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A SERIES OF BALANCED INCOMPLETE BLOCK DESIGNS

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1. Introduction. The available literature on BIBD, shows that the series with parameters

(1) $v = 4t, \quad r = v - 1, \quad k = 4, \quad \lambda = 3, \quad b = v(v - 1)/4,$

seem not to have been considered with the following constraint:

I(a) Each block consists of 2 pairs of elements that oppose one another.

(b) Each element appears with every other element once in a pair and twice in opposed pairs.

Our object is therefore to determine its solution with I for $v = p^n + 1$ where p denotes an odd prime in the form $12\tau + 7$ and $12\tau - 1$.

It is interesting to note that the proposed solution (see R. C. Bose [1]) is applicable to designing ordinary doubles tournaments where each player associates with every other player once as a teammate and twice as an opponent.

The application is left to the reader.

2. Design for $v = p^n + 1$ where $t = 3\tau + 2$. We identify $v - 1$ elements with elements of $GF(p^n)$ and let

(2) $$B_i = (P_i, Q_i; R_i, S_i), \qquad i = 0, 1, \cdots, t - 1,$$

generate all the blocks of a design, viz.,

(3) $$B_{0a} = (P_0 + a, v; R_0 + a, S_0 + a);$$

$$B_{ia} = (P_i + a, Q_i + a; R_i + a, S_i + a)$$

$$i = 1, 2, \cdots, t - 1, \qquad a = 0, 1, \cdots, 4t - 2.$$

This system of blocks will not produce the desired design unless 6 sets of numbers called the differences of elements in pairs and differences of elements in opposed pairs satisfy certain conditions. They are designated as:

Differences of elements in pairs:

$$P_iQ_i \text{ differences} = \pm(P_i - Q_i), \qquad R_iS_i \text{ differences} = \pm(R_i - S_i).$$

Differences of elements in opposed pairs:

$$P_iR_i \text{ differences} = \pm(P_i - R_i), \qquad P_iS_i \text{ differences} = \pm(P_i - S_i),$$

$$Q_iR_i \text{ differences} = \pm(Q_i - R_i), \qquad Q_iS_i \text{ differences} = \pm(Q_i - S_i).$$

These differences must satisfy the hypothesis of the following theorem.

THEOREM a. *A set of t initial blocks $B_0, B_1, \cdots, B_{t-1}$ generate the design of this*

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section in the above described manner, if and only if

(A) *the* P_iQ_i *differences are equal to the* $R_{i+t-1}S_{i+t-1}$ *differences,*

(B) *the* Q_iS_i, P_iR_i, P_iS_i, Q_iR_i *differences are equal to the* $P_{i+\tau}S_{i+\tau}$, $Q_{i+\tau+1}R_{i+\tau+1}$, $P_{i+2\tau+1}R_{i+2\tau+1}$, $Q_{i+2\tau+1}S_{i+2\tau+1}$ *differences respectively.*

PROOF. For the considered set of initial blocks, (A) and (B) would imply that the differences of elements in pairs are unequal and the differences of elements in opposed pairs are equal in distinct pairs but for the following equal differences, viz.,

P_0Q_0 difference $=$ $R_{t-1}S_{t-1}$ difference;

Q_0S_0 difference $=$ $P_\tau S_\tau$ difference $=$ $P_{t-1}R_{t-1}$ difference;

Q_0R_0 difference $=$ $Q_{2\tau+1}S_{2\tau+1}$ difference $=$ $P_{t-1}S_{t-1}$ difference.

These equal differences on replacing Q_0 by v give

P_0v difference $\neq$ $R_{t-1}S_{t-1}$ difference;

vS_0 difference $\neq$ $P_\tau S_\tau$ difference $=$ $P_{t-1}R_{t-1}$ difference and;

vR_0 difference $\neq$ $Q_{2\tau+1}S_{2\tau+1}$ difference $=$ $P_{t-1}S_{t-1}$ difference.

Hence the proof that the design of this section satisfies I follows.

We proceed to construct a set of t initial blocks satisfying Theorem a. Let

$$B_i = (x^{0+i}, x^{t-1+i}; x^{t+i}, x^{2t-1+i}), \quad i = 0, 1, \cdots, t-1, \tag{4}$$

be a set of t initial blocks where $x^0 - x^t = 2x^{\tau-t+1}$, $x^0 - x = \pm 2$ and x is a primitive element in $GF(p^n)$.

This set of generators satisfies the following equalities:

$$\begin{aligned} &(\text{a}) \quad x^0 - x^{t-1} = -x^{t-1}(x^t - x^{2t-1});\\ (5)\quad &(\text{b}) \quad x^{t-1} - x^{2t-1} = x^\tau(x^0 - x^{2t-1}) = x^{t-1}(x^0 - x^t);\\ &(\text{c}) \quad x^{t-1} - x^t = \pm x^{2\tau+1}(x^{t-1} - x^{2t-1}) = \pm x^{t-1}(x^0 - x^{2t-1}). \end{aligned}$$

For the proof of (5) it is sufficient to note that 5(a) simplifies to $x^{2t-1} = -1$ and 5(b) to 5(c) are verifiable by a suitable use of $x^0 - x^t = 2x^{\tau-t+1}$ and $x^0 - x = \pm 2$.

ILLUSTRATIONS. Set of initial blocks for some v where $t = 3\tau + 2$ and $12\tau + 7 = p^n$

v	initial blocks
20	(1, 20; 15, 18), (3, 15; 7, 16), (9, 7; 2, 10), (8, 2; 6, 11), (5, 6; 18, 14)
32	(1, 32; 20, 30), (3, 20; 29, 28), (9, 29; 25, 22), (27, 25; 13, 4), (19, 13; 8, 12), (26, 8; 24, 5), (16, 24; 10, 15), (17, 10; 30, 14)

3. Design for $v = p^n + 1$ where $t = 3\tau$. This section will apply the method of generators described in Section 2. A system of t initial blocks

$$B_i = (P_i, Q_i; R_i, S_i), \quad i = 0, (t-1), 2(t-1), \cdots, (t-1)^2, \tag{6}$$

will generate all the blocks of a design, viz.,

$$(7)\quad B_{0a} = (P_0 + a, v; R_0 + a, S_0 + a); B_{ia} = (P_i + a, Q_i + a; R_i + a, S_i + a),$$
$$i = (t-1), 2(t-1), \cdots, (t-1)^2, \; a = 0, 1, \cdots, 4t-2.$$

We give the following theorem:

THEOREM b. *A set of t initial blocks* $B_0, B_{t-1}, B_{2(t-1)}, \cdots, B_{(t-1)^2}$ *generate the design of this section in the above described manner, if and only if*

(A) *the* R_iS_i *differences are equal to the* $P_{i+(t-1)^2}Q_{i+(t-1)^2}$ *differences or the* P_iQ_i *differences are equal to the* $R_{i+(t-1)^2}S_{i+(t-1)^2}$ *differences*

(B) *the* P_iS_i, Q_iS_i, P_iR_i, Q_iR_i *differences are equal to the* $P_{i+\tau(t-1)}R_{i+\tau(t-1)}$, $Q_{i+\tau(t-1)}R_{i+\tau(t-1)}$, $Q_{i+2\tau(t-1)}S_{i+2\tau(t-1)}$, $P_{i+(2\tau-1)(t-1)}S_{i+(2\tau-1)(t-1)}$ *differences respectively.*

The proof is left to the reader.

A set of generators satisfying Theorem b will follow. Let

$$(8)\quad B_i = (x^{0+i}, x^{2(t-1)^2+i}; x^{(t-1)^2+i}, x^{(t-1)^2+2t-1+1}),$$
$$i = 0, (t-1), 2(t-1), \cdots, (t-1)^2,$$

be a set of t initial blocks where $x^0 + x^{(t-1)^2} = x^{\tau(t-1)}(x^0 - x^{(t-1)^2})$, $(1 - x^{2(t-1)^2}) = \pm 2$ or $1 - x^{2(t-1)^2} = \pm 2x^{2(t-1)^2}$ and x denotes a primitive element in $GF(p^n)$.

This set of initial blocks satisfies the following equalities:

$$\text{(a)}\quad x^{(t-1)^2} - x^{(t-1)^2+2t-1} = \pm\, x^{(t-1)^2}(x^0 - x^{2(t-1)^2}) \text{ or}$$
$$x^0 - x^{2(t-1)^2} = \pm\, x^{(t-1)^2}(x^{(t-1)^2} - x^{(t-1)^2+2t-1});$$
$$(9)\quad \text{(b)}\quad x^0 - x^{(t-1)^2+2t-1} = x^{\tau(t-1)}(x^0 - x^{(t-1)^2});$$
$$\text{(c)}\quad x^{2(t-1)^2} - x^{(t-1)^2+2t-1} = -\, x^{\tau(t-1)}(x^{2(t-1)^2} - x^{(t-1)^2});$$
$$\text{(d)}\quad x^0 - x^{(t-1)^2} = \pm x^{2\tau(t-1)}(x^{2(t-1)^2} - x^{(t-1)^2+2t-1});$$
$$\text{(e)}\quad x^{2(t-1)^2} - x^{(t-1)^2} = \pm\, x^{(2\tau-1)(t-1)}(x^0 - x^{(t-1)^2+2t-1}).$$

The proof of (9) is left to the reader.

ILLUSTRATIONS. Set of initial blocks for some v where $v = 3\tau$ and $12\tau - 1 = p^n$

v	initial blocks
12	(1, 12; 5, 6), (4, 1; 9, 2), (5, 4; 3, 8)
24	(1, 24; 10, 13), (20, 22; 16, 7), (9, 3; 21, 2), (19, 14; 6, 17), (12, 4; 5, 18); (10, 11; 8, 15)

REFERENCES

[1] BOSE, R. C. (1942). On some new series of balanced incomplete block designs. *Calcutta Math. Soc.* **34** 17–31.

[2] SCHEID, FRANCIS. (1960). A tournament problem. *Amer. Math. Monthly* **67** 39–41.