/l(\lambda(j)) \leqq (1 + e)x$, which means

$$(\lambda(j)/l(\lambda(j)) - x) \leqq ex.$$

Thus, from (4.10),

$$1 - F_j(x) \geqq (\gamma_j - I_j(x))(ex)^{-1}, \qquad r^* < j \leqq s^*.$$

From this result and (4.8) we infer

$$\begin{aligned} R_n(x^{-1}) &< \tfrac{1}{2} - (2ex)^{-1}\textstyle\sum_1^{r^*} \gamma_j - (2ex)^{-1}\sum_{r^*+1}^n I_j(x) \\ &\quad - (2ex)^{-1}\textstyle\sum_{r^*+1}^n (\gamma_j - I_j(x)) \\ &= \tfrac{1}{2} - (2ex)^{-1}\textstyle\sum_1^n \gamma_j < \tfrac{1}{2} - \delta_1\lambda(n)(2ex)^{-1}, \end{aligned} \tag{4.11}$$

by (4.2). Therefore, from (4.7) we have

$$G_n(x) < \exp\{\tfrac{1}{2} - \delta_1\lambda(n)(2ex)^{-1}\}, \qquad r^* < n \leqq s^*. \tag{4.12}$$

From (4.6) and (4.12) we may thus infer that

$$\begin{aligned} \textstyle\sum_{n=C\Lambda(x)+1}^{s^*} n^{\alpha-1}G_n(x) &< e^{\frac{1}{2}}\textstyle\sum_{C\Lambda(x)}^{\infty} n^{\alpha-1} \exp[-(\delta_1\lambda(n)/2ex)]. \\ &< e^{\frac{1}{2}}\textstyle\sum_{C\Lambda(x)}^{\infty} n^{\alpha-1}(2ex/\delta_1\lambda(n)), \end{aligned}$$

if we choose C sufficiently large so that the inequality $e^{-\zeta} < \zeta^{-1}$ is valid for $\zeta = (\delta_1\lambda(n)/2ex)$. The argument used near the end of Section 3 will then show that

$$\limsup_{x\to\infty} [\Lambda(x)]^{-\alpha} \textstyle\sum_{n>C\Lambda(x)}^{s^*(x)} n^{\alpha-1}G_n(x) \tag{4.13}$$

can be made arbitrarily small, by choice of C.

(iii) $s^*(x) < n$. Arguments similar to those for case (ii) show that (4.7) holds with (4.8) replaced by

$$\begin{aligned} R_n(x^{-1}) &< \tfrac{1}{2} - (2ex)^{-1}\textstyle\sum_1^{s^*} \gamma_j - (2ex)^{-1}\sum_{s^*+1}^n I_j(x) \\ &\quad - \tfrac{1}{2}\textstyle\sum_{s^*+1}^n \{(\gamma_j - I_j(x))/[\lambda(j)/l(\lambda(j)) - x]\}. \end{aligned} \tag{4.14}$$

However, when $j > s^*$, $\lambda(j)/l(\lambda(j)) > (1 + e)x$ and one can infer that

$$(\lambda(j)/l(\lambda(j)) - x) > ex$$

and hence deduce that

(4.15) $R_n(x^{-1}) < \frac{1}{2} - (2ex)^{-1}\sum_1^{s^*}\gamma_j - \frac{1}{2}\sum_{s^*+1}^{n}\{\gamma_j/[\lambda(j)/l(\lambda(j)) - x]\}.$

Write

$$T_n = \sum_{s^*+1}^{n}\{\gamma_j/[\lambda(j)/l(\lambda(j)) - x]\}$$

and $\Gamma_j = \gamma_1 + \gamma_2 + \cdots + \gamma_j$. Then

$$T_n = \Gamma_n[\lambda(n)/l(\lambda(j)) - x]^{-1} - \Gamma_{s^*}[\lambda(s^* + 1)/l(\lambda(s^* + 1) - x]^{-1}$$
$$+ \sum_{s^*+1}^{n-1}\Gamma_j\{[\lambda(j)/l(\lambda(j)) - x]^{-1} - [\lambda(j + 1)/l(\lambda(j + 1)) - x]^{-1}\}.$$

We may assume $s^*(x)$ to be large, so that $\Gamma_j > \delta_1\lambda(j)$ for all j in the range. Thus

$$T_n > \delta_1\lambda(n)[\lambda(n)/l(\lambda(n)) - x]^{-1} - \Gamma_{s^*}[\lambda(s^* + 1)/l(\lambda(s^* + 1)) - x]^{-1}$$
$$+ \delta_1\sum_{s^*+1}^{n-1}\lambda(j)\{[\lambda(j)/l(\lambda(j)) - x]^{-1} - [\lambda(j + 1)/l(\lambda(j + 1)) - x]^{-1}\}$$
$$= [\delta_1\lambda(s^* + 1) - \Gamma_{s^*}][\lambda(s^* + 1)/l(\lambda(s^* + 1)) - x]^{-1}$$
$$+ \delta_1\sum_{s^*+2}^{n}\{[\lambda(j) - \lambda(j - 1)][\lambda(j)/l(\lambda(j)) - x]^{-1}\}.$$

If we use this last inequality in (4.14) we find

(4.16) $$2R_n(x^{-1}) < 1 - \delta_1\lambda(s^* + 1)[\lambda(s^* + 1)/l(\lambda(s^* + 1)) - x]^{-1}$$
$$- \delta_1 l(\lambda(s^* + 2))\sum_{s^*+2}^{n}\{(\lambda(j) - \lambda(j - 1))/\lambda(j)\}.$$

If $|x - j| < 1$ then $\lambda(x)/\lambda(j) \to 1$ as x and j increase. Thus

$$\sum_{s^*+2}^{n}\{(\lambda(j) - \lambda(j - 1))/\lambda(j)\} \sim \int_{s^*+1}^{n} d\lambda(x)/\lambda(x) = \log(\lambda(n)/\lambda(s^* + 1)).$$

Hence, for some δ_2, $0 < \delta_2 < \delta_1$, we have from (4.16) that, for all large x,

$$2R_n(x^{-1}) < 1 - \delta_1\lambda(s^* + 1)[\lambda(s^* + 1)/l(\lambda(s^* + 1)) - x]^{-1}.$$
$$- 2\delta_2 l(\lambda(s^* + 1))\log(\lambda(n)/\lambda(s^* + 1)).$$

We must thus conclude from (4.7) that, when $n > s^*$,

$$G_n(x) < [\lambda(s^* + 1)/\lambda(n)]^{\delta_2 l(\lambda(s^*+1))}e^{\psi(x)},$$

where

(4.17) $\psi(x) = \frac{1}{2} - \delta_1\lambda(s^*(x) + 1)/2[\lambda(s^* + 1)/l(\lambda(s^* + 1)) - x].$

For sufficiently large x, $\delta_2 l(\lambda(s^*(x) + 1) > \chi > \alpha\beta$, and so

$$\sum_{s^*+1}^{\infty} n^{\alpha-1}G_n(x) < e^{\psi(x)}[\lambda(s^* + 1)]^{\chi}\sum_{s^*+1}^{\infty} n^{\alpha-1}[\lambda(n)]^{-\chi}.$$

But $[\lambda(n)]^{-\chi}$ varies regularly with exponent $-\chi/\beta$, and so, from Lemma 3.4 we have, as $x \to \infty$,

$$\sum_{s^*+1}^{\infty} n^{\alpha-1}[\lambda(n)]^{-\chi} \sim \beta(s^* + 1)^{\alpha}/[\lambda(s^* + 1)]^{\chi}|\chi - \alpha\beta|.$$

Therefore

$$\sum_{s^*+1}^{\infty} n^{\alpha-1} G_n(x) = O((s^* + 1)^{\alpha} e^{\psi(x)}). \tag{4.18}$$

Since

$$\begin{aligned} \lambda(s^* + 1)/l(\lambda(s^* + 1)) &= (\lambda(s^* + 1)/\lambda(s^*))(l(\lambda(s^*))/l(\lambda(s^* + 1)))(\lambda(s^*)/\lambda(l(s^*))) \\ &\leqq (\lambda(s^* + 1)/\lambda(s^*))(1 + e)x, \end{aligned}$$

it is clear that for large x

$$\lambda(s^* + 1)/l(\lambda(s^* + 1) \leqq (1 + 2e)x.$$

Therefore, from (4.17),

$$\psi(x) < \tfrac{1}{2} - \delta_1 \lambda(s^* + 1)(4ex)^{-1}. \tag{4.19}$$

Furthermore, if we represent $\lambda(x) = x^{1/\beta} L(x)$, where $L(x)$ is a function of slow growth,

$$[(s^* + 1)/\Lambda(x)]^{1/\beta} = \lambda(s^* + 1)x^{-1} \cdot L(\Lambda(x))/L(s^* + 1). \tag{4.20}$$

But, from the familiar canonical representation of $L(x)$ due to Karamata (1930),

$$L(x) = a(x) \exp [\textstyle\int_1^x \mathcal{E}(u) u^{-1} \, du],$$

where $a(x) \to 1$ and $\mathcal{E}(x) \to 0$ as $x \to \infty$. Thus, for some finite $A_1 > 0$

$$\begin{aligned} L(\Lambda(x))/L(s^* + 1) &\leqq A_1 \exp \{ -\textstyle\int_{\Lambda(x)}^{s^*+1} \mathcal{E}(u) u^{-1} \, du \} \\ &\leqq A_1((s^* + 1)/\Lambda(x))^{\epsilon/\beta}, \end{aligned}$$

for all large x and arbitrary $\epsilon > 0$. Thus, from (4.20),

$$((s^* + 1)/\Lambda(x))^{(1-\epsilon)/\beta} \leqq A_1(\lambda(s^* + 1)/x). \tag{4.21}$$

From (4.18), (4.19), and (4.21) we then deduce

$$[\Lambda(x)]^{-\alpha} \sum_{s^*+1}^{\infty} n^{\alpha-1} G_n(x) = O([\omega(x)]^{\alpha\beta/(1-\epsilon)} \exp [-\delta_1 \omega(x)/4e]), \tag{4.22}$$

say, where $\omega(x) = \lambda(s^*(x) + 1)/x$. But $\lambda(s^* + 1) > (1 + e)x\, l(\lambda(s^* + 1))$, and so $\omega(x) \to \infty$ as $x \to \infty$. Thus we have, finally, from (4.22) that

$$[\Lambda(x)]^{-\alpha} \sum_{s^*+1}^{\infty} n^{\alpha-1} G_n(x) \to 0, \quad \text{as} \quad x \to \infty.$$

This result, coupled with the arbitrary smallness of (4.13), completes the proof of the theorem.

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