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ABSTRACTS OF PAPERS

(Abstract of a paper presented at the Western Regional meeting, Missoula, Montana, June 15–17, 1967. Additional abstracts appeared in earlier issues.)

19. Methods for statistical analysis and testing in practical metric spaces.

J. MacQueen, Western Management Science Institute, University of California, Los Angeles. (Invited)

Let $M_1, M_2, \cdots M_N$ be compact metric spaces with distances $\rho_1, \rho_2, \cdots \rho_N$ respectively. Let M be the product space with distance $\rho = \sum_i \alpha_i \rho_i$, $\alpha_i \geqq 0$. For a probability distribution P on M, a centroid of order r is a point in the non-empty class C_r of points x such that $\int \rho^r(x, y)\, dP(y) = \inf_z \int \rho^r(z, y)\, dP(y)$; the latter quantity itself is termed the variation of order r of P. For a sample $x_1, x_2, \cdots, x_n$, $x_1 \varepsilon M$, a sample centroid minimizes $\sum_i \rho^r(z, x_i)/n$; the minimum value is the sample variation. Methods for statistical analysis based on these simple concepts are described. They are applicable to 'practical' metric spaces, which is to say, spaces where the necessary operations can be carried out on a computer. In addition to Euclidean space, three other spaces have been treated and a computer program developed which permits various kinds of statistical analysis in any of these spaces, or in products of various combinations of these spaces. For example, statistical tests with prescribed type I error can be performed to test whether or not several random variables with values in such spaces are independent or to test whether or not two such random variables have the same distribution. These tests are accomplished by sampling from the appropriate randomization distribution. The special domain of application of these methods is in connection with complex data problems, where the natural descriptions involve a variety of different kinds of information, e.g., medical case records or cross cultural surveys. Large samples are easily handled. The methods make use of an efficient algorithm for finding a partition of a sample of points in M which has low within-class variation. This algorithm can also be used for clustering, automatic file construction, and certain types of pattern recognition. (Received 4 December 1967.)

(Abstracts of papers to be presented at the Annual meeting, Washington, D. C., December 27–30, 1967. Additional abstracts appeared in the June, August, October, and December issues.)

89. $O(c)$-Bayes procedures in sequential design of experiments.

Gary Lorden, Northwestern University.

A sequential design procedure is $O(c)$-Bayes for a given prior distribution if its integrated risk exceeds the Bayes risk by $O(c)$ as c, the cost per observation, approaches zero. Kiefer and Sacks [*Ann. Math. Statist.* **34** 705–750] considered a very general formulation of the sequential design problem and gave simple procedures using non-randomized choices of designs whose integrated risk exceeds the Bayes risk (of order $c|\log c|$) by $o(c|\log c|)$. Their procedures, like all procedures requiring at most one switch of designs are, however, not $O(c)$-Bayes. The present investigation is restricted to a class of problems involving a finite number of states of nature. Simple $O(c)$-Bayes procedures are given, using only non-randomized choices of designs. They are similar to the procedures of Chernoff [*Ann. Math. Statist.* **30** 755–770], although the latter are $O(c)$-Bayes only for problems in which they specify non-randomized design choices. (Received 6 November 1967.)

90. An extension of Paulson's selection procedure. M. MAZUMDAR, Westinghouse Research Laboratories.

In this paper we consider the problem of ranking k populations belonging to the same Koopman-Darmois family distributed according to the density $dP(\theta) = \exp\{P(x)Q(\theta) + R(x) + S(\theta)\}$. A sequential procedure is given for selecting the population with the largest value of the Koopman-Darmois parameter $\tau = Q(\theta)$ so that the probability of making the correct selection exceeds a specified value when the largest τ exceeds all the others by at least a specified amount. This procedure is based on that suggested by Paulson [*Ann. Math. Statist.* **35** 174–180] for selecting the normal population with the greatest mean and retains the latter rule's special feature of elimination of non-contending populations from the sampling scheme. Numerical methods based on relaxation techniques are given for determining the exact probability of correct selection (PCS) and average sample number (ASN) in one special case, namely, when $k = 3$ and the populations are Poisson processes. The results indicate that this procedure results in considerable overprotection. An *ad hoc* modification for reducing the overprotection and the ASN is suggested. (Received 16 October 1967.)

91. On conditional rank-order tests for experimental designs. K. L. MEHRA, Iowa State University.

Consider an experimental design with K treatments, n blocks and m observations per cell. For testing the hypothesis H_0 concerning the equality of the treatment effects, Hodges and Lehmann had proposed [*Ann. Math. Statist.* **33** (1962) 482–497] certain conditional rank tests based on interblock comparison of the observations after "alignment". In Mehra and Sarangi [*Ann. Math. Statist.* **38** (1967) 90–107], the asymptotic efficiency $e_{L,\mathfrak{F}}$ of these conditional tests relative to the $\mathfrak{F}$-tests was shown to be never less than $3/\pi$ under normality. The purpose of the present paper is to extend the results of these papers to general rank-order tests: It is shown that, under very general conditions, the convergence in (conditional) distribution, under H_0, of the test functions is again uniform in the "configuration" $\mathcal{E}$ (where the "configuration" $\mathcal{E}$ stands for the observed distribution of ranks over the blocks). For the normal-score versions of the conditional rank-order tests, the asymptotic efficiency $e_{L,\mathfrak{F}}$ obtains the value one under normality and is bounded below by one for a large class of distributions. (Received 13 November 1967.)

92. Combination of unbiased estimators of the mean which consider inequality of unknown variances. J. S. MEHTA and JOHN GURLAND, University of Wisconsin.

The problem considered is how to combine estimators of the common mean from two samples corresponding to normal populations with different unknown variances. Attention is confined to the case where it is known that the variance of one specific population exceeds that of the other. Five classes of unbiased estimators are presented, two of which are based on a preliminary test of significance regarding the ratio of the population variances. The gain achieved by utilizing the knowledge that the ratio of variances exceeds one is investigated by comparing the efficiencies of these estimators with an estimator of Graybill and Deal in which no restriction on the ratio of variances is present. (Received 6 November 1967.)

93. Method of images in the plane. RICHARD W. MENSING and H. T. DAVID, Iowa State University and University of Minnesota.

Absorption probabilities for certain boundaries are generalized to the plane for tied and untied Wiener processes and random walks. Work is based on planar versions of the method of images and the invariance principle. (Received 7 November 1967.)

94. Some common sharp bounds for many probabilities. Govind S. Mudholkar, University of Rochester.

Let $X_1, X_2, \cdots, X_n$ be n jointly distributed rv's with $EX_i = 0$, $EX_i^2 = \sigma_i^2$, $i = 1, 2, \cdots, n$. Let φ denote a symmetric gauge function of n variables, that is, let $\varphi(\mathbf{t}) = \varphi(t_1, t_2, \cdots, t_n)$ satisfy (i) $\varphi(\mathbf{t}) \geqq 0$ with equality iff $\mathbf{t} = \mathbf{0}$, (ii) $\varphi(c\mathbf{t}) = |c|\varphi(\mathbf{t})$ for any real c, (iii) $\varphi(\mathbf{s} + \mathbf{t}) \leqq \varphi(\mathbf{s}) + \varphi(\mathbf{t})$, (iv) $\varphi(\epsilon_1 t_{i_1}, \epsilon_2 t_{i_2}, \cdots, \epsilon_n t_{i_n}) = \varphi(\mathbf{t})$, where $\epsilon_i = \pm 1$ and $(i_1, i_2, \cdots, i_n)$ is a permutation of $(1, 2, \cdots, n)$, and (v) $\varphi(1, 0, 0, \cdots, 0) = 1$. The main purpose of this paper is to show that the bound $P[\varphi(X_1^2, X_2^2, \cdots, X_n^2) \geqq 1] \leqq \sum_{i=1}^{n} \sigma_i^2$ is valid and is sharp for any φ. Furthermore, if the n columns of a $(p \times n)$ random matrix $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n]$ satisfy $E\mathbf{u}_i = \mathbf{0}$, $E\mathbf{u}_i\mathbf{u}_i' = \mathbf{\Sigma}_i$, $\mathbf{\Sigma}_1 + \mathbf{\Sigma}_2 + \cdots + \mathbf{\Sigma}_n = \mathbf{\Sigma}$, then it is shown that for any symmetric gauge function ψ of p variables the bound $P[\|\mathbf{UU}'\|_\psi \geqq 1] \leqq \operatorname{tr} \mathbf{\Sigma}$, where $\|\mathbf{UU}'\|_\psi$ denotes the symmetric gauge function ψ of the p eigenvalues of $\mathbf{UU}'$, is valid and sharp. (Received 23 October 1967.)

95. A one dimensional random space-filling problem. John P. Mullooly' National Institutes of Health.

Consider an interval of the real line $(0, x)$, $x > 0$; and place in it a random subinterval $S(x)$ defined by the random variables X_x and Y_x, the position of the center of $S(x)$ and the length of $S(x)$. The set $(0, x) - S(x)$ consists of two intervals of length δ and η. Let $a > 0$ be a fixed constant. If $\delta \geqq a$, then a random interval $S(\delta)$ defined by X_δ, Y_δ is placed in the interval of length δ. If $\delta < a$, the placement of the second interval is not made. The same is done for the interval of length η. Continue to place non-intersecting random subintervals in $(0, x)$, and require that the lengths of all the random subintervals be $\geqq a$. The process terminates after a finite number of steps when all the segments of $(0, x)$ uncovered by random subintervals are of length $< a$. At this stage, we say that $(0, x)$ is saturated. Define $N(a, x)$ as the number of random subintervals that have been placed when the process terminates. We are interested in the asymptotic behavior of the moments of $N(a, x)$ for large x. Our main result is that when Y_x is uniformly distributed over (a, x), the first moment of $N(a, x)$, $E[N(a, x)] \sim C(a)x^{2^{1/2}-1}$, for large x; where $C(a)$ is a positive constant independent of x. (Received 31 October 1967.)

96. Two queues in series with a finite, intermediate waitingroom. Marcel F. Neuts, Purdue University.

An $M|G|1$ queue in series with a finite state Markovian service unit is considered. The second unit has a maximum allowable number of customers and the first unit "blocks" whenever this saturated state is reached. By considering a general class of operating rules during the blocked intervals, we may give a unified treatment of many queues of practical interest. The time dependent features of this queueing system may be studied in terms of an imbedded semi-Markov process, which has a formal analogy with multitype branching processes. We discuss the time dependence, the equilibrium conditions and the asymptotic distributions for this system. (Received 10 October 1967.)

97. Stationary point processes invariant under superposition (preliminary report). David S. Newman, The Boeing Company, Seattle, Washington.

Many investigators have used stationary point processes as a model for data representing a series of discrete events in time. This paper describes a class of stationary point processes with two features which are very desirable for such applications: First, the processes are completely defined by their first and second moment density functions. Second, the superposition of two independent processes in this class is another member of the class. The class is defined as follows: Let $\varphi^{(m)}(t_1, \cdots, t_m)$ be the moment density of order m. Let $\varphi^{(1)}(t) = \lambda$, $\varphi^{(2)}(t_1, t_2) = \lambda^2 b(t_2 - t_1)$ for convenience. First it is shown that there is one and only one function which is (a) symmetric in $t_1, \cdots, t_m$, (b) a multilinear form of degree $[m/2]$ in $b_{ij} = b(t_i - t_j)$, (c) satisfying the "ergodic consistency" condition $\varphi^{(m)}(t_1, \cdots, t_m) \rightarrow \varphi^{(m-1)}(t_1, \cdots, t_{m-1})$ when $t_m \rightarrow \infty$ (and $t_1, \cdots, t_{m-1}$) remain fixed). Then it is shown that if N_1 and N_2 are independent point processes with moment density functions satisfying (a)–(c) for each m, the superposition $N_{12} = N_1 + N_2$ has moment densities satisfying the same conditions. (Received 7 November 1967.)

98. A test for the homogeneity of variances in regression models.
WALID A. NURI, University of Southwestern Louisiana.

Consider the regression model $Y_j = x_j'\alpha + \epsilon_j$, where $x_j' = (x_{1j}, x_{2j}, \cdots, x_{kj})$ is a known vector; $\alpha = \alpha(k \times 1)$ is a certain unknown parameter vector; and the ϵ_j's are independent and normally distributed with zero means and variances σ_j^2 ($j = 0, \pm 1, \pm 2, \cdots$). Herbst [*J. Roy. Statist. Soc. Ser. B.* **25** 442–450] has used Fourier methods to study the variance fluctuations in time series analysis. In this paper a short Fourier expansion is assumed for the set of variances in a sample of size n to construct a test procedure for the hypothesis that $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2$ where it is assumed that $\sigma_{n+k}^2 = \sigma_k^2$, for all k. The test statistic proposed is a function of $I_u(p) = n^{-2}|\sum_{j=1}^{n} u_j^2 \exp(2\pi ijp/n)|^2$, ($p = 0, 1, 2, \cdots, m$), where $i = -1^{\frac{1}{2}}$, $u_j = Y_j - x_j'\hat{\alpha}$, $m = [n/2]$ and $\hat{\alpha}$ is the least square estimate of α. Theorems are proved to obtain an asymptotic distribution of the test statistic under the null hypothesis and artificial examples are given for the support of the theoretical results. (Received 1 November 1967.)

99. A nonparametric test for the multiple correlation.
KANTILAL M. PATEL, University of Georgia.

Consider a $(p + 1)$-variate continuous population with corresponding $(p + 1)$-element variate vector X having the first element Y and the remaining elements $X^{(i)}$, $i = 1, \cdots, p$. Let ρ designate the population multiple correlation between Y and the set $X^{(i)}, \cdots, X^{(p)}$. Also, let a sample of n independent observation vectors X_α on X, $\alpha = 1, \cdots, n$. We consider a nonparametric test of the null hypothesis: $\rho = 0$ against alternatives $\rho \neq 0$. The following test is proposed and discussed: compute $B^2 = (n - 1)r'R^{-1}r$ where the vector r has p elements, the sample product-moment correlations between Y and $X^{(i)}$, $i = 1, \cdots, p$, and R is the sample product-moment correlation matrix of $X^{(1)}, \cdots, X^{(p)}$. Reject the null hypothesis if B^2 is too large. Under the null hypothesis, we use a permutation procedure to develop the null distribution of B^2 that assumes that $n!$ permutations of the Y_α are equally likely. The null distribution of B^2 is approximated by a beta distribution by equating the corresponding mean and variance. It has been shown that the test is consistent and the limiting null distribution of B^2 is chi-square with p degrees of freedom. We also replace each X_α by its corresponding vector of ranks (normal scores), observations on each variate having been ranked (assigned normal scores) separately in the sample. The resulting tests are also shown to be consistent. The test procedures are illustrated by numerical examples. (Received 2 November 1967.)

100. A bivariate distribution.
S. A. PATIL, Tennessee Technological University.

A bivariate distribution of $(2U_1 - 1)Y$, $(2U_2 - 1)$, where U_1U_2, Y are independent. U_1, U_2 each have beta distribution with parameters $\frac{1}{2}(m - 1)\frac{1}{2}(m - 1)$ and Y has $(\chi_m^2)^{\frac{1}{2}}$ distribution. The joint distribution function is studied. When m is odd the joint-density function is expressed in terms of $\phi(\cdot)$ and $\Phi(\cdot)$, where $\phi(\cdot)$ is the density function and

$\Phi(\cdot)$ is a cumulative distribution function of the standardized normal distribution. The marginal distribution, the conditional distribution, the moments, the moment generating functions of the joint distribution and conditional distribution are obtained. The regression for the square of the random variables is found to be linear. The distribution is useful in finding variance of the minimum variance unbiased estimator of $\Phi(\mu - \tau)$, in the case when μ, τ are known and σ unknown, also when τ is known and both μ and σ are unknown. (Received 9 October 1967.)

101. Sequential interval estimation for the means of normal populations. EDWARD PAULSON, Queens College, City University of New York.

Interesting results on sequential estimation for means of normal populations are obtained if the requirement that the width of the confidence interval should be $\leqq w$ is relaxed when the confidence limits are favorably situated with respect to some standard value (or with respect to the limits for the mean of another population). Let $\{X_r\}$ $(r = 1, 2, \cdots)$ denote a sequence of independent and normally distributed variables with mean m and variance σ^2. Let $\bar{x}_r = \sum_{i=1}^{r} X_i/r$, let $Z(p)$ satisfy $(2\pi)^{-\frac{1}{2}}\int_{Z(p)}^{\infty} \exp(-t^2/2)\, dt = p$, let T denote the smallest integer $\geqq 4\sigma^2[Z(\alpha/4)]^2/w^2$, let $A = wT/2$, let $U_r = \min (1 \leqq j \leqq r)$ $[x_j + A/j]$ and $L_r = \max (1 \leqq j \leqq r)[\bar{x}_j - A/j]$. It can be shown that $P[L_r \leqq m \leqq U_r$ for all r, $1 \leqq r \leqq T] \geqq 1 - \alpha$. When X_r $(r = 1, 2, \cdots)$ represents the difference between measurements with two experimental categories, and w, α, and c are constants fixed in advance, a reasonable procedure for estimating m with confidence coefficient $1 - \alpha$ (when σ is known) is to stop the experiment and decide that $L_n < m < U_n$ as soon as either (1) $U_n - L_n \leqq w$ or (2) $L_n > 0$ and $Un - L_n < cL_n + w$ or (3) $U_n < 0$ and $U_n - L_n < -cU_n + w$. This can be extended to deal with the case when σ is unknown, and also to find simultaneous confidence limits for the means of k populations. (Received 20 October 1967.)

102. The modified compound Poisson process with normal compounding. S. JAMES PRESS, University of Chicago.

The compound Poisson process in which the compounding variables are normally distributed is considered as a special case of a slightly more general process. The model is suggested for studying the behavior of security prices in the stock market. Distributional properties of the process are developed and it is shown that the distribution of the increments is skew-symmetric, peaked, "fat tailed," and generally, multimodal. A cumulant matching method for estimating the parameters is suggested. (Received 25 October 1967.)

103. On unbiased density estimators based on one observation. JOSEPH PUTTER, University of Wisconsin.

For a complete family of probability densities $\{p(x; \theta)\}$, no estimator of p based on a single observation can be unbiased for all x in a set of positive probability. For a shift or scale family, this implies the non-existence of such an unbiased estimator for any x. In the normal case, the density of $\mathfrak{N}(\theta, \sigma^2)$ can be unbiasedly estimated using a $\mathfrak{N}(\theta, \tau^2)$ (σ^2 and τ^2 known) if and only if $\tau^2 < \sigma^2$. An application is given, in which a "selection bias" parameter cannot be unbiasedly estimated regardless of sample size. (Received 27 October 1967.)

104. A mathematical setting for the inference problem (preliminary report). CHARLES H. RANDALL, University of Massachusetts. (Introduced by G. B. OAKLAND)

The principal result of this paper is a multilevel logic system that admits both objective probabilities and related subjective probabilities. The system is constructed by initially adopting a descriptive logic Q (conventionally a countably complete Boolean algebra) of experimental propositions; each asserting that a physical operation has been carried out and in that realization a certain result was observed. States (or probabilities in the frequency sense) can then be defined on Q in a natural way. Certain subsets of the space of all such states amount to propositions concerning the statistical laws that may obtain in the described empirical universe of discourse; these form in a natural way a countably complete field of sets. As a consequence, it is possible to define the set M of all probability measures on this field and to interpret these as credibilities (or subjective probabilities). In such a setting, it can be argued that an inference procedure, or strategy, ought to correspond to a function $I:Q \times M \to M$. If an f in Q is interpreted as 'hard data' and an m in M as an '*a priori* credibility,' then the resulting $I\ (f,\ m)$ can be regarded as a 'posterior credibility.' (Received 30 October 1967.)

105. Toward assumption-free inference in time-sequence data analysis. ROBERT H. RIFFENBURGH and RICHARD H. LAVOIE, University of Connecticut.

Consider an ordered sequence of observations y_t subject to null-expectationed random error e_t where $E(y_t)$ is an unknown function, say, f_t. Suppose there is permissible only a subset, which may be null, of the common assumptions. (For example, y_t's may not occur at equal intervals and/or e_t's may not be independent and/or var (e_t) may be a function of t and/or f_t may contain jumps, etc.) Further, suppose it is desired to inductively infer whatever is possible about f_t by generalizing from the sole specific of data values rather than to more deductively infer a specific conclusion from a general assumption-set-plus-data. Developed in this paper are: (a) the general philosophy of reduced-assumption and assumption-free analyses, (b) the concept of and some methods for simultaneous testing and estimation, or "testimation," arising from a certain logical quandary, (c) the theory of one method of reduced-assumption analysis, (d) the theory for the first branches of the assumption-free analysis tree, (e) the analysis of probability behavior of a sequence of testimations, and (f) illustrations of both analyses as applied to the study of ocean temperature recordings. (Received 3 November 1967.)

106. An optimal ranking procedure for M-ordered densities. M. HASEEB RIZVI, Stanford University.

The paper presents a constructive method for obtaining a ranking procedure for M-ordered densities (defined below) and gives alternative simple proofs for some results of Hall (1959), Lehmann (1966) and Eaton (1967) in the special case of the ranking problem when one is interested in selecting $t(<k)$ largest parameters. Let $X = (X_1, \cdots, X_k)$ have density $f(x, \theta)$ and let $\theta_{[1]} \leqq \cdots \leqq \theta_{[k]}$ denote the ordered values of the components of the parameter $\theta = (\theta_1, \cdots, \theta_k)$. The density $f(x, \theta)$ is defined to be *M-ordered* if for each $i, j\,(i \neq j)$, $x_i \geqq x_j$ and $\theta_i \geqq \theta_j$ implies that $f(x, \theta) \geqq f(x, (i, j)\theta)$ where $(i, j)\theta$ is the vector θ with the components θ_i and θ_j interchanged. A search for the Bayes decision rule with respect to the simple loss function and the prior distribution which puts equal mass at each of the $\binom{k}{t}$ points obtained by permuting the components of θ^*, where θ^* has t components all equal to $\theta_{[k-t+1]}$ and the rest all equal to $\theta_{[k-t]}$, leads to the natural ranking procedure ϕ^* based on X as the solution, provided $f(x, \theta)$ is M-ordered. Furthermore, if the risk of ϕ^* is maximized at every permutation of θ^*, the procedure ϕ^* is both minimax and admissible. This in turn implies that ϕ^* is most economical in the usual indifference-zone formulation of the ranking problem. (Received 7 November 1967.)

107. Bayesian estimation of mixing distributions. JOHN E. ROLPH, Columbia University.

Let $\mathcal{Q}$ be a parameterized family of probability distributions and $X_1, \cdots, X_n$ be integer valued independent identically distributed random variables whose distributions are mixtures over $\mathfrak{Q}$. Thus for $(q_1(t), q_2(t), \cdots) = Q(t)$ in $\mathfrak{Q}$ and t in the unit interval, $P(X = x \mid t) = q_x(t); x = 1, 2, \cdots$. If G is a distribution function on [0, 1], the distribution of X is a G mixture over $\mathfrak{Q}$ if $P_G(X = x) = \int_0^1 q_x(t)\, dG(t); x = 1, 2, \cdots$. The family $\mathfrak{Q}$ is assumed to be identifiable, so that $P_G = P_{G'}$ implies $G = G'$. The problem is to estimate G from $X_1, \cdots, X_n$. The method is to put a prior distribution on the space of mixing distributions via their moments and thus construct Bayes estimates of G. The posterior distribution is calculated and its consistency proven. Consistent estimates of G based on the posterior distribution, but slightly different than the usual Bayes estimates are proposed. The estimates are extended to cover the case where $q_x(t)$ need only be a positive continuous function of t on [0, 1]. The results hold for a fairly rich class of prior distributions. (Received 27 October 1967.)

108. The existence of certain stopping times on Brownian motion. DAVID ROOT, University of Washington.

Let Ω be the space of continuous functions $\omega:[0, \infty) \rightarrow (-\infty, +\infty)$ such that $\omega(0) = 0$. Let $X_t(\omega) = \omega(t)$ be Brownian motion on Ω. For a stopping time τ define $X_\tau(\omega) = \omega(\tau(\omega))$. THEOREM: *For every* rv X *such that* $E(X) = 0$, $E(X^2) < \infty$ *there exists a* τ *such that* X *and* X_τ *are equal in law. Moreover* $E(\tau^n) < \infty$ *if and only if* $E(X^{2n}) < \infty$. τ is a stopping time with respect to the increasing σ-fields $\mathfrak{B}_t = \sigma(X_s, s \leqq t)$ and does not depend upon an auxillary rv as do those in a similar theorem of Skorohod. (Skorohod (1965) *Studies in the Theory of Random Processes*, Chap. 7). This theorem can be applied to construct families of random walks on Ω whose sample paths will almost surely converge uniformly on bounded time intervals to the sample paths of X_t. (Received 21 November 1967.)

109. Non-discounted denumerable Markovian decision models. SHELDON M. ROSS, Stanford University.

Countable state, finite action Markovian decision processes are studied under the average cost criterion. The problem is studied by using the known results for the discounted-cost problem. Sufficient conditions are given for the existence of an optimal rule which is of the stationary deterministic type. This rule is shown to be, in some sense, a limit point of the optimal discounted-cost rules. Sufficient conditions are also given for the optimal discounted-cost rules to be ϵ-optimal with respect to the average cost criterion. It is shown that if there is a replacement action then there exists an optimal rule but it may not be of the stationary deterministic type. It is also shown how, in a special case, the average cost criterion can be reduced to the discounted cost criterion. (Received 16 October 1967.)

110. Nonparametric estimation of the transition distribution function of a Markov process. GEORGE G. ROUSSAS, University of Wisconsin.

Let $\{X_n\}$, $n = 1, 2, \cdots$, be a (strictly) stationary Markov process having initial, 2-dimensional, and transition densities denoted by p, q, and t, respectively. Let K be a probability density on the real line, R, into itself, and let $\{h(n)\} = \{h\}$ be a sequence of positive numbers converging to zero, as $n \rightarrow \infty$. Set: $p_n(x) = (nh)^{-1}\sum_{j=1}^n K[(x - X_j)h^{-1}]$, $q_n(y) = (nh^2)^{-1}\sum_{j=1}^n K[x - X_j)h^{-1}]K[(x - X_{j+1})h^{-1}]$, and $t_n(x' \mid x) = q_n(y)/p_n(x)$, where x, $x' \varepsilon R$, $y = (x, x')$. Furthermore, let F be the initial distribution function of the process and $G(\cdot \mid x)$, $x \varepsilon R$ be its transition distribution function. Define F_n and $G_n(\cdot \mid x)$ by: $F_n(x) = \int_{-\infty}^x p_n(z)\, dz$, $G_n(z \mid x) = \int_{-\infty}^z t_n(dx' \mid x)$, $x \varepsilon R$. Then, under suitable conditions on the process, the density K, and the sequence $\{h(n)\}$, the following theorems are proved. THEOREM 1. (Glivenko-Cantelli). sup $\{|F_n(x) - F(x)|; x \varepsilon R\} \rightarrow 0$ a.s., *as* $n \rightarrow \infty$. THEOREM 2. *For* $x \varepsilon R$, sup $\{|G_n(z \mid x) - G(z \mid x)|; z \varepsilon R\} \rightarrow 0$ *in probability, as* $n \rightarrow \infty$. THEOREM 3. *For* $x \varepsilon R$ *and some positive integer* r, *let* $m(r; x) = E(X_2^r \mid X_1 = x) = \int_{-\infty}^{\infty} z^r G(dz \mid x)$, *and set* $m_n^*(r; x) = \int_{-\infty}^{\infty} z^r t_n(dz \mid x)$ *and* $m_n(r; x) = (nh)^{-1}p_n^{-1}(x)\sum_{j=1}^n X_{j+1}^r K[(x - X_j)h^{-1}]$.

Then for $k = 1, \cdots, r$, *we have*: (i) $m_n^*(k, x) - m_n(k; x) \to 0$ *in probability; as* $n \to \infty$. (ii) $m_n(k; x) \to m(k; x)$ *in probability, as* $n \to \infty$. THEOREM 4. *For* $x \varepsilon R$, *let* $\xi(p, x) = \xi$ *be the* pth *quantile* $(0 < p < 1)$ *of* $G(\cdot \mid x)$ *and define* $\xi_n(p, x)$ *to be the "smallest" root of*: $G_n(z \mid x) = p$. *Then* $\xi_n(p, x) \to \xi(p, x)$ *in probability, as* $n \to \infty$. Finally, THEOREM 5. *For* $x \varepsilon R$, *we have*: $(nh)^{\frac{1}{2}}[\xi_n(p, x) - \xi(p, x)]$ *converges in law to* $N(0, \tau^2(\xi, x))$, *as* $n \to \infty$, *where* $\tau^2(\xi, x) = t^{-2}(\xi \mid x)G^2(\xi \mid x)\int_{-\infty}^{\infty} K^2(z)\, dz$. (Received 20 October 1967.)

111. An old approach to finite population sampling theory. RICHARD M. ROYALL, The Johns Hopkins University.

The objective of this work is to relate classical inference and sampling theory (and current sampling practice) to recent advances toward formalizing and unifying the theory of sampling from finite populations. It is pointed out that adherence to the traditional definitions of simple random sampling, stratified sampling, and post-stratification (stratification after sampling) leads to the solution of such riddles which appear in the more recent theory as the failure of the "likelihood principle" to sanction any non-trivial inference whatever. In particular, a general version of the "likelihood principle" is shown to support new estimates for the population mean when the three aforementioned sampling plans are employed. Completeness theorems are proved which imply that in the most frequently encountered sampling problems conventional unbiased estimators of population parameters have the happy property of being the unique unbiased estimators. (Received 1 November 1967.)

112. More support for a conjecture. S. M. SAMUELS, University of California, Santa Cruz.

Extensions of the results in my paper "On a Chebyshev-type inequality for sums of independent random variables" *Ann. Math. Stat.* **36** (1966) 1272–1278 are presented utilizing a new method first described at the Central Regional Meeting March 23–25, 1967. Among the extensions presented are a proof of the conjecture for $n = 4$ and a new proof of Theorem 5.1, one which hopefully can be generalized. (Received 23 October 1967.)

113. Nonparametric two-sample tests for circular distributions. SIEGFRIED SCHACH, Stanford University.

Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be independent samples from continuous circular distributions. A satisfactory definition of the rank of X_i cannot be given, since it would depend on an arbitrary cut-off point. But an analogue of a rank order statistic can be defined by requiring that such a statistic T_n be invariant under the group of all homeomorphic mappings of the circle onto itself. Statistics invariant under this group are nonparametric. The locally most powerful invariant test against rotation alternatives has a critical region of the form $T_n = \sum_{j=1}^{n} \sum_{i=1}^{n} h_n(R_i - R_j) > c$, where $h_n(\cdot)$ is symmetric with respect to 0 and periodic with period $2n$, and R_i is the rank of ith observation of the X-sample, defined with respect to an arbitrary but fixed cut-off point. The large sample distribution of T_n is obtained under the hypothesis of equality as well as under rotation alternatives. Using Bahadur's (*Ann. Math. Statist.* **31** 276–295) concept of approximate relative efficiency, it can be shown that under certain regularity conditions an asymptotically efficient invariant test exists. (Received 1 November 1967.)

114. Probabilities of conditional events (preliminary report). GEZA SCHAY, JR., University of Massachusetts.

In probability theory conditional probabilities of events are defined, rather than probabilities of conditional events, although one speaks of the latter and uses the notation

$P(A \mid B)$ as if they were defined. For any two sets A and B in a given Boolean algebra of sets the author defined the symbol $A \mid B$ as the indicator function of A restricted to B, and gave a natural structure to the set of all such functions (to be published; abstracted in *Amer. Math. Soc.* Notices 14, Oct. 1967 p. 818). This algebra of conditional events conforms to the intuitive notion as used loosely by probabilists. In this paper probabilities are defined on this algebra by the usual axioms $P(A \mid B) \geqq 0$, $P(A \mid A) = 1$ if $A \neq \emptyset$, and $P(\cup A_i \mid B) = \sum P(A_i \mid B)$ if the A_i are pairwise disjoint. If the fourth axiom $P(A \cap B \mid C) = P(A \mid B \cap C)P(B \mid C)$ is added then we obtain the classical situation, although with slight extensions. Examples can be given, however, in which it is desirable that the fourth axiom should not hold. Thus we examine some of the consequences of dropping it. (Received 6 November 1967.)

115. Non-parametric estimation of a regression function. EUGENE F. SCHUSTER, University of Arizona.

$(X_1, Y_1), \cdots, (X_n, Y_n), \cdots$ are independent samples from a continuous bivariate population with pdf f. g_1 is the pdf of a univariate normal and g_2 is the pdf of a non-singular bivariate normal. $\{a_n\}$, $\{b_n\}$ and $\{k_n\}$ are sequences of positive numbers tending to 0, 0 and ∞ respectively such that $a_n/k_n = n^{-\frac{1}{2}+\delta}$ for some $\delta > 0$. $f_n(x) = \sum_{i=1}^n g_1((x - X_i)/a_n)/na_n$ and $f_n(x, y) = \sum_{i=1}^n I_n(Y_i)g_2((x - X_i)/a_n, (y - Y_i)/a_nb_n\sum_{i=1}^n I_n(Y_i)$ if $\sum_{i=1}^n I_n(U_i) > 0$ and $g_2(x, y)$ if $\sum_{i=1}^n I_n(Y_i) = 0$ where I_n is the indicator function of $[-k_n, k_n]$. $m_n(x) = \int_{-\infty}^{\infty} yf_n(x, y)\, dy/f_n(x)$. The following convergence property of $m_n(x)$ is proved: If there exists an open interval containing $[-a, a]$ on which $\int_{-\infty}^{\infty} f(x, y)\, dy$ is continuous, and bounded away from 0 and on which $\int_{-\infty}^{\infty} yf(x, y)\, dy$ is continuous and uniformly convergent, and if $E_f[Y]$ exists, then $m_n(x)$ converges to $E_f[Y \mid X = x]$ uniformly on $[-a, a]$ with probability one. (Received 19 October 1967.)

116. On the rate of uniform convergence of estimated probability density functions. EUGENE F. SCHUSTER, University of Arizona. (By title).

$X_1, X_2, \cdots, X_3, \cdots$ are independent continuous random variables with common pdf g. $\{a_n\}$ is a sequence of positive numbers converging to zero, f is a pdf with $\int_{-\infty}^{\infty} |u|f(u)\, du$ finite and $f^{(s)}$ of bounded variation for $s = 0, 1, \cdots, r$. Define $g_n(x) = \sum_{i=1}^n f((x - X_i)/a_n)/na_n$. This note establishes the following theorem: *If g and its first $r + 1$ derivatives are bounded, then with $a_n = n^{-1/(2r+4)}$ and* $0 < c < 1/(2r + 4)$, $\sup_{-\infty<x<\infty} n^c|g_n^{(r)}(x) - g^{(r)}(x)|$ *converges to* 0 a.s. This theorem extends results by Nadaraya [*Theor. Prob. Appl.* **10** (1965) 186–190] and Bhattacharya [To be published in *Sankhya Ser. A* **29** Part 4] in two different directions. (Received 19 October 1967.)

117. Estimating the parameters of a sum of two independent nonidentically distributed random variables (preliminary report). STANLEY L. SCLOVE and JOHN VAN RYZIN, Stanford University and University of Wisconsin.

We observe $Z_1, Z_2, \cdots, Z_n$, independent random variables distributed as $Z = X + Y$, where X and Y are independent and have distributions from two different identifiable families. Conditions for identifiability of the family of distributions for Z are given. The conditions are in terms of the existence of functions which can be used as are the sample moments in the method of moments. It is shown that if X has only a location parameter and Y only a scale parameter, then the family of distributions for Z is identifiable. Some examples are treated; methods of moments estimators are given for the cases in which Y is normal with mean zero (case of additive normal noise) and X is binomial or Poisson. The parameters of the asymptotic normal distributions of these estimators are computed. (Received 7 November 1967.)

118. Some estimation problems related to a geometric distribution (preliminary report). STANLEY L. SCLOVE and JOHN VAN RYZIN, Stanford University and University of Wisconson. (By title)

We observe n independent random variables distributed according to the geometric distribution on 1, 2, 3, $\cdots$. Exact expressions for the first two moments of the maximum likelihood estimator for the probability p of "heads" are obtained. This estimator has a bias of order $1/n$. The jackknife version of the maximum likelihood estimator is discussed. There is a minimum variance unbiased bounded estimator; its variance is given. Instead of observing the geometrically distributed random variable, one may observe only a linear function of it. In this case one wishes to estimate the parameters of the linear function. The method of maximum likelihood fails, and the method of moments suggests itself as a workable alternative. The estimation problem is further complicated by assuming that the observations contain additive normal noise. In this case, one also wishes to estimate the variance of the noise distribution.

Method of moments estimators are obtained for the various problems. These estimators have asymptotic normal distributions; the parameters of these distributions are computed for some of the estimators. (Received 7 November 1967.)

119. Power of the Lawley-Hotelling trace test in multivariate analysis is a function of the population trace. J. M. SRIVASTAVA, Colorado State University.

Consider a multivariate analysis of variance model with p responses. The standard tests for the general multivariate linear hypothesis are based on the characteristic roots c_1, c_2, $\cdots$, c_p of $(S_h S_e^{-1})$, where $S_h(p \times p)$ and $S_e(p \times p)$ are the sum of product matrices due to hypothesis and error, respectively. It is well known that the distribution of $c_1, \cdots, c_p$ under the non-null hypothesis involves certain noncentrality parameters, say $\gamma_1, \cdots, \gamma_t$, these being the characteristic roots of a certain matrix involving various population parameters. Lawley and Hotelling have proposed the trace test, namely one whose acceptance region is of the form $c_1 + c_2 + \cdots + c_p \leqq \mu$, where μ is a constant. In this paper, it is shown that the power of the trace-test depends on $\gamma_1, \cdots, \gamma_t$ only through the function $\gamma_1 + \cdots + \gamma_t$, which may be called the population trace. Thus it is shown that the power of the trace test depends on a *single noncentrality parameter*, namely, the population trace. (Received 13 November 1967.)

120. Balanced optimal 2^m fractional factorial designs of resolution V, $m \leqq 6$. J. N. SRIVASTAVA, and D. V. CHOPRA, Colorado State University and Wichita State University.

A balanced 2^m fractional factorial design T of resolution V is identical with a partially balanced (PB) array T with 2 symbols and strength 4. The latter is defined as a (0, 1) matrix T $(m \times N)$, (where N denotes the number of treatment-combinations in the design), such that if T_0 is *any* $(4 \times N)$ submatrix of T, and $\mathbf{u}$ is *any* vector with 4 elements (out of which i elements are nonzero, $i = 0, 1, \cdots, 4$), then $\mathbf{u}$ occurs exactly μ_i times as a column of T_0, where the μ_i are nonnegative integers not depending on the chosen T_0. Given any design T, we assume that interest lies in the estimation of the general mean, main effects and 2-factor interactions, under the assumption that the higher order interactions are zero. Let $(V)_T$

be the variance matrix of the usual estimate of the parameter vector under the design T. Then we want to choose T such that (i) T is balanced, i.e. $(V)_T$ is invariant under a permutation of factor symbols, and (ii) tr $(V)_T$ is a minimum. In this paper, for $4 \leqq m \leqq 6$, and various practical values of N arrays satisfying (i) and (ii) are obtained. The parameters $\mathbf{u}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ of these arrays are given below: (i) $m = 4$: (1, 0, 1, 1, 0), (10111), (20111), (01110), (11110), (11111), (21111), (21112), (31112), (12111), (12112), (21122), (12120), (12121), (12122), (22122), (22211), (22212); (ii) $m = 5$: (11111), (21111), (21112), (31112), (31113), (22111), (22112), (22113), (12121), (12122), (22122), (12212), (22212), (32212), (12221), (22221), (22222); (iii) $m = 6$: (31121), (21123), (31123), (32114), (32115), (22211), (22122), (32212), (32213), (22221), (22222), (32222), (32223), (32224), (42224), (33222), (33223), (43223), and (43224); the actual arrays will be given in the paper. (Received 6 November 1967.)

121. On a class of non-parametric tests for regression parameters. M. S. Srivastava, University of Toronto.

For testing the hypothesis $H:\beta_1 = \beta_2 = \cdots = \beta_p = 0$ in the linear regression model $Y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots \beta_p x_{pi} + \epsilon_i$, a class of rank score tests is proposed under weaker conditions than Hájek (1962) and Adichie (1967). The limiting distribution of the test statistics is shown to be central χ_p^2, under H, and non-central χ_p^2, under a sequence of alternatives tending to the hypothesis at a suitable rate. The Pitman efficiency of the proposed tests, relative to the classical F-test, is proved to be the same as the efficiency of the corresponding rank score tests relative to the t-test in the two sample problem. (Received 1 November 1967.)

122. Linear functions of order statistics. Stephen M. Stigler, University of Wisconsin.

Conditions are found on the weights and underlying distribution under which a linear combination of order statistics has a limiting normal distribution. The approach used is to approximate the linear combination of order statistics in mean square by a sum of independent random variables, show that the difference tends to zero in mean square, and prove asymptotic normality for the sum of independent variables. In the course of the investigation, asymptotic expressions are derived for the investigation, asymptotic expressions are derived for the covariance of two order statistics and for the variance of a linear combination of order statistics which strengthen previous results. (Received 20 October 1967.)

123. Comparison of Rao-Hartley-Cochran estimate in unequal probability sampling with ratio estimate and its application to sampling on successive occasions. B. V. Sukhatme and M. S. Avadhani, Iowa State University and Indian Society of Agricultural Statistics. (Introduced by T. A. Bancroft.)

The paper discusses the problem of estimating the population mean and considers the estimate proposed by Rao, Hartley and Cochran [*J. Roy. Statist. Soc. Ser. B* (1962)] when the units are selected with unequal probabilities and without replacement and the usual ratio estimate in simple random sampling. A finite population model is proposed and conditions given under which the ratio estimate is more efficient than the estimate in the case of sampling with unequal probabilities and without replacement. The paper then considers the problem of sampling on successive occasions and discusses the relative merits of the corresponding estimators in successive sampling both from the point of view of applicability and efficiency. (Received 16 October 1967.)

124. Some new conditions for the strong law. HENRY TEICHER, Columbia University.

If $\{X_n, n \geqq 1\}$ are independent random variables with $EX_n = 0$, $EX_n^2 = \sigma_n^2$, a sufficient condition for the strong law of large numbers (SLLN), i.e., $n^{-1}\sum_{i=1}^{n} X_i \rightarrow_{a.s.} 0$ is the convergence of the series $\sum_{n=1}^{\infty} n^{-2}\sigma_n^2$ [Kolmogorov, *C. R. Acad. Sci.* **191** (1930) 910–912]. The Kolmogorov series may be envisaged as the first in an infinite hierarchy of series of decreasing severity, the convergence of any one of which, in conjunction with two other conditions (automatically fulfilled in the Kolmogorov case) implies the SLLN. In certain cases, e.g., $\sigma_n^2 = n (\log n)^{-1}$, the resulting theorems permit a relaxation of the Prohorov condition $|X_n| = o(n/\log\log n)$ for the SLLN [Prohorov, *Izvest. Akad. Nauk SSSR* **14** (1950) 523–536.] (Received 25 October 1967.)

125. Some properties of countable spaces with applications. B. J. TRAWINSKI, University of Alabama. (By title.)

Let 0 be a subset of a countable set C in E^k consisting of elements $y \varepsilon C$ with ordered coordinates, and let $y_{r+1} - y_r = \delta_r$ $(\geqq 0)$. An element $y \varepsilon C$ is said to be *left* [*right*] *reflexive* to $y \varepsilon C$ and of order $k - |\tau|$ $(\tau = 0, \pm 1, \pm 2, \cdots, \pm(k-3))$, if $\tau \leqq 0$ $[\tau > 0]$ and $\delta_r = \bar{\delta}_{k+\tau-r}$ for $r = 1, 2, \cdots, k + \tau - 1$ $[r = \tau + 1, \cdots, k - 1]$, provided $y_r = \bar{y}_r$ for $r = k + \tau + 1, \cdots, k$ $[r = 1, 2, \cdots, \tau]$; a self, left [right] reflexive element $y \varepsilon C$ is *left* [*right*] *symmetric*. These definitions (modified when appropriate) lead to the establishment of: (i) equivalence classes in C which are related to number theoretic properties of the elements in 0; (ii) dual subspaces in $E^k \cap C$ which are also maps of spaces of similar structure in lower dimension $(<k)$. Applications are directed mainly to design of experiments employing rankings and to the study of relationships between points in sample spaces and properties of associated graphs (not necessarily antisymmetric), their paths and circuits. (Received 6 November 1967.)

126. Admissibility and distribution of some probabilistic functions of discrete finite state Markov chains. CHIA KUEI TSAO, Wayne State University.

Let $\mathfrak{X}_s = \{z_1, \cdots, z_s\}$ be a finite state space consisting of s states $z_i = (\delta_{i1}, \cdots, \delta_{is})$, where δ_{ij} is the Kronecker delta. Let $Z_0, Z_1, \cdots, Z_T$, be a Markov chain defined on $\mathfrak{X}_s$: $Z_t = (Z_{t1}, \cdots, Z_{ts})$. (1) Let $2 \leqq h \leqq s$, and, for each u $(u = 1, \cdots, s)$ and each j $(j = 1, \cdots, h)$, let $0 = M_{u0} < M_{u1} < \cdots < M_{uh} = s$ and $Y_{ujt} = \sum_v Z_{t-1u} Z_{ta(u,v)}$, where $a(u, 1), \cdots, a(u, s)$ represent a permutation of $1, \cdots, s$ and the summation $\sum$ is taken over $v = M_{u(j-1)} + 1, \cdots, M_{uj}$. Let $X_1, \cdots, X_T$ be T vector-valued probabilistic functions defined by $X_t = (X_{t1}, \cdots, X_{th})$, $X_{tj} = Y_{1jt} + \cdots + Y_{sjt}$. If $E(Y_{ujt})/E(Z_{t-1u}) = p_j > 0$, then $Z_1, \cdots, X_T$ are independently, identically distributed multinomial random vectors with the common distribution $(p_1, \cdots, p_h)$. (2) For the case $h = 2$, under suitable conditions, the binomial random vector $X_1 + \cdots + X_T$ is a UMP unbiased test statistic for testing hypotheses about p_1. Furthermore, the well-known "sign" test criterion and the "total number of runs" can be considered as special cases of this statistic. (Received 25 October 1967.)

127. Some tests of independence for stationary multivariate time series. GRACE WAHBA, University of Wisconsin.

Let $X(t)$, $t = \cdots -1, 0, 1, \cdots 1$, be a P dimensional stationary Gaussian time series, $X(t) = (X, (t), X_2(t), \cdots X_P(t)$, with strictly positive definite spectral density matrix

$F(\omega)$ whose determinant is bounded over ω. Tests for the following types of independence, based on observing a record of length T, are considered: (1) $X_i(t)$ independent of $X_j(s)$, all $i \neq j$, all s, t. (2) $X_1(t)$ independent of $X_j(s), j = 2, 3, \cdots P$, all s, t. (3) $X_1(t)$ independent of $X_1(s)$, all $s \neq t$. In cases (1) and (2) approximate likelihood ratio test statistics are derived by a multivariate circularization of the process. Exact null and asymptotic alternative densities are given for the circularized process. The statistics are complex analogues of statistics commonly occurring in ordinary multivariate analysis. An approximate likelihood ratio test statistic for case (3) follows from a univariate circularization of the process and is equivalent to Bartlett's test for homogeneity of variances. A series expression for the characteristic function under nearby alternatives is given, together with exact (series) and approximate expressions for the first two moments. (Received 7 November 1967.)

128. An application of optimal stopping rules to certain problems of allocation of resources. M. E. WALKER, G. E. SWINSON and P. H. RANDOLPH, Braddock, Dunn and McDonald and New Mexico Srate University. (By Title.)

In this paper a nonlinear programming model is presented to solve a rather wide class of problems in the optimum allocation of a fixed stockpile of resources. By the application of successive approximations in dynamic programming a relative optimum payoff is obtained. If the initial allocation is chosen randomly, a set of relative optima will be observed, one of which may be the absolute maximum. By means of a Baysian stopping rule in which the cost of sampling is weighed against the improvement in payoff, a rule is obtained for determining the optimal point at which to stop sampling and to choose the maximum payoff observed to that point. (Received 20 November 1967.)

129. A characterization for the linear exponential family. J. K. WANI, Saint Mary's University.

The probability density function of the exponential family can be expressed by $p(x) = a(x) \exp [\sum_1^s \omega_j B_j(x)]/f(\omega)$ where ω stands for the s-vector $(\omega_1, \omega_2, \cdots, \omega_s)$. If we take the exponent in the above expression to be linear in the random variable we get $p(x) = a(x) \exp (\omega x)/f(\omega)$ where x and ω are scalars or s-vectors depending upon whether the distribution is univariate or multivariate. The last form of the probability density function is called the linear exponential family. If $m_r(\omega)$ stands for the rth crude moment of the linear exponential family then under certain regularity conditions it is shown that $m_{r+1}(\omega)f(\omega) = (d/d\omega)[m_r(\omega)f(\omega)]$ for $r = 0, 1, 2, \cdots$, characterises the linear exponential family. Also if $f(\omega)$ is known the exact distribution can be written down. This is illustrated with the examples of the normal, the binomial, the Poisson and the logarithmic series distributions. The multivariate extension of the result follows by considering the relation $m_{r_1, \cdots, r_i+1, \cdots, r_s}(\omega)f(\omega) = (\partial/\partial\omega_i)[m_{r_1, \cdots, r_i, \cdots, r_s}(\omega)f(\omega)]$. (Received 30 October 1967.)

130. Sum of lifetimes in age dependent branching processes (preliminary report). H. WEINER, University of California, Davis.

In a Bellman-Harris age-dependent branching process, let $G(t)$ be the cell lifetime distribution, and let k cells be born to a parent cell with probability p_k. Let $h(s) = \sum_{k=0}^{\infty} p_k s^k$, and $h'(1) \equiv m$, and $h''(1) < \infty$. Let $Z_1(t)$ and $Z_2(t)$ denote the sum of lifetimes of cells alive at t and of all cells born by t, respectively. For $m > 1$, $E[Z_i(t)] \sim n_i \exp(\alpha t)$, $i = 1, 2$, as $t \to \infty$, where $\alpha : m\int_0^{\infty} \exp(-\alpha u)\, dG(u) = 1$. The variables $W_i(t) \equiv Z_i(t)/E[Z_i(t)]$ converge in quadratic mean to the same Bellman-Harris random variable W. Monotonicity of $E[Z_i(t)] \exp(-\alpha t)$ is discussed. For $m = 1$, and all moments of G exist, the conditional distribution of $Z_1(t)/t$ given that the population is alive at t approaches an exponential law. $E[Z_2^n(t)] \sim c_n t^{2n-1}$, where the c_n are characterized. For $m < 1$, $Z_2(t) \uparrow Z_2$ and if $h^{(n)}(1) < \infty$ for all n, the moments of Z_2 may be recursively obtained. Similar results hold for increasing functionals of cell lifetimes and to a sister cell correlation model, with application to cell secretion determinations. (Received 16 October 1967.)

131. On choosing a delta-sequence (preliminary report). MICHAEL WOODROOFE, Carnegie-Mellon University.

One way to estimate the density, f, from which a random sample, $X_1, \cdots, X_n$, has been drawn is by $f_n(x; h) = (1/h)\int K((x - y)/h)\, dF_n(y)$ where K is a suitable "kernel" or "window," F_n is the sample distribution function, and h is small. The optimal choice, h^0, of h (in the sense of minimizing the mean square error, $E((f_n(x; h) - f(x))^2))$ depends on the smoothness of f near x and is therefore unknown to the statistician. In order to attain asymptotically the minimum mean square error, estimates of the following type are considered: first form $f_n(x; h_1)$ and $f_n^{(r)}(x; h_2)$, the rth derivative of f_n with respect to x, where h_1 and h_2 are specified in advance; next estimate h^0 by $\hat{h} = \hat{h}(f_n, f_n^{(r)})$; and finally estimate $f(x)$ by $f_n(x; \hat{h})$. It is shown that $E((f_n(x; \hat{h}) - f(x))^2) \sim E((f_n(x; h^0) - f(x))^2)$ as $n \to \infty$ under quite general assumptions on the behavior of f near x. (Received 1 November 1967.)

(Abstract of a paper presented at the East Lansing meeting, Michigan, March 18–20, 1968. Additional abstracts will appear in the June issue.)

1. Asymptotically nearly efficient estimators of multivariate location parameters. DAVID S. MOORE, Purdue University.

Ogawa (*Osaka Math. J.* **3** (1951) 175–213) found the asymptotically best linear unbiased estimator (ABLUE) for estimation of a univariate location parameter from a chosen set of sample quantiles. This estimator is asymptotically nearly efficient (ANE), i.e., it approaches asymptotic efficiency as larger sets of more closely spaced quantiles are chosen for use. We give three ANE estimators for multivariate location parameters. The first two are linear and represent multivariate analogs of Ogawa's estimators. (1) Choose a set of marginal sample quantiles in each direction from a continuous k-variate location parameter distribution. These quantiles form a random partition of k-space and for $k > 1$ the observed cell frequencies contain additional information. The ABLUE's in terms of the chosen quantiles and the cell frequencies are ANE. (2) Simpler linear ANE estimators are obtained by choosing a single sample quantile in each direction, partitioning k-space by marking off fixed distances from these, and proceeding as in (1). (3) ANE estimators can be obtained from RBAN estimators for a sequence of multinomial problems related to the given location parameter family. (Received 9 January 1968.)

(Abstracts of papers to be presented at the Eastern Regional meeting, Blacksburg, Virginia, April 8–10, 1968. Additional abstracts will appear in future issues.)

1. Exact confidence intervals for the scale parameter of the exponential distribution based on optimally chosen order statistics (preliminary report). KENNETH S. KAMINSKY, Rutgers-The State University.

Let $X' = (X_1, X_2, \cdots, X_n)$ denote an ordered random sample from the exponential distribution $f(x) = (1/\beta) \exp[-(x - \alpha)/\beta]$, $x > \alpha, \beta > 0$, and let $1 \leqq k \leqq n$, k an integer. BLUE's for α and β, based on the optimum choice of k order statistics, have been found by

Harter [*Ann. Math. Statist.* **32** (1961) 1078–1090] and Kulldorff [*Ann. Math. Statist.* **34** (1963) 1419–1431], (based on complete samples), and by Saleh [*Technometrics* **9** (1967) 279–292], (based on samples Type II censored on the right), and others. In this paper, the author finds exact confidence intervals for the scale parameter, β, (with α known [unknown]), for $k = 1(1)5$, $n = k(1)100$, based on the distribution of the BLUE for β (see for example Likeš [*J. Amer. Statist. Assoc.* **62** (1967) 259–271]), for complete samples and samples Type II censored on the right. These intervals are compared with the corresponding intervals obtained from complete samples (or the uncensored portion of the Type II censored sample, whichever is appropriate), based on the ratio of expected length and the ratio of expected squared length. Attained confidence coefficients of the large sample confidence intervals for β based on the ABLUE's for β are also obtained for $k = 1(1)5$, $n = k(1)100$ (see, for example Sarhan and Greenberg [*Contributions to Order Statistics*, Wiley, New York, pp. 380, 381]). (Received 20 December 1967.)

2. On the distribution of the maximum of a semi-Markov process. LAWRENCE D. STONE, DANIEL H. WAGNER, Associates, Paoli.

This paper deals with the problem of finding the distribution of $z(t) = \sup [x(s):0 \leqq s \leqq t]$ for a wide class of semi-Markov processes $\{x(t):t \geqq 0\}$. Semi-Markov processes are analyzed by considering the time and place of the first jump in order to obtain the main result which is a recurrence relation for the distribution of $z(t)$. Define $m_{ij}(s)$ to be the Laplace transform of $P[Z(t) = j \mid X(0) = i]$ and let $m(s) = (m_{ij}(s))$. The recurrence relation which is obtained for $m(s)$ is $m(s)=g(s) + (q(s)m(s))^\sigma$ where $q(s)$ and $g(s)$ are matrices whose elements are Laplace transforms of distributions which occur in the definition of the semi-Markov process and σ is an operator on matrices. Under a condition on the matrix $q(s)$ which guarantees that the process makes only a finite number of jumps in any finite interval of time, $m(s)$ is the unique solution of the above equation. In the case of spatial homogeneity, it is possible to solve the recurrence relation by a Wiener-Hopf type factorization. (Received 17 January 1968.)

(Abstract of a paper to be presented at the European meeting, Amsterdam, Netherlands, September 2–7, 1968. Additional abstracts will appear in future issues.)

1. A numerical procedure for a general class of Markov-processes with discrete time parameter and dependent increments. P. VAN DER LAAN and R. P. ADRIAANSE, N.V. Philips I.S.A.R., Eindhoven.

We consider a class of Markov-processes with discrete time parameter and dependent increments $\{X_n ; n = 0, 1, 2, \cdots\}$ where the transitions from one state to another are determined by $\Pr [X_{n+1} = h_j(x) \mid X_n = x] = p_j$ $(j = 1, 2, \cdots, k;\ \sum_{j=1}^k p_j = 1,\ p_j > 0)$ with $-\infty < a \leqq X_0 \leqq b < \infty$. The functions $h_j(\cdot)$ are strictly increasing functions on the interval $[a, b]$. There are the following assumptions: (i) $h_j(a) \geqq a$ and $h_j(b) \leqq b$ for all j (ii) for each $x(a < x < b)$ there exists at least one j for which $h_j(x) > x$ and at least one j for which $h_j(x) < x$. Defining the cdf $F_n(x) = \Pr [X_n \leqq x]$, it follows from the model that these functions have to satisfy the following functional equation $(n = 0, 1, 2, \cdots)$:

$$\begin{aligned} F_{n+1}(x) &= 0 && \text{for } x < a \\ &= \textstyle\sum_{j=1}^k p_j F_n(h_j^{-1}(x)) && \text{for } a \leqq x < b \\ &= 1 && \text{for } b \leqq x \end{aligned}$$

(with some special conditions on the inverse functions $h_j^{-1}(\cdot)$) where $F_0(\cdot)$ is an arbitrary initial cdf with $F_0(x) = 0$ for $x < a$ and $= 1$ for $x \geqq b$. A numerical procedure is given bv which two non-decreasing functions $G_{n_0}(x)$ and $H_{n_0}(x)$ can be constructed for given $n_0(n_0 \geqq 0)$, such that for all $x:0 \leqq H_{n_0}(x) \leqq F_n(x) \leqq G_{n_0}(x) \leqq 1$ for all $n \geqq n_0$. Using these inequalities upper and lower bounds for percentiles of $F_n(x)(n \geqq n_0)$ can be determined. Some numerical examples are given from which can be seen that the upper and lower bounds for $F_n(x)$ can lie very close together. Next an application, determination of an alarm-level in a flame failure control system, has been discussed. (Received 9 January 1968.)

(Abstracts of papers not connected with any meeting of the Institute.)

1. A sequential rank test for some k-sample slippage problems (preliminary report). CHARLES E. ANTONIAK, Naval Electronics Laboratory Center and University of California, Los Angeles.

Let $X_1, X_2, \cdots, X_k$ be independent random variables with continuous distribution functions $F_1(x), F_2(x), \cdots, F_k(x)$. Consider the problem of testing the null hypothesis that the X_i are identically distributed, against the alternative that exactly one unspecified X_j has a different distribution. A sample of size one is taken from each random variable, and the observations are ranked and then replaced by their ranks. Under the null hypothesis there are $k!$ different, equally likely k-tuples that can occur, and the observation from any particular X_i is equally likely to have any of the ranks 1 through k. Under the alternative hypothesis, we assume that a sample from the random variable with the different distribution will have a non-uniform distribution of ranks. A Wald sequential probability ratio test is applied to successive k-tuples generated in this way, to test these hypotheses. For cases where specific assumptions can be made about the distributions involved, the exact formula for the likelihood ratio to be used in the Wald test is derived. In particular, for normal slippage alternatives and for certain Lehman alternatives, the robustness of the rank test is compared with that of its parametric counterpart, and their relative efficiencies are studied. The test is shown to be distribution-free under very general conditions, and a sequentially adaptive version is derived for situations where no assumptions can be made about the distributions. (Received 2 December 1967.)

2. Approximation to Bayes risk in sequences of non-finite games. DENNIS C. GILLILAND, Michigan State University.

We consider a two-person game where player I chooses $\epsilon \varepsilon M$ and player II chooses $\delta \varepsilon N$ with loss $L(\epsilon, \delta) \geqq 0$. The Bayes envelope is defined by $R(P) = \inf \{\int L(\epsilon, \delta)P\cdot(d\epsilon) \mid \delta \varepsilon N\}$ where P is a probability measure on M. Suppose that this game occurs repeatedly, ϵ_i represents player I's move at the ith stage, and G_{i-1}, the empirical distribution of $\epsilon_1, \cdots, \epsilon_{i-1}$, is known to player II before he makes his move at the ith stage, $i \geqq 2$. In this paper sequence strategies $\boldsymbol{\delta} = (\delta_1, \delta_2, \cdots)$ for player II are given, where δ_i depends upon G_{i-1}, $i \geqq 2$, and some artificial randomization, such that, $n^{-1} \sum_1^n E(L(\epsilon_i, \delta_i)) - R(G_n) \rightarrow 0$ as $n \rightarrow \infty$ uniformly in ε under certain compactness conditions on the component game. The results provide a generalization of some of the finite M results of Hannan [*Contributions to the Theory of Games* **3** (1957) 97–139]. The sequence strategies are analogous to those developed by Hannan for the finite M case. (Received November 1967.)

3. Sequential compound estimation for squared error loss and certain exponential families. DENNIS C. GILLILAND, Michigan State University.

Consider a statistical decision problem where $X \sim P_\theta$, $\theta \varepsilon \Omega$, and an action $A \varepsilon \mathfrak{A}$ is to be taken with A allowed to depend upon the random variable X. A loss function $L \geqq 0$ is defined on $\Omega \times \mathcal{A}$, and a non-randomized procedure φ incurs a risk $R(\theta, \varphi) = \int L(\theta, \varphi)\, dP_\theta$. Suppose this decision problem occurs n times with $\boldsymbol{\theta}_n = (\theta_1, \cdots, \theta_n) \varepsilon \Omega^n$ and $\mathbf{X}_n = (X_1, \cdots, X_n) \sim P_{\theta_1} \times \cdots \times P_{\theta_n}$. A sequential compound rule $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots)$ is such that for each i, φ_i is the means by which the ith action is taken and $L(\theta, \varphi_i)$ is $\mathbf{X}_i$ measurable for each θ. The compound risk up to stage n is taken to be the average of the component risks, $R_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) = n^{-1} \sum_1^n R(\boldsymbol{\theta}_i, \varphi_i)$ where $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots)$. The modified regret $D_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) = R_n(\boldsymbol{\theta}, \boldsymbol{\varphi}) - R(G_n)$ with G_n the empirical distribution of $\boldsymbol{\theta}_n$ and $R(G)$ the Bayes risk at G in the component problem was introduced by Robbins as a standard for compound procedures. In this paper estimation is treated and it is shown that $|D_n(\boldsymbol{\theta}, \boldsymbol{\varphi})| \leqq Kn^{-\frac{1}{2}}$ for squared error loss, certain discrete exponential families including the Poisson and negative binomial, and "natural" sequential compound procedures $\boldsymbol{\varphi}$. The constant K is independent of $\boldsymbol{\theta}$ and n. The rate $O(n^{-\frac{1}{2}})$ is a significant improvement over previously established one-sided rates [Samuel, *Ann. Math. Statist.* **36** (1965) 879–889] and [Swain, Stanford Tech. Report No. 81 (1965)] and is in line with the rate $O(n^{-\frac{1}{2}})$ established for the general finite $m \times n$ decision problem by Van Ryzin [*Ann. Math. Statist.* **37** (1966) 954–975]. The rate $O(n^{1/6})$ is established for a family of $N(\theta, 1)$ distributions. (Received 1 November 1967.)

4. Inequalities for the integral of a bell-shaped function over a symmetric non-convex set. DENNIS C. GILLILAND, Michigan State University.

In what follows $x = (x_1, \cdots, x_n)$ is a generic point of Euclidean n-space R^n, T is a symmetric subset of R^n in the sense that $x \varepsilon T$ if $-x \varepsilon T$, and f is a non-negative integrable (Lebesgue) function with $\int f(x)\, dx > 0$. The problem treated in this paper is that of maximizing $\nu_a(T) = \int_T f(x - a)\, dx$ by choice of a. A solution to the maximization problem for convex T and symmetric, unimodal f follows from a theorem of Anderson [*Proc. Amer. Math. Soc.* **6** (1955) 170–176] which states that if (i) $f(x) = f(-x)$, (ii) $\{x \mid f(x) \geqq u\}$ is convex for every $u \geqq 0$, and (iii) T is convex, then $\nu_{ka}(T) \geqq \nu_a(T)$ for all $0 \leqq k \leqq 1$, $a \varepsilon R^n$. It is obvious that the convexity assumption concerning T can not be removed from the hypothesis of Anderson's theorem without adding more conditions on f and the extent of T. In this paper we obtain monotonicity results for bounded, non-convex T and smooth bell-shaped densities f. The theorems generalize the normal theory results reported by the author [*Ann. Math. Statist.* **34** (1964) 441–442] and [*Amer. Math. Monthly* **73** (1966) 713–716]. (Received 1 November 1967.)

5. On some tests concerning dispersion matrices of multivariate normal populations. SOMESH DAS GUPTA, University of Minnesota.

It has been shown that the likelihood ratio test for the hypothesis $\Sigma = \Sigma_0$, a known p.d. matrix, against the alternatives $\Sigma \neq \Sigma_0$ based on a random sample from $N_p(\mu, \Sigma)$ is not unbiased; however, the power function of Bartlett's modified likelihood ratio test has the usual monotonicity property. It is further shown that the likelihood ratio test based on random samples from $N_p(\mu, \Sigma)$ is unbiased for each of the following testing problems: (a) $H: \Sigma = \sigma^2 I_p$, σ^2 is unknown; K: not H; (b) $H: \mu = \mu_0$, $\Sigma = \Sigma_0$; K: not H. The likelihood ratio test for testing the equality of covariance matrices of two p-variate normal distributions is shown to be biased when the sample sizes are unequal but it is unbiased when the sample sizes are equal. (Received 14 December 1967.)

6. Estimation by the method of ranks in multiple linear regression. HIRA LAL KOUL, University of California, Berkeley.

In the multiple linear regression model $Y = \boldsymbol{\theta}'X + Z$ the problem of estimating the vector parameter by the method of ranks is considered. The estimator is defined as the center of gravity of a certain confidence region which is based on the test statistic M which in turn is a quadratic combination of the Wilcoxon type linear rank statistics. If the underlying distribution of Z is symmetric about 0, then the estimator is shown to be a symmetric unbiased estimator of the true parameter. Under some suitable conditions on the regression scores and the underlying distribution, the estimator is shown to be asymptotically normally distributed. Its asymptotic efficiency, when defined as an inverse ratio of the generalized limiting variances, relative to the least squares estimator, is shown to be the same as that of the corresponding test statistic relative to the classical F-statistic. Finally, a consistent estimator of the functional $\int_{-\infty}^{+\infty} f^2(x)\, dx$ is defined where f is the density of the underlying distribution. (Received 13 November 1967.)

7. Group-testing to classify all units in a multinomial sample. S. KUMAR, University of Wisconsin, Milwaukee.

The problem is to classify each of the N given units into one of the k disjoint categories by means of group testing. We shall label the k categories as 'the best', 'the 2nd-best', $\cdots$, 'the kth-best'. In group testing, a set of x units is tested simultaneously as a group with one of the k possible outcomes: At least one of the x units belongs to the ith-best category and none of the x units belongs to the jth-best category for $j > i$ $(i = 1, 2, \cdots, k)$. It is assumed that the N units can be represented by independent multinomial random variables with a known probability q_i of any unit belonging to the ith-best category for $i = 1, 2, \cdots, k$ where $q_i \geqq 0$ and $q_1 + q_2 + \cdots + q_k = 1$. The problem is to devise a procedure which minimizes the expected number of tests $E(T)$ required to classify all the N units into one of the k categories. A procedure, which is optimal in a certain class of procedures is proposed. Furthermore, we prove the following theorem: *For $k \geqq 4$ and $N \geqq 2$ the optimal group test plan among all procedures has the following properties*: (i) *if* $\sum_{j=k-1}^{k} q_j \left(\sum_{i=1}^{j} q_i\right) + \sum_{j=2}^{k-2} q_j(q_1 + q_j) > q_1^2$ *then* $ET = N$ *and the units are tested one at a time.* (ii) *if* $\sum_{i=2}^{k} q_j\left(\sum_{i=1}^{j} q_j\right) < q_1^2$, *then* $ET < N$. (Received 12 December 1967.)

8. Conditionally distribution-free tests for interactions. K. L. MEHRA, Iowa State University.

Consider for simplicity, a three-way layout linear model $X_{ijkl} = \alpha_i^A + \alpha_j^B + \alpha_k^C + \alpha_{ij}^{AB} + \alpha_{jk}^{BC} + \alpha_{ik}^{AC} + \alpha_{ijk}^{ABC} + e_{ijkl}$, with n $(\geqq 2)$ observations per cell, $l \leqq i \leqq I, l \leqq j \leqq J, 1 \leqq k \leqq K, 1 \leqq l \leqq n$, and where the errors e_{ijkl} are independently and identically distributed according to a continuous distribution F. Assuming a two-way layout additive linear model, the author had earlier investigated conditional rank-order tests for testing main effects, based on suitable alignment of observations (Stanford University, Technical Report 1967). The purpose of the present paper is to show how the arguments of the above paper can be extended to construct conditionally distribution-free tests for testing successively the hypotheses of no-interactions in two or higher way layouts. It is shown, under some very mild assumptions, that the convergence of the conditional null distribution, given a 'configuration', is again uniform in the configuration. The asymptotic efficiency of the proposed test-statistics relative to the classical $\mathfrak{F}$-test is obtained, and it is shown that the Wilcoxon and the normal-score versions of the proposed tests, besides possessing a high degree of efficiency: $e_{W,\mathfrak{F}} \geqq 3/\pi$, $e_{\text{N.S.},\mathfrak{F}} = 1$ under normality, are also fairly robust in general (relative to the $\mathfrak{F}$-test) when F is not normal. (Received 3 January 1968.)

9. Interval estimation of the largest mean of k normal populations. K. M. LAL SAXENA and YUNG LIANG TONG, University of Nebraska.

Let $\Pi_1, \Pi_2, \cdots, \Pi_k (k \geqq 1)$ be k normal populations with means μ_i $(i = 1, 2, \cdots, k)$ and a common known variance σ^2. Let $\mu_{[k]} = \max_{1 \leqq i \leqq k} \mu_i$ and we assume there is no *a priori* knowledge about $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_k)$. For preassigned constants $d > 0$ and $\alpha \varepsilon (0, 1)$, the following is a single-stage procedure for providing a confidence interval I of length $2d$ such that $P[\mu_{[k]} \varepsilon I \mid \boldsymbol{\mu}] \geqq \alpha$ for every $\boldsymbol{\mu}$: Take n observations from each one of the k populations, compute $\bar{X}_i = n^{-1} \sum_{j=1}^{n} X_{ij}$ for $i = 1, 2, \cdots, k$ and assert that $I = [\bar{X}_{[k]} - d, \bar{X}_{[k]} + d]$ where $\bar{X}_{[k]} = \max_{1 \leqq i \leqq k} \bar{X}_i$. Under this procedure (1) $\inf_{\boldsymbol{\mu}} P[\mu_{[k]} \varepsilon I \mid \boldsymbol{\mu}] = P[\mu_{[k]} \varepsilon I \mid \boldsymbol{\mu}^0] = \Phi^k(dn^{\frac{1}{2}}/\sigma) - \Phi^k(-dn^{\frac{1}{2}}/\sigma) = G(k, \sigma, n, d)$ say, where $\boldsymbol{\mu}^0$ is such $\mu_1 = \mu_2 = \cdots = \mu_k$ and $\Phi(\cdot)$ is the standard normal cdf; (2) The sample size required is the smallest integer n such that $G(k, \sigma, n, d) \geqq \alpha$. It is clear that if n is preassigned rather than d, the smallest value of d can similarly be obtained. An obvious modification of the above procedure can provide a confidence interval for the smallest mean, and can also provide a solution for the case of known but unequal variances. The authors are considering the case when the variances are not known. (Received 27 December 1967.)

10. Multivariate multisample analogues of some one sample tests. HAROLD D. SHANE, Courant Institute of Mathematical Sciences, New York University.

Consider t treatments in an experiment involving paired comparisons, and suppose that for the pair (i, j) of treatments $(1 \leqq i < j \leqq t)$, the N_{ij} encounters yield the random variables $\mathbf{X}_{ij,l} = (X_{ij,l}^{(1)}, \cdots, X_{ij,l}^{(p)})$, $l = 1, \cdots, N_{ij}$, which are independent and identically distributed according to an absolutely continuous p-variate cumulative distribution function $F_{ij}(\mathbf{x}) = F_{ij}(x_1, \cdots, x_p)$. Then for testing the hypothesis that each $F_{ij}(\mathbf{x})$ is diagonally symmetric with respect to the origin and that furthermore all the $t(t-1)/2$ cumulative distribution functions are identical, a class of nonparametric tests based on the ranks of the observations are proposed and studied. These tests are functions of the Chernoff-Savage type of test statistics $W_{ij} = (T_{ij}^{(1)}, \cdots, T_{ij}^{(p)})$, $1 \leqq i < j \leqq t$; and include among other tests, the normal scores and the rank sum analogues of the corresponding tests studied by Mehra and Puri [*Ann. Math. Statist.* **38** (1967) 523–549], for the case $p = 1$, and, Sen and Puri [*Ann. Math. Statist.* **38** (1967) 1216–1228] for the case $t = 2$; sufficient conditions are given for the joint asymptotic normality of $\{W_{ij}, 1 \leqq i < j \leqq t\}$. Under certain contiguous alternatives the proposed test statistics have limiting noncentral chi square distributions. Finally the asymptotic relative efficiencies of the proposed tests relative to some of its nonparametric as well as parametric competitors are studied. The results obtained are the generalizations of the results of Mehra and Puri, and Sen and Puri cited above. (Received 12 January 1968.)

11. On the central and non-central distribution of indefinite quadratic form in normal samples. B. K. SHAH, Yale University.

Let $y_1, y_2, \cdots, y_n$ be independently and normally distributed random variables with means ν_i $(i = 1, 2, \cdots, n)$ and variances one. And let A be any square matrix so that $y'Ay$ can be written as $X = \sum_1^{n_1} \lambda_{1i} x_i^2 - \sum_{n_1+1}^{n_2} \lambda_{2i} x_i^2$ where $n = n_1 + n_2$, λ_{1i} and λ_{2i} are all positive and x_i's are now independent and normally distributed random variable with mean μ_i $(i = 1, 2, \cdots, n)$ and variances one. Shah (*Ann. Math. Statist.* **34** 186–190) gave a very complicated result for the distribution function of the random variable X. In this paper we are showing that the distribution of the non-central indefinite quadratic form can be expressed as that of the central indefinite quadratic form except for the change in definitions of the coefficients a's and b's in Gurland's paper (*Ann. Math. Statist.* **26** 122–127, corrections in the *Ann. Math. Statist.* **33** 813). Using these results we have prepared tables for the distribution function of central and non-central indefinite quadratic form for n_1 even and for interesting values of the parameters λ_1's, λ_2's and μ's. (Received 10 November 1967.)

12. Tables of positive definite quadratic form in non-central normal variables. B. K. SHAH, Yale University.

Let $z_1, z_2, \cdots, z_n$ be independent and normally distributed random variables with means ν_i and variances one. Let A be a positive definite quadratic form. Then $z'Az$ can be written as $\sum \lambda_i y_i^2$ where λ_i's are the positive characteristic roots of the matrix A and y_i's are the independent normally distributed random variables with means μ_i. We have expressed the distribution function of $X = \sum \lambda y_i^2$ as same as that of the Gurland's result (*Ann. Math. Statist.* **26** 122-127, Corrections in the *Ann. Math. Statist.* **33** 813). Using this result we have prepared the tables of the distribution function of the random variable X for interesting values of λ's ($\sum \lambda_i = 1$) and μ's. A general computer program for the distribution function of the noncentral quadratic form is written in Fortran IV for IBM 7040–7094 series. We found some algorithms about the number of terms required to converge to any given number of significant decimals places. From this one can prepare tables for the central positive definite quadratic form by putting $\mu_i = 0$ for all i as input variables. We have checked the results (namely for the central case) with existing tables published by H. Solomon and G. Marsaglia. (Received 10 November 1967.)

13. On the cost of not knowing the variance when making a fixed-width confidence interval for the mean. GORDON SIMONS, Stanford University.

Let $X_1, X_2, \cdots$ be iid normal $N(\mu, \sigma^2)$ random variables. Let $C \equiv a^2\sigma^2/d^2$ for given positive constants a, d. Let $\alpha \equiv \int_{-a}^{a} (2\pi)^{-\frac{1}{2}} e^{-u^2/2}\, du$. Let $\bar{X}_n \equiv n^{-1} \sum_1^n X_i$, $s_n^2 \equiv (n-1)^{-1} \sum_1^n (X_i - \bar{X}_n)^2$. For fixed sample size $N \geqq C$, it is easily shown that

$$(1) \qquad P\{|\mu - \bar{X}_N| < d\} \geqq \alpha \quad \text{for all values of } \mu.$$

When σ^2 is unknown, C is unknown and sequential procedures suggest themselves. Let $\hat{C}_n \equiv a^2 s_n^2/d^2$ and define stopping variable $L \equiv$ least index $n \geqq 3$ for which $n \geqq \hat{C}_n$. We show there exists a positive integer valued constant k (depending on a but not μ, σ or d) for which (1) holds for $N \equiv L + k$. We show also that $EN \leqq C + k + 3$, and hence the cost of ignorance concerning σ^2 is a finite number of observations. Currently, some work is in progress to determine k explicitly and to examine some similar but presumably better ways of defining N. (Received 17 November 1967.)

14. A note on the distribution of quadratic forms for mixtures of normal distributions (preliminary report). K. SUBRAHMANIAN, University of Manitoba.

We consider the distribution of quadratic forms $\mathbf{x}'A\mathbf{x}$, $\mathbf{x}'B\mathbf{x}$ when $(x_1, \cdots, x_n)$ are a sample of size n from the population $p\phi_1(x; \mu_1, \sigma_1^2) + (1-p)\phi_2(x; \mu_2, \sigma_2^2)$ where ϕ_i stands for the normal pdf with mean μ_i and variance σ_i^2. Hyrenius [*Biometrika* (1950) 429–442 and *Biometrika* (1952) 238–246] has considered distribution of t and F-variables for populations of this type. We consider the distribution of quadratic forms and their ratios. Some examples will also be examined. (Received 7 November 1967.)

15. Optimal stopping of averaged Brownian motion. HOWARD M. TAYLOR, Cornell University.

Let $(B_s; s \geqq 0)$ be Brownian motion. We show there exists a stopping time S^* which maximizes over all stopping times S the expected averaged return $E[B_S/(1+S)]$. An equivalent problem is shown to be: find a stopping time T^* which maximizes over all stopping times T the expected discounted return $E[e^{-T}X_T]$ where $(X_t; t \geqq 0)$ is a Uhlenbeck process. The solution is $T^* = \inf \{t: X_t \geqq b\}$ or $S^* = \inf \{s: B_s \geqq b(1+s)^{\frac{1}{2}}\}$ where $b = 0.839 \cdots$ is the unique solution to $(1 - b^2)\Phi(b) - b\phi(b) = 0$ with Φ and ϕ the standard normal distribution and density functions, respectively. (Received 17 November 1968.)

16. The moments of log-Weibull order statistics. JOHN S. WHITE, General Motors Corp.

Let $X_{1n} \leqq X_{2n} \leqq \cdots \leqq X_{nn}$ be the order statistics of a random sample of size n. For any integrable function $g(x)$ define $F(i, n) = E(g(X_{in}))$ and $M(n) = E(1, n) = E(g(X_{1n}))$.

A number of formulae expressing $E(i, n)$ in terms of $M(j), j \leqq n$, are developed. For example, $E(i, n) = \sum_{j=0}^{j=i-1} \binom{n}{j} \Delta^j M(n - j)$. These results are applied to obtain the means and variances of the order statistics of a log-Weibull distribution ($F(x) = 1 - \exp(-\exp x)$). Tables of these means and variances are given for $1 \leqq i \leqq n$, $n = 1$ (1) 50 (5) 100. The computations were made using a set of 100 decimal place logarithms of integers. Examples of the use of these tables in obtaining weighted least squares estimates from censored samples from a Weibull distribution are also given. (Received 21 November 1967.)

17. Equivalence of a k-sample rank test for censored data and Kruskal's H*.

HANS K. URY, San Francisco Medical Computer Center, University of California and California State Department of Public Health, Berkeley.

Let X_{ij} ($j = 1, 2, \cdots, n_i$; $i = 1, 2, \cdots, k$) be k independent samples of sizes n_i from k populations with continuous cumulative distribution functions F_i. Let all the $N = \sum n_i$ observations be combined and ordered to form a single sequence and suppose that only the first r ordered observations are available. For this (right) censored sample situation of total size r (A. P. Basu, *Ann. Math. Statist.* **38** (1967) 1520–1535) proposed a statistic to test $H_0: F_1 = \cdots = F_k$ against H_1: not all F_i equal. He showed that for the no-censoring case, $r = N$, his statistic is identical with Kruskal's H-statistic. In this note it is shown that for any r, Basu's statistic is identical with the H-statistic computed under the following tie situation: a tie of size $N - r$ exists for the maximum rank and the mid-rank $(N + r + 1)/2$ is assigned to all members of this tie. Thus Basu's statistic is equivalent to the (conditional) statistic H^* (section 9 of *Ann Math. Statist.* **23** (1952) 525–540). (Received 13 November 1967.)

18. On the inadmissibility of translation and scale invariant estimators of the within component of variance in a one-way Model II ANOVA.

S. ZACKS, Kansas State University.

Consider the one-way layout Model II ANOVA, in which n-replications of r treatments are observed. Let Y_{ij} ($i = 1, \cdots, r; j = 1, \cdots, n$) be the observed random variables. The statistical model assumes that $Y_{ij} = \mu + a_i + e_{ij}$, where e_{ij} are iid like $N(0, \sigma_e^2)$ and a_i are iid like $N(0, \sigma_a^2)$, and independent of e_{ij}. The minimal sufficient statistic is $(S_e^2, S_a^2, \bar{Y})$ where: $S_e^2 = \sum_{i=1}^r \sum_{j=1}^n (Y_{ij} - \bar{Y}_{i\cdot})^2$, $\bar{Y}_{i\cdot} = \sum_{j=1}^n Y_{ij}/n$; $S_a^2 = n \sum_{i=1}^r (\bar{Y}_{i\cdot} - \bar{Y})^2$, and $\bar{Y} = \sum_{i=1}^r \sum_{j=1}^n Y_{ij}/nr$. It is simple to show that every translation and scale sufficiently invariant estimator of the within component of variance, σ_e^2, is of the form $S_e^2 f(S_a^2/S_e^2)$, where f is an appropriate measurable function. Extending a theorem of Stein we show that any estimator of σ_e^2 of the form: $W\psi(S_a^2/W, S_e^2/W)$ where $W = S_e^2 + S_a^2 + nr\bar{Y}^2$, is inadmissible with respect to the square-error loss function. In particular, all the sufficiently invariant estimators are inadmissible, and hence all translation and scale invariant estimators are inadmissible. Such a general result could not be proven for translation and scale invariant estimators of the between component of variance, σ_a^2. But, one can show that several invariant estimators in common use, such as the maximum likelihood estimator, are inadmissible. (Received 11 December 1967.)

19. A representation of Bayes invariant procedures in terms of Haar measure.

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Although fairly general conditions have been given under which minimax invariant procedures are minimax, there do not seem to be any such conditions for determining whether or not Bayes invariant procedures are Bayes. Although the answer seems to be affirmative only when the group is compact, it is shown that under fairly general conditions the Bayes invariant procedure is at least a formal Bayes procedure with respect to a prior measure constructed from the right Haar measure on the group and the specified prior. The method involved is a generalization of that used in proving the well known result that if the group operating on the parameter space does so in a simply transitive manner, the "best invariant" procedure is the formal Bayes or Bayes procedure with respect to the prior measure induced on the parameter space by the right Haar measure of the group. The result is in a useable form and several examples are given. One of these involves the two action problem and another generalizes a result given by Studden, On selecting a subset of k populations containing the best, *Ann. Math. Statist.* **38** (1967) 1072–1078. (Received 29 December 1967.)