The Annals of Mathematical Statistics
1968, Vol. 39, No. 1, 123–133

CONVERGENCE OF SUMS OF SQUARES OF MARTINGALE DIFFERENCES[1]

BY Y. S. CHOW

Purdue University

1. Introduction and notation. Let $(\Omega, \mathfrak{F}, P)$ be a probability space. A stochastic basis $(\mathfrak{F}_n, n \geqq 1)$ is a monotonically increasing sequence of σ-fields of measurable sets. A stochastic sequence $(y_n, \mathfrak{F}_n, n \geqq 1)$ consists of a stochastic basis $(\mathfrak{F}_n, n \geqq 1)$ and a sequence of random variables $(y_n, n \geqq 1)$ such that y_n is $\mathfrak{F}_n$-measurable. For a stochastic sequence $(x_n, \mathfrak{F}_n, n \geqq 1)$, we put (here as well as in following sections)

$$x_0 = 0, \mathfrak{F}_0 = \{\Phi, \Omega\}, d_n = x_n - x_{n-1} \quad \text{for} \quad n \geqq 1, s_n = (\textstyle\sum_{k=1}^n d_k^2)^{\frac{1}{2}},$$

$$x^* = \sup_{n \geqq 1} |x_n|, d^* = \sup_{n \geqq 1} |d_n|, s = \lim_{n \to \infty} s_n,$$

and I_A = indicator function of set A. If $(x_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale, then $(d_n, \mathfrak{F}_n, n \geqq 1)$ is called a martingale difference sequence. For a given stochastic basis $(\mathfrak{F}_n, n \geqq 1)$, a stopping time t is an extended positive integral valued measurable function such that $[t = n] \varepsilon \mathfrak{F}_n$ for each n. For a stopping time t and a measurable function y, $E_t y$ is defined as $\int_{[t<\infty]} y \, dP$ (or $\int_{[t<\infty]} y$, in short), if it exists.

Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale. Austin [1] recently proves that if $\sup_{n \geqq 1} E|x_n| < \infty$, $s < \infty$ a.e.; also Burkholder [2] proves that if $Es < \infty$ x_n converges a.e. and that if $\sup_{n \geqq 1} E|x_n| < \infty$, then $\sum_{k=1}^\infty \varphi_k d_k$ converges a.e. for every stochastic sequence $(\varphi_k, \mathfrak{F}_{k-1}, n \geqq 1)$ for which $\sup_{n \geqq 1} |\varphi_n| < \infty$ a.e.; Gundy [8] proves that if $(d_n, n \geqq 1)$ is an orthonormal sequence such that each d_n assumes at most two non-zero values with positive probability, and if the σ-field generated by $d_1, \cdots, d_n$ consists of exactly n atoms, such that

$$\inf_{n \geqq 1} \min(P[d_n > 0], \quad P[d_n < 0])/P[d_n \neq 0] > 0,$$

then for every sequence a_n of real numbers, $\sum_{n=1}^\infty a_n^2 d_n^2 < \infty$ if and only if $\sum_{n=1}^\infty a_n d_n$ converges.

Let $(\mathfrak{F}_n, n \geqq 1)$ be a stochastic basis. If for each n, $\mathfrak{F}_n$ is generated by atoms of $\mathfrak{F}_n$, then $(\mathfrak{F}_n, n \geqq 1)$ is said to be atomic. For a σ-field $\mathcal{G}$ of measurable sets and $A \varepsilon \mathfrak{F}$, a $\mathcal{G}$-measurable cover of A is a set $C \varepsilon \mathcal{G}$ such that $P(A - C) = 0$ and that if $B \varepsilon \mathcal{G}$ and $P(A - B) = 0$, then $P(C - B) = 0$. For $A \varepsilon \mathfrak{F}$, let $C_n(A)$ be the $\mathfrak{F}_n$-measurable cover of A. If there exists $M > 0$ such that $PC_n(A) \leqq MPA$ for every $A \varepsilon \mathfrak{F}_{n+1}, n = 1, 2, \cdots$, then $(\mathfrak{F}_n, n \geqq 1)$ is said to be regular.

Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a submartingale and $E|x_n| < \infty$ for each n. If $(\mathfrak{F}_n,$

Received 3 March 1967; revised 5 September 1967.

[1] This research was supported by the National Science Foundation under Grant No. GP-06073.

$n \geqq 1$) is an atomic, regular stoachastic basis, then [3] x_n converges a.e. on the set [$\sup x_n < \infty$]. In [7], Doob extends this result to the non-atomic cases: if for $K > 0$, there exist $M > K$ and $\delta > 0$ such that

$$P\{[\max_{k \leqq n} x_k < K] - ([P(x_{n+1} \geqq M \mid \mathfrak{F}_n) = 0] \cup [P(x_{n+1} \geqq K \mid \mathfrak{F}_n) > \delta])\} = 0,$$

then x_n converges a.e. on the set [$\sup_{n \geqq 1} x_n < K$].

In this paper, we will give new proofs of those theorems mentioned above and in some cases extend them, by method of stopping times. The results of Gundy, Austin and Burkholder are unified into Theorems 3 and 5. Theorem 4 extends a result of Doob [6] 320, to regular stochastic basis.

2. Some new proofs.

THEOREM 1. *If* $(x_n, \mathfrak{F}_n, n \geqq 1)$ *is a martingale with difference sequence* d_n *satisfying*

$$(1) \qquad E_t|d_t| \leqq MK, \qquad E_\tau x_\tau < \infty,$$

for some $M > 0$ *and every* $K > 1$, *where* $t = \inf\{n \mid x_n^2 \geqq K^2\}$ *and* $\tau = \inf\{n \mid x_n \geqq K\}$, *then* $s < \infty$ a.e. *on the set* [$\sup x_n < \infty$].

PROOF. By a proof of Doob ([6], 322), under the second condition of (1), x_n converges a.e. on [$\sup x_n < \infty$]. Hence we need only to prove that $s < \infty$ a.e. on the set $A = [x_n$ converges].

For $\delta > 0$ and $K > 3$, put

$$g_0 = 1, \quad g_n = \prod_1^n (1 + d_k/K) \quad \text{for} \quad n \geqq 1$$

$$t = t_K = \inf\{n \mid |g_n| \geqq 1 + \delta \quad \text{or} \quad |x_n| \geqq \log K\}.$$

Set $h_n = g_{\min(t,n)}$. Then

$$|h_n| \leqq (1 + \delta)I_{[t>n]} + (1 + \delta)(1 + |d_t|/K)I_{[t \leqq n]}.$$

For every stopping time $t' \leqq \inf\{n \mid x_n^2 \geqq K^2\}$, the first condition of (1) implies that $E_{t'}|d_{t'}| < MK$. Therefore $E_t|d_t| \leqq M \log K$. Hence $Eh^* < \infty$ and $(h_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale. By Doob's martingale convergence theorem ([6]; 319), h_n tends to h_∞ a.e. and in L_1. Hence $\int_A h_\infty = \int_A h_n + \epsilon_n = PA + \epsilon_{n,K} + \epsilon_n$, where $\lim_n \epsilon_n = \lim_K \epsilon_{n,K} = 0$.

$$PA + \epsilon_{n,K} + \epsilon_n = \int_A h_\infty = \int_{A[t<\infty]} g_t + \int_{A[t=\infty]} h_\infty$$
$$\leqq (1 + \delta) \int_{A[t<\infty]} (1 + |d_t|/K) + (1 + \delta)PA[t = \infty, h_\infty > 0].$$

Let $\epsilon > 0$. Since $E_t|d_t| \leqq M \log K \leqq \epsilon K$ for all large K, $(1 + \delta)PA[t = \infty, h_\infty \leqq 0] \leqq \delta PA + (2 + \delta)\epsilon$. Since x_n converges on A, it follows that $\lim_n g_n = g_\infty$ exists on A, $\lim_{K\to\infty} PA[t_K = \infty] = PA$, and on A, $g_\infty > 0$ if and only if $s < \infty$. Hence $PA[g_\infty \leqq 0] \leqq \delta + 2\epsilon$ and $PA[s = \infty] \leqq \delta + 3\epsilon$, if K is large enough. Therefore $s < \infty$ a.e. on A.

If $(x_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale with $\sup E|x_n| \leqq M < \infty$, then (1) is satisfied, since for every $K > 1$, $t = \inf\{n \mid x_n^2 \geqq K^2\}$, and $\tau = \inf\{n \mid x_n \geqq K\}$,

$$E_t|d_t| = \lim_n \int_{[t \leqq n]} |dt| \leqq \lim_n \int_{[t \leqq n]} (|x_t| + |x_{t-1}|) \leqq M + K \leqq K \max(M, 1),$$

$$E_\tau|x_\tau| \leqq \lim_n E|x_{\min(\tau,n)}| \leqq \lim_n E|x_n|.$$

Hence Theorem 1 implies Austin's result [1] mentioned before. If $(x_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale with $E\, d^* < \infty$, then (1) is satisfied and hence $s < \infty$ a.e. on the set $[\sup x_n < \infty]$. The last result is due to Burkholder [2], and his proof is based on Austin's theorem. (With a slight modification, Burkholder's proof yields Theorem 1 also.)

The following theorem is suggested by a recent work of Gundy [9] on the decomposition of L_1-bounded martingales.

THEOREM 2. *Let $r \geqq 1$ and $(d_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale difference sequence satisfying $E|d_n| < \infty$. Put*

$$\sigma = \left(\sum_1^\infty |d_n|^r \right)^{1/r}. \tag{2}$$

Then for every $K > 0$, we can decompose

$$d_n = a_n + b_n + c_n, \qquad n \geqq 1, \tag{3}$$

where a_n, b_n and c_n are martingale difference sequences satisfying

$$E \sum_1^\infty |a_n|^r \leqq 2^r E \min(\sigma, K)^r \leqq 2^r K^{r-1} E\sigma \quad \text{for} \quad r \geqq 1, \tag{4}$$

$$E \sum_1^\infty a_n^2 \leqq E \min(\sigma, K)^2 \leqq KE\sigma \quad \text{for} \quad r = 2;$$

$$E \sum_1^\infty |b_n| \leqq 2E\, d^* \leqq 2E\sigma; \tag{5}$$

$$[c^* > 0] \subset [\sigma > K], \quad P[c^* > 0] \leqq E\sigma/K, \quad E\left(\sum_1^\infty |c_n|^r\right)^{1/r} \leqq E\sigma. \tag{6}$$

PROOF. Put $\sigma_n = \left(\sum_1^n |d_k|^r\right)^{1/r}$. Then $\lim \sigma_n = \sigma$ a.e. For $K > 0$, define

$$t = \inf\{n \mid \sigma_n > K\}. \tag{7}$$

Put $a_n = d_n I_{[t>n]} - E(d_n I_{[t>n]} \mid \mathfrak{F}_{n-1})$, $b_n = d_n I_{[t=n]} - E(d_n I_{[t=n]} \mid \mathfrak{F}_{n-1})$ and $c_n = d_n I_{[t<n]}$. Clearly, (3) is satisfied, and a_n, b_n and c_n are martingale difference sequences. Then for $r \geqq 1$,

$$\begin{aligned} E \sum_1^\infty |a_n|^r &\leqq 2^r \sum_1^\infty E|d_n|^r I_{[t>n]} \\ &\leqq 2^r E \sum_1^{t-1} |d_n|^r \\ &\leqq 2^r E \min(\sigma, K)^r, \end{aligned}$$

and for $r = 2$, $E \sum_1^\infty a_n^2 \leqq \sum_1^\infty E\, d_n^2 I_{[t>n]} \leqq E \sum_1^{t-1} d_n^2 \leqq E \min(\sigma, K)^2$, which yield (4). Now

$$E \sum_1^\infty |b_n| \leqq 2E \sum_1^\infty |d_n| I_{[t=n]} \leqq 2E\, d^* \leqq 2E\sigma,$$

$$[c^* > 0] \subset [\sigma > K],\ P[c^* > 0] \leqq P[\sigma \geqq K] \leqq E\sigma/K,$$

$$E\left(\sum_1^\infty |c_n|^r\right)^{1/r} \leqq E\left(\sum_1^\infty |d_n|^r\right)^{1/r} = E\sigma.$$

Therefore (5) and (6) hold and the proof is completed.

COROLLARY 1 (Burkholder [2]). *Suppose that* $(x_n, \mathfrak{F}_n, n \geqq 1)$ *is a martingale with difference sequence* d_n *and that* $(g_n, \mathfrak{F}_{n-1}, n \geqq 1)$ *is a stochastic sequence such that* $g^* < \infty$ a.e. (i) *If* $Ed^* < \infty$, x_n *converges* a.e. *on* $[s < \infty]$. (ii) *If* sup $E|x_n| < \infty$, $\sum g_n d_n$ *converges* a.e.

PROOF. (i) By Theorem 2, for $K > 0$ we can write $d_n = a_n + b_n + c_n$, where a_n, b_n and c_n are martingale difference sequences such that $\sum_1^\infty Ea_n^2 < \infty$, $\sum_1^\infty E|b_n| < \infty$ and $[c^* > 0] \subset [s \geqq K]$. Hence x_n converges a.e. on $[s < K]$. Since K is arbitrary, x_n converges a.e. on $[s < \infty]$.

(ii) By Austin's theorem [1], $s < \infty$ a.e. Note that without loss of generality we can assume that $|g_n| \leqq 1$ a.e. for each n. Then $\sum_1^\infty g_k^2 d_k^2 < \infty$ a.e. For $K > 0$, put $t = t_K = \inf \{n \mid |x_n| \geqq K\}$, $e_k = I_{[t \geqq k]} g_k d_k$ and $z_n = \sum_1^n e_k$. Then $(z_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale and $\sum_1^\infty e_k^2 < \infty$ a.e. Since $|e_k| \leqq I_{[t \geqq k]}|x_k - x_{k-1}| \leqq 2K + |x_t| I_{[t<\infty]}$, $E \sup |e_k| < \infty$. By (i), $\sum_1^\infty e_k$ converges a.e. and hence $\sum_1^\infty g_k d_k$ converges a.e. on $[t = \infty]$. Since $\lim_K P[t_K = \infty] = 1$, $\sum_1^\infty g_k d_k$ converges a.e.

The following corollary is due to Burkholder [2] in the case $r = 2$, and due to Stout [11] in the case $r > 2$.

COROLLARY 2. *Suppose that* $(x_n, \mathfrak{F}_n\ n \geqq 1)$ *is a martingale with difference sequence* e_n. *Let* $\beta_n > 0$ *be a sequence of constants such that* $\sum_1^\infty \beta_n < \infty$ *and*

$$E\left(\sum_1^\infty |e_n|^r \beta_n^{1-r/2}\right)^{1/r} < \infty \quad \textit{for some} \quad r \geqq 2, \tag{8}$$

then x_n *converges* a.e.

PROOF. Put $d_n = e_n \beta_n^{1/r - \frac{1}{2}}$. Then d_n is a martingale difference sequence and $E(\sum_1^\infty |d_n|^r)^{1/r} < \infty$. By Theorem 2, we can decompose $d_n = a_n + b_n + c_n$, where a_n, b_n and c_n are martingale differences sequences satisfying (4), (5) and (6). By a result of [4], (4) implies that $\sum_1^\infty (e_n I_{[t>n]} - E(e_n I_{[t>n]} \mid \mathfrak{F}_{n-1}))$ converges a.e. Now (5) implies that $\sum_1^\infty \beta_n^{1/r - \frac{1}{2}} | e_n I_{[t=n]} - E(e_n I_{[t=n]} \mid \mathfrak{F}_{n-1})| < \infty$ a.e. Hence $\sum_1^\infty |e_n I_{[t=n]} - E(e_n I_{[t=n]} \mid \mathfrak{F}_{n-1})| < \infty$ a.e. Since

$$P[c^* > 0] = P[\sup |e_n| I_{[t<n]} > 0] \leqq E\sigma/K \leqq \epsilon$$

for a given $\epsilon > 0$ if K is large enough, x_n converges a.e.

In [5], the following strong law of large numbers has been proved. If a martingale difference sequence e_n satisfying

$$E \sum_1^\infty |e_n|^r n^{-r/2-1} < \infty$$

for some $r \geqq 2$, then $\lim_n \sum_1^n e_k/n = 0$ a.e. This result can be extended to:

COROLLARY 3. *Let* e_n *be a martingale difference sequence and* $r \geqq 2$. *If*

$$E\left(\sum_1^\infty |e_n|^r n^{-r/2-1}\right)^{1/r} < \infty, \tag{9}$$

then $\lim_n \sum_1^n e_k/n = 0$ a.e.

PROOF. Put $d_n = e_n n^{-\frac{1}{2}-1/r}$. Then (2) is satisfied. For $\epsilon > 0$, according to Theorem 2, we can decompose $d_n = a_n + b_n + c_n$, where a_n, b_n and c_n are martingale difference sequences, $\sum_1^\infty b_n$ converges a.e., $\sum_1^\infty c_n$ converges, except on a set of measure less than ϵ, and a_n satisfies (4). Put $a_n' = a_n n^{\frac{1}{2}+1/r}$, $b_n' =$

$b_n n^{\frac{1}{2}+1/r}$ and $c_n' = c_n n^{\frac{1}{2}+1/r}$. Then $e_n = a_n' + b_n' + c_n'$, $\sum_1^\infty b_n' n^{-\frac{1}{2}-1/r}$ converges a.e., $\sum_1^\infty c_n' n^{-\frac{1}{2}-1/r}$ converges, except on a set of measure less than ϵ, and

$$E \sum_1^\infty |a_n'|^r n^{-r/2-1} < \infty.$$

By the result of [5], $\lim \sum_1^n a_k'/n = 0$ a.e. Since by Kronecker lemma $\lim \sum_1^n (b_k' + c_k')/n = 0$, except on a set of measure less than ϵ, $\lim \sum_1^n e_k/n = 0$ a.e.

In [11], Stout gives a weaker version of Corollary 2 as follows: *If e_n is a martingale difference sequence satisfying* (9), *then* $\lim_n \sum_1^n (e_k - e_k')/n = 0$ a.e., *where*

$$e_n' = E(e_n I_{[|e_n|^{2r} \leqq n^{r+1}]} \mid \mathfrak{F}_{n-1}).$$

The following corollary is due to Burkholder [2]. Gundy [9] has another proof by applying his decomposition of L_1-bounded martingales.

Corollary 4. *If $(x_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale with difference sequence d_n satisfying*

$$Es = E(\sum_1^\infty d_n^2)^{\frac{1}{2}} < \infty,$$

then for $\lambda > 0$,

$$\lambda P[x^* \geqq \lambda] \leqq 9Es.$$

Proof. By Theorem 2, for $\lambda > 0$, we can write $d_n = a_n + b_n + c_n$, where a_n, b_n and c_n are martingale difference sequences satisfying $E \sum_1^\infty a_n^2 \leqq \lambda Es$, $E \sum_1^\infty |b_n| \leqq 2Es$, and $P[c^* > 0] \leqq Es/\lambda$. Hence

$$P[x^* \geqq \lambda] \leqq P[\sup_n |\sum_1^n a_k| \geqq \lambda/2] + P[\sum_1^\infty |b_n| \geqq \lambda/2] + P[c^* > 0] \leqq 9Es/\lambda.$$

3. Some preliminary remarks. In this section, we will make some remarks about the assumptions in the following sections.

For a martingale $(x_n, \mathfrak{F}_n, n \geqq 1)$ with difference sequence d_n satisfying $E \sup_n d_n < \infty$, Doob ([6], p. 322) proves that x_n converges a.e. on $[\sup_n x_n < \infty]$ in the following way:

For $K > 0$ define $t = \inf\{n \mid x_n > K\}$. Then $y_n = x_{\min(t,n)}$ forms a martingale and $\sup E|y_n| < \infty$. Hence y_n converges a.e. and x_n converges a.e. on $[t = \infty] = [\sup x_n \leqq K]$. To insure that the stopped martingale y_n have nice properties, one is forced to assume some conditions on the sequence d_n. However, if for every $\epsilon > 0$, we have another stopping time t^* such that $t^* \leqq t$, $t^* < t$ on $[m < t < \infty]$ for some $m \geqq 1$, and $P[t^* < t = \infty] \leqq \epsilon$, then $\sup E|x_{\min(t^*,n)}| < \infty$ and hence x_n converges a.e. on $[t = \infty]$. This approach leads us to the following concept of "induced" stopping times.

Let t be a stopping time relative to a stochastic basis $(\mathfrak{F}_n, n \geqq 1)$ and $(B_n, n \geqq 1)$ be a sequence of measurable sets such that $B_n \varepsilon \mathfrak{F}_n$ for each n. Let C_n be the $\mathfrak{F}_n$-measurable cover of $B_n[t = n + 1]$. For $m = 1, 2, \cdots$, define $\tau = \tau_m = \inf\{n \geqq m \mid \omega \varepsilon C_n\}$ and

$$t^* = t_m^* = \min(t, \tau). \tag{10}$$

Then the stopping times t_m^* are said to be "induced" by $\{t, (B_n, n \geqq 1), m\}$. If $\lim_m P[t_m^* < t = \infty] = 0$, then the sequence B_n is said to be t-regular.

LEMMA 1. *For a stopping time t and a sequence $(B_n, n \geqq 1)$ of measurable sets such that $B_n \in \mathfrak{F}_n$, let C_n be the $\mathfrak{F}_n$-measurable cover of $B_n[t = n + 1]$ and define $t^* = t_m^*$ by (10). Then t^* is a stopping time, $t^* \leqq t$* a.e.,

$$[t^* = t = k] \subset [t = k] - B_{k-1} \tag{11}$$

for $m < k < \infty$, and if

$$P[C_n, \text{ i.o.}] = \lim_n P \textstyle\bigcup_n^\infty C_k = 0, \tag{12}$$

then $(B_n, n \geqq 1)$ is t-regular, that is

$$\lim_m P[t_m^* < t = \infty] = 0. \tag{13}$$

PROOF. Clearly, t^* is a stopping time and $t^* \leqq t$ a.e. Since $[t > n] \supset C_n$, $\tau < t$ on $[\tau < \infty]$. For $m < k < \infty$, if $t^*(\omega) = t(\omega) = k$, $\omega \notin C_{k-1} \supset B_{k-1}[t = k]$ and $\omega \in [t = k] - B_{k-1}$. If (12) holds, then

$$\lim_m P[t_m^* < t = \infty] \leqq \lim P[\tau_m < t] = \lim P \textstyle\bigcup_m^\infty C_k = 0,$$

which yields (13).

In most applications of Lemma 1, we put either $B_n = \Phi$ for every n or $B_n = \Omega$ for every n. In the former case, (12) is automatically satisfied; in the latter case, if $(\mathfrak{F}_n, n \geqq 1)$ is regular, (13) is satisfied for every t, since for some $M > 0$,

$$P \textstyle\bigcup_n^\infty C_k \leqq \sum_n^\infty PC_k \leqq M \sum_n^\infty PB_k[t = n + 1] \leqq MP[n < t < \infty].$$

Therefore the sequence $B_n = \Phi$ is t-regular for every stopping time t, and if $(\mathfrak{F}_n, n \geqq 1)$ is regular, every sequence $B_n \in \mathfrak{F}_n$ is t-regular for every stopping time t. These are the trivial cases. In order to obtain some non-trivial examples of t-regular sequences B_n for a given t, we prove the following lemma first.

LEMMA 2. *Let $\mathcal{G}$ be a σ-field of measurable sets and let C be the $\mathcal{G}$-measurable cover of a measurable set A. Then $C = [P(A \mid \mathcal{G}) > 0]$.*

PROOF. First,

$$P(A - C) = PA - PAC = PA - \textstyle\int_C P(A \mid \mathcal{G})$$
$$= PA - EP(A \mid \mathcal{G}) = 0.$$

Now, let $B \in \mathcal{G}$ and $P(A - B) = 0$. Then $P(A - ABC) = 0$ and $PA(C - B) = 0$. Hence $\int_{C-B} P(A \mid \mathcal{G}) = PA(C - B) = 0$. Since $P(A \mid \mathcal{G}) > 0$ a.e. on C, $P(C - B) = 0$. The proof is completed.

LEMMA 3. *Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a stochastic sequence with difference sequence d_n, and for $K > 0$, let*

$$t = \inf\{n \mid x_n \geqq K\}, \qquad \tau = \inf\{n \mid s_n \geqq K\}. \tag{14}$$

If for some $M \geqq K$ and $\delta > 0$,

$$P([t > n] - [P(x_{n+1} \geqq M \mid \mathfrak{F}_n) = 0 \quad or \quad P(x_{n+1} \geqq K \mid \mathfrak{F}_n) \geqq \delta]) = 0, \tag{15}$$

then $B_n = [E(I_{[t=n+1]}(x_t - M) \mid \mathfrak{F}_n) > 0]$ *is t-regular; and if for some* $M \geqq K$ *and* $\delta > 0$,

$$(16) \quad P([\tau > n] - [P(s_{n+1} \geqq M \mid \mathfrak{F}_n) = 0 \quad or \quad P(s_{n+1} \geqq K \mid \mathfrak{F}_n) \geqq \delta]) = 0,$$

then $B_n = [E(I_{[\tau=n+1]}(s_\tau - M) \mid \mathfrak{F}_n) > 0]$ *is τ-regular.*

PROOF. We will only prove the first half; the proof of the second half is similar. By Lemma 2,

$$\begin{aligned} PC_n &= P[P(B_n[t = n + 1] \mid \mathfrak{F}_n) > 0] \\ &= P[t > n, P(x_{n+1} \geqq K \mid \mathfrak{F}_n) > 0, E(I_{[x_{n+1} \geqq K]}(x_{n+1} - M) \mid \mathfrak{F}_n) > 0] \\ &\leqq P[t > n, E(I_{[x_{n+1} \geqq M]}(x_{n+1} - M) \mid \mathfrak{F}_n) > 0]. \end{aligned}$$

Since $E|x_{n+1}| < \infty$, by monotone convergence theorem for conditional expectations,

$$[E(I_{[x_{n+1} \geqq M]}(x_{n+1} - M) \mid \mathfrak{F}_n) > 0] \subset [P(x_{n+1} \geqq M \mid \mathfrak{F}_n) > 0].$$

Hence

$$\begin{aligned} PC_n &\leqq P[t > n, \quad P(x_{n+1} \geqq M \mid \mathfrak{F}_n) > 0] \\ &\leqq P[t > n, \quad P(x_{n+1} \geqq K \mid \mathfrak{F}_n) \geqq \delta] \\ &\leqq \textstyle\int_{[t>n]} P(x_{n+1} \geqq K \mid \mathfrak{F}_n)/\delta = P[t = n + 1]/\delta. \end{aligned}$$

Therefore as $n \to \infty$,

$$P(\textstyle\bigcup_n^\infty C_k) \leqq \sum_n^\infty PC_k \leqq \delta^{-1} P[n < t < \infty] \to 0,$$

and B_n is t-regular.

The condition (15) was first introduced by Doob [7]. He noted that if $(\mathfrak{F}_n, n \geqq 1)$ is atomic and regular, then (15) is satisfied by every stochastic sequence $(x_n, \mathfrak{F}_n, n \geqq 1)$.

4. Some extensions.

THEOREM 3. *Let* $(x_n, \mathfrak{F}_n, n \geqq 1)$ *be a martingale,* $E|x_n| < \infty$ *and for* $K > 0$, *put* $t = \inf\{n \mid x_n \geqq K\}$. *If there exist a stochastic sequence* $y_n \geqq 0$ *and a sequence* $B_n \in \mathfrak{F}_n$ *such that* B_n *is t-regular and that*

$$(17) \quad B_n \supset [E(I_{[t=n+1]}(x_t - y_t) \mid \mathfrak{F}_n) > 0], \qquad \sum_2^\infty \int_{[t=k]-B_{k-1}} y_t = M < \infty,$$

then

$$(18) \qquad P[s = \infty, \sup x_n < K] = 0.$$

PROOF. Let $d_n = x_n - x_{n-1}$ for $n \geqq 1$. For $m = 1, 2, \cdots$, define $t^* = t_m^*$ by (10) and $z_n = \sum_1^n d_k I_{[t^* \geqq k]}$. Then $(z_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale and

$$\begin{aligned} z_n &\leqq K, && \text{if } t^* > n \text{ or } t > t^* < n, \\ &= x_t > 0, && \text{if } m < t^* = t = k \leqq n, \\ &\leqq |d_1| + \cdots + |d_m|, && \text{if } t^* \leqq m. \end{aligned}$$

Since

$$\sum_{m+1}^{\infty} \int_{[t^*=t=k]} x_t \leqq \sum_{m+1}^{\infty} \int_{[t=k]-B_{k-1}} ((x_t - y_t) + y_t)$$
$$\leqq M + \sum_{m+1}^{\infty} \int_{\Omega - B_{k-1}} E(I_{[t=k]}(x_t - y_t) \mid \mathfrak{F}_{k-1}) \leqq M,$$

then $\sup Ez_n^+ < \infty$ and thus $\sup E|z_n| < \infty$. By Austin's theorem,

$$\sum_1^\infty d_k^2 I_{[t^* \geqq k]} < \infty \quad \text{a.e.}$$

Therefore $P[s = \infty, t_m^* = \infty] = 0$. Since B_n is t-regular, $\lim_m P[t_m^* < t = \infty] = 0$ and hence $P[s = \infty, t = \infty] = 0$, which completes the proof.

Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale. If (i) $\sup E|x_n| < \infty$, or (ii) $E\, d^* < \infty$, then (17) is satisfied with $y_n = x_n$ and $B_n = \Phi$. If (iii) $(\mathfrak{F}_n, n \geqq 1)$ is regular and $E|x_n| < \infty$, then (17) is satisfied with $y_n = x_n$ and $B_n = \Omega$. Therefore under the conditions (i), (ii) or (iii), $s < \infty$ a.e. on $[\sup x_n < \infty]$. Hence Theorem 3 combines some results of Austin [1], Burkholder [2] and Gundy [8].

THEOREM 4. *Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale, $Ex_n^2 < \infty$, and for $K > 0$, put $t = \inf\{n \mid |x_n| \geqq K\}$. If there exist a stochastic sequence $y_n \geqq 0$ and a sequence $B_n \varepsilon \mathfrak{F}_n$ such that B_n is t-regular and that*

$$(19) \quad B_n \supset [E(I_{[t=n+1]}(x_t^2 - y_t) \mid \mathfrak{F}_n) > 0], \quad \sum_2^\infty \int_{[t=k]-B_{k-1}} y_t = M < \infty,$$

then

$$(20) \qquad P\left[\sum_1^\infty E(d_{k+1}^2 \mid \mathfrak{F}_k) = \infty, \quad x^* < K\right] = 0,$$

$$(21) \qquad P\left[x_n \text{ diverges}, \quad \sum_1^\infty E(d_{k+1}^2 \mid \mathfrak{F}_k) < \infty\right] = 0.$$

PROOF. Let $d_n = x_n - x_{n-1}$ for $n \geqq 1$. For $m = 1, 2, \cdots$, define $t^* = t_m^*$ by (10) and $z_n = \sum_1^n d_k I_{[t^* \geqq k]}$. Then $(z_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale and

$$\begin{aligned} |z_n| &\leqq K, && \text{if} \quad t^* > n \quad \text{or} \quad t > t^* < n, \\ &= |x_k|, && \text{if} \quad m < t^* = t = k \leqq n, \\ &\leqq |d_1| + \cdots + |d_m|, && \text{if} \quad t^* \leqq m. \end{aligned}$$

As in the proof of Theorem 3, we have

$$\sum_{m+1}^\infty \int_{[t^*=t=k]} x_t^2 \leqq \sum_{m+1}^\infty \int_{[t=k]-B_{k-1}} ((x_t^2 - y_t) + y_t) \leqq M;$$

hence $\sup Ez_n^2 < \infty$ and $E \sum_1^\infty d_k^2 I_{[t^* \geqq k]} < \infty$. Therefore

$\sum_1^\infty E(d_k^2 \mid \mathfrak{F}_{k-1}) I_{[t^* \geqq k]} < \infty$ a.e. and

$$P\left[\sum_1^\infty E(d_k^2 \mid \mathfrak{F}_{k-1}) = \infty, \quad t_m^* = \infty\right] = 0.$$

Since B_n is t-regular, $\lim_m P[t_m^* < t = \infty] = 0$ and hence (20) holds. In ([6]; 320), Doob stated that if $(x_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale and $E(d^*)^2 < \infty$, then x_n converges if and only if $\sum_1^\infty E(d_{k+1}^2 \mid \mathfrak{F}_k) < \infty$. However, his proof of the "if" part requires only the assumption that $E(d_{k+1}^2 \mid \mathfrak{F}_k) < \infty$ a.e. for each k. Hence (21) is a special case of Doob's theorem. The proof is completed.

Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale. If $E(d^*)^2 < \infty$, then (19) is satisfied with $y_n = x_n^2$ and $B_n = \Phi$. If $(\mathfrak{F}_n, n \geqq 1)$ is regular and $Ex_n^2 < \infty$, then (19) is satisfied with $y_n = x_n^2$ and $B_n = \Omega$. Hence Theorem 4 extends the above cited Doob's theorem to regular stochastic basis.

THEOREM 5. *Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale, $E|x_n| < \infty$, and for $K > 0$, put $t = \inf\{n \mid s_n \geqq K\}$. If there exist a stochastic sequence $y_n \geqq 0$ and a sequence $B_n \varepsilon \mathfrak{F}_n$ such that B_n is t-regular and that*

$$(22) \qquad B_n \supset [E(I_{[t=n+1]}(s_t - y_t) \mid \mathfrak{F}_n) > 0], \quad \sum_2^\infty \int_{[t=k]-B_{k-1}} y_t = M < \infty,$$

then

$$(23) \qquad P[x_n \textit{ diverges}, \quad s < K] = 0.$$

PROOF. Let $d_n = x_n - x_{n-1}$ for $n \geqq 1$. For $m = 1, 2, \cdots$, define $t^* = t_m^*$ by (10). Put $e_k = d_k I_{[t^* \geqq k]}$ and $z_n = \sum_1^n e_k$. Then $(z_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale, and

$$E(\sum_1^n e_k^2)^{\frac{1}{2}} \leqq \int_{[t^*>n]} s_n + \int_{[t^* \leqq m]} s_m + \sum_{m+1}^\infty \int_{[t^*=k \leqq t]} s_k$$
$$\leqq K + Es_m + \sum_{m+1}^\infty \int_{[t^*=t=k]} s_k .$$

As in the proof of Theorem 3,

$$\sum_{m+1}^\infty \int_{[t^*=t=k]} s_k \leqq \sum_{m+1}^\infty \int_{[t=k]-B_{k-1}} (s_t - y_t + y_t) \leqq M.$$

Hence $E(\sum_1^\infty e_k^2)^{\frac{1}{2}} < \infty$, and by Corollary 1, z_n converges a.e. Hence $P[x_n$ diverges, $t_m^* = \infty] = 0$. Since B_n is t-regular, $\lim_m P[t_m^* < t = \infty] = 0$ and hence (23) holds. Thus we complete the proof.

Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale. If (i) $E\,d^* < \infty$, then (22) is satisfied with $y_n = s_n$ and $B_n = \Phi$. If (ii) $(\mathfrak{F}_n, n \geqq 1)$ is regular and $E|x_n| < \infty$, then (22) is satisfied with $y_n = s_n$ and $B_n = \Omega$. Therefore under the conditions (i) or (ii), x_n converges a.e. on $[s < \infty]$. Hence Theorem 5 combines some results of Burkholder [2] and Gundy [8].

5. Application and corollaries.

THEOREM 6. *Let $(x_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale with $E|x_n| < \infty$.*

(i) *If $(\mathfrak{F}_n, n \geqq 1)$ is a regular stochastic basis, then except on a null set*

$$(24) \qquad s < \infty,$$

$$(25) \qquad x_n \textit{ converges},$$

are equivalent, and if $Ex_n^2 < \infty$, then except on a null set, (25) is equivalent to

$$(26) \qquad \sum_{k=1}^\infty E(d_{k+1}^2 \mid \mathfrak{F}_k) < \infty.$$

(ii) *If $E_t\, d_t^2 < \infty$ for every stopping time t of the form $t = \inf\{n \mid |x_n| \geqq K\}$, then, except on a null set, (25) and (26) are equivalent.*

(iii) *For $K > 0$, put $t = \inf\{n \mid x_n \geqq K\}$ and $\tau = \inf\{n \mid s_n \geqq K\}$. If $E_t x_t < \infty$, then $P[s = \infty, \sup x_n < K] = 0$, and if $E_\tau s_\tau < \infty$, then $P[x_n$ diverges, $s < K] = 0$.*

In particular, if $E_\sigma|d_\sigma| < \infty$ *for every stopping time* σ, *then, except on a null set,* (24) *and* (25) *are equivalent.*

(iv) *For* $K > 0$, *put* $t = \inf\{n \mid x_n \geqq K\}$ *and* $\tau = \inf\{n \mid s_n \geqq K\}$. *Let* $M \geqq K$ *and* $\delta > 0$. *If*

$$(27) \quad P\{[t > n] - ([P(x_{n+1} \geqq M \mid \mathfrak{F}_n) = 0] \cup [P(x_{n+1} \geqq K \mid \mathfrak{F}_n) \geqq \delta])\} = 0,$$

then $P[s = \infty, \sup x_n < K] = 0$, *and if*

$$(28) \quad P\{[s > n] - ([P(s_{n+1} \geqq M \mid \mathfrak{F}_n) = 0] \cup [P(s_{n+1} \geqq K \mid \mathfrak{F}_n) \geqq \delta])\} = 0,$$

then $P[x_n$ *diverges,* $s < K] = 0$.

Proof. (i) Put $y_n = 0$ and $B_n = \Omega$. Then B_n is t-regular for every stopping time t and (17), (19) and (22) are satisfied.

(ii) Let $A = [\sup |x_n| < K]$, $t = \inf\{n \mid |x_n| \geqq K\}$ and $z_n = x_{\min(t,n)}$. Then z_n is a martingale, $A \subset [t = \infty]$, and $Ez_n^2 \leqq 2(K^2 + E_t d_t^2) < \infty$. Hence $\sum_1^\infty E(d_k^2 \mid \mathfrak{F}_{k-1}) I_{[t \geqq k]} < \infty$ a.e. Therefore $\sum_1^\infty E(d_k^2 \mid \mathfrak{F}_{k-1}) < \infty$ a.e. on A. Since K is arbitrary, (25) implies (26). Conversely, Doob's proof of ([6]; 323) implies that (25) holds when (26) is true (even without the condition $E_t d_t^2 < \infty$).

(iii) Assume that $E_t x_t < \infty$. Put $y_n = \max(0, x_n)$ and $B_n = \Phi$. Then B_n is t-regular and (17) is satisfied. By Theorem 3, $P[s < \infty, \sup x_n < K] = 0$.

Assume that $E_\tau s_\tau < \infty$. Put $y_n = s_n$ and $B_n = \Phi$. Then B_n is τ-regular and (22) is satisfied. By Theorem 5, $P[x_n$ diverges, $s < K] = 0$.

(iv) Put $y_n = M$ and $B_n = [E(I_{[t=n+1]}(x_t - M) \mid \mathfrak{F}_n) > 0]$. Obviously (17) is satisfied. By Lemma 3, and B_n is t-regular. Hence the first part follows from Theorem 3. Similarly, the second part follows from Lemma 3 and Theorem 5.

As an application of the "induced" stopping times, we prove the following submartingale convergence theorem, which includes, by Lemma 3, a result of Doob [7].

Theorem 7. *Let* $(x_n, \mathfrak{F}_n, n \geqq 1)$ *be a submartingale,* $E|x_n| < \infty$, *and for* $K > 0$, *define* $t = \inf\{n \mid x_n \geqq K\}$. *If there exist a stochastic sequence* $y_n \geqq 0$ *and a sequence* $B_n \in \mathfrak{F}_n$ *such that* B_n *is t-regular and* (17) *is satisfied, then* $P[x_n$ *diverges,* $\sup x_n < K] = 0$.

Proof. Let $d_n = x_n - x_{n-1}$ for $n \geqq 1$. For $m = 1, 2, \cdots$, define $t^* = t_m^*$ by (10) and $z_n = \sum_1^n d_k I_{[t^* \geqq k]}$. Then z_n is a submartingale. As in the proof of Theorem 3, we have $\sup Ez_n^+ < \infty$. Since $Ez_1 > -\infty$, $\sup E|z_n| < \infty$. By Doob's submartingale convergence theorem, z_n converges a.e. Hence $P[x_n$ diverges, $t_m^* = \infty] = 0$. Since B_n is t-regular, $\lim_m P[t_m^* < t = \infty] = 0$. Therefore $P[x_n$ diverges, $t = \infty] = 0$, which completes the proof.

If $(\mathfrak{F}_n, n \geqq 1)$ is a regular stochastic basis, then $B_n = \Omega$ is t-regular and (17) is satisfied trivially. Hence we have a simple proof of the following corollary.

Corollary 5. *Let* $(x_n, \mathfrak{F}_n, n \geqq 1)$ *be a submartingale and* $E|x_n| < \infty$. *If* $(\mathfrak{F}_n, n \geqq 1)$ *is regular, then* x_n *converges* a.e. *on the set* $[\sup x_n < \infty]$.

Under the condition that $(\mathfrak{F}_n, n \geqq 1)$ is regular and atomic, Corollary 5 has been proved in [3] and [7] by different methods.

REFERENCES

[1] Austin, D. G. (1966). A sample function property of martingales. *Ann. Math. Statist.* **37** 1396–1397.

[2] Burkholder, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504.

[3] Chow, Y. S. (1960). Martingales in a σ-finite measure space indexed by directed sets. *Trans. Amer. Math. Soc.* **97** 254–285.

[4] Chow, Y. S. (1965). Local convergence of martingales and the law of large numbers. *Ann. Math. Statist.* **36** 552–558.

[5] Chow, Y. S. (1967). On a strong law of large numbers for martingales. *Ann. Math. Statist.* **38** 610.

[6] Doob, J. L. (1953). *Stochastic processes.* Wiley, New York.

[7] Doob, J. L. (1961). Notes on martingale theory. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **2** 95–102. Univ. of California Press.

[8] Gundy, R. F. (1966). Martingale theory and pointwise convergence of certain orthogonal series. *Trans. Amer. Math. Soc.* **124** 228–248.

[9] Gundy, R. F. (1968). A decomposition for L^1-bounded martingales. Ann. Math. Stat. **39** 134–138.

[10] Hardy, G. H., Littlewood, J. E. and Polyá, G. (1934). *Inequalities.* Univ. of Cambridge Press.

[11] Stout, W. F. (1967). Some results on almost sure and complete convergence in the independent and martingale cases. Thesis, Purdue University.