The Annals of Mathematical Statistics
1968, Vol. 39, No. 2, 531–546

A COMPARISON OF THE MOST STRINGENT AND THE MOST STRINGENT SOMEWHERE MOST POWERFUL TEST FOR CERTAIN PROBLEMS WITH RESTRICTED ALTERNATIVE

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0. Summary. The paper studies hypothesis testing problems (H, K_1) for the mean of a vector variate having a multivariate normal distribution with known covariance matrix, in cases where the alternative K_1 is restricted by a number of linear inequalities (Section 2). We describe a method for obtaining the *most stringent size-α test* φ^* for Problem (H, K_1) (Section 3). This method is used for constructing φ^* for a number of special problems (H, K_1) (Sections 5, 6 and 7) where (i) K_1' is symmetrical and (ii) Ψ_0 is sufficiently small. Thus for these special problems, we can compare φ^* with the *most stringent* SMP *size-α* test φ_0 that can be obtained by applying the general methods of [9] and [10] (described in Section 2). It turns out (for the special problems with $\alpha = .05$ or $.01$) that φ^* does not provide a worth-while improvement upon $\varphi_0 : \varphi^*$ has a smaller maximum shortcoming but φ_0 is better than φ^* from other over-all points of view. This supports the opinion that generally no serious objections can be made to the use of the MSSMP tests constructed in [10] Part 2 for problems from actual practice. This is a fortunate circumstance, for these MSSMP tests require only *simple calculations*.

By the way we prove a theorem characterizing the MS size-α test for a very general problem $(H, K_{(m)})$ where the hypothesis H may be composite while $K_{(m)}$ consists of a *finite* number (m) of parameters $\theta_i (i = 1, \cdots, m)$ (Section 4).

1. Introduction. Let (H, K) be a testing problem with hypothesis H and alternative K. The *shortcoming* $\gamma_{\varphi,D}(\theta)$ in $\theta(\varepsilon K)$ of test φ with respect to the class D of tests (test functions) is defined by

$$\gamma_{\varphi,D}(\theta) = \beta_D^*(\theta) - \beta_\varphi(\theta) \tag{1}$$

where $\beta_D^*(\theta) = \sup_{\varphi \varepsilon D} \beta_\varphi(\theta)$ denotes the envelope power in θ with respect to D and $\beta_\varphi(\theta) = E_\theta\{\varphi(X)\}$ is the *power* in θ of test (function) φ. A test φ^* is said to be MS(D) (*most stringent* with respect to the class D) for Problem (H, K) if (i) $\varphi^* \varepsilon D$ and (ii)

$$\sup_{\theta \varepsilon K} \gamma_{\varphi^*,D}(\theta) = \inf_{\varphi \varepsilon D} \sup_{\theta \varepsilon K} \gamma_{\varphi,D}(\theta). \tag{2}$$

In case D is the class of size-α tests, φ^* is said to be MS size-α.

A test φ is called SMP(D) (*somewhere most powerful* with respect to D) if (i) $\varphi \varepsilon D$ and (ii) $\gamma_{\varphi,D}(\theta) = 0$ for some $\theta \varepsilon K$. Let C denote the class of all SMP(D) tests for Problem (H, K). A test φ_0 is said to be MSSMP(D)

Received 2 February 1967; revised 29 May 1967.

(*most stringent somewhere most powerful* with respect to D) if (i) $\varphi_0 \varepsilon C$ and (ii)

$$\sup_{\theta \varepsilon K} \gamma_{\varphi_0, D}(\theta) = \inf_{\varphi \varepsilon C} \sup_{\theta \varepsilon K} \gamma_{\varphi, D}(\theta). \tag{3}$$

In case D is the class of size-α tests, φ_0 is said to be MSSMP size-α.

In [9] and [10] Chapter 2 we described methods for obtaining MSSMP(D) tests for large classes of testing problems with a restricted alternative. Applications to problems from actual practice were given in [10] Part 2.

Though MSSMP(D) tests are admissible (at least if D is sufficiently large) they are not satisfactory for each problem with a restricted alternative: in [10] Section 2.15 we described a family of testing problems (H, K) where K is defined by *one* inequality and where a worth-while improvement upon the MSSMP(D) tests turned out to be possible.

Obviously we are in want of a criterion characterizing the problems for which the MSSMP(D) tests cannot be improved upon to a worth-while extent (a provisional criterion has been described in [10] Section 2.13). For that purpose the MSSMP(D) test should be compared with all other (admissible) size-α tests, for a large number of problems.

In this paper we confine our attention to Problems (H, K_1) (to be formulated in Section 2) which are of the form $\{(H, K_1), \sigma^2 = 1\}$ described in [9] Section 3 and [10] Section 2.4. For such problems we try to obtain the MS size-α test φ^* by means of a method (Section 3) which resembles the method of *van Zwet and Oosterhoff* (1967) for obtaining MS size-α tests for combination of tests problems (H, K_1) where K_1 is a positive orthant with vertex H (in our notation H and K denote also the corresponding sets of parameters Ω_H and Ω_K). Next we formulate a rule of thumb characterizing the problems for which the MS size-α test φ^* does not provide a worth-while improvement upon the MSSMP size-α test φ_0. Assuming that for such problems no other size-α test will improve upon φ_0 to a worth-while extent, we may postulate the rule of thumb as the criterion wanted. Thus we obtained the opinion mentioned at the end of the Summary.

REMARK. Regarded as test-functions over the sample space, φ_0 and φ^* are obviously not uniquely determined. In this paper we "identify" tests when their power-functions over K_1 are identical.

2. The formulation of problem (H, K_1). In accordance with [9] Section 3 and [10] Section 2.4, let $X = (X_1, \cdots, X_n)$ have the multivariate normal $N(\xi, \Sigma)$ distribution with *known* nonsingular covariance matrix $\Sigma = A^{-1}$ where A is the $n \times n$ matrix (a^{ij}). The outcomes x of X and the admitted vectors ξ of means $(\xi_1, \cdots, \xi_n)$ are regarded as points in the same n dimensional space R^n. The vector ξ of means is known to lie in the parameter space $\Omega = H \cup K_1$ defined by the r inequalities

$$\textstyle\sum_{i=1}^n b^{ih} \xi_i \geqq 0, \qquad (h = n - s + 1, \cdots, n - s + r) \tag{4}$$

in the subspace R^s defined by the $n - s$ equalities,

$$\textstyle\sum_{i=1}^n b^{ih} \xi_i = 0, \qquad (h = 1, \cdots, n - s); \tag{5}$$

$2 \leqq r \leqq s \leqq n$; (b^{ih}) $(i = 1, \cdots, n; h = 1, \cdots, n - s + r)$ is supposed to be a matrix of rank $n - s + r$.

The hypothesis H is to be tested that ξ lies in the subspace R^{s-r} in R^s, which is obtained when all $\geqq$ signs in (4) are replaced by $=$ signs. The alternative K_1 corresponds with the subset of R^s which is defined by (4) where at least one inequality is strong.

In R^n an inner product is defined by means of the bilinear form

$$(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} a^{ij} x_i y_j$$

and accordingly (i) orthogonality $x \perp y$ is defined by $(x, y) = 0$, (ii) the norm $\|x\|$ by $\|x\|^2 = (x, x)$ and (iii) if x spans the half-line l (l is the set of all points θx with $\theta > 0$) and y spans the half-line m then $\cos\{\Psi(l, m)\} = \|x\|^{-1} \|y\|^{-1}(x, y)$, where $\Psi(l, m)$ is the angle between l and $m (0 \leqq \Psi(l, m) \leqq \pi)$.

In [9] Section 4 and [10] Section 2.6, we constructed the MSSMP size-α test φ_0 for Problem (H, K_1). Let R^r be the linear subspace in R^s perpendicular to the R^{s-r} defined by H and let K_1' denote the intersection of K_1 and R^r. K_1' is a polyhedral angle with edges $e_1, \cdots, e_r$ which are obtained when all inequalities (4) but one are replaced by the corresponding equalities. Next let the angle Ψ_0 and the half-line l_0 in K_1' be defined by

$$(6) \qquad \Psi_0 = \inf_{l \subset K_1'} \sup_{m \subset K_1'} \Psi(l, m) = \sup_{m \subset K_1'} \Psi(l_0, m),$$

and let X^I be the projection of the sample point X on the R^1 spanned by l_0; further $0X^I = \|X^I\|$ (or $-\|X^I\|$) in case $X^I \in$ (or $\notin$) l_0. The MSSMP size-α test φ_0 for problem (H, K_1) rejects if and only if $0X^I \geqq u_\alpha$ where $u_\alpha = (\Phi^\times)^{-1}(\alpha)$ and $\Phi^\times(x) = P(U \geqq x)$ when U has the normal $N(0, 1)$ distribution.

D being the class of size-α tests for Problem (H, K_1), the envelope power in a point $\xi \in K_1'$ is equal to $\beta_D^*(\xi) = \Phi^\times(u_\alpha - \|\xi\|)$. The maximum of the shortcoming of φ_0 over an arbitrary half-line l in K_1' is a strictly increasing function of the angle $\Psi(l, l_0)$:

$$(7) \quad \gamma_{\varphi_0, D}(l) = \sup_{\|\xi\|>0} [\Phi^\times(u_\alpha - \|\xi\|) - \Phi^\times [u_\alpha - \|\xi\| \cos\{\Psi(l, l_0)\}]].$$

Tables of $\gamma_{\varphi_0,D}(l)$ as a function of $\Psi = \Psi(l, l_0)$ and as a function of $\cos \Psi$ will be given in [8]. We shall compare the MSSMP size-α test φ_0 with the MS size-α test φ^* by considering graphs of $\gamma_{\varphi,D}(l)$ for $\varphi = \varphi_0$ and $\varphi = \varphi^*$ and l varying over (subsets of) K_1' (see the Figures 3, 4, 5, 8, 9 and 10).

Remark. It may be proved easily that the (power function of the) MSSMP size-α test φ_0 for Problem (H, K_1) is uniquely determined.

3. A general method for obtaining the MS size-α test φ^* for Problem (H, K_1). Our method is based on the following two lemmas:

First let $(H, K_{(m)})$ be a testing problem for the variate X over the sample space $\mathfrak{X}$, such that $K_{(m)}$ consists of exactly m parameterpoints θ_i defining pdf's $f_{\theta_i}(x)$ over $\mathfrak{X} (i = 1, \cdots, m)$. Let φ be MP size-α for testing H against the simple alternative that X has the pdf $\sum p_i f_{\theta_i}(x)$ over $\mathfrak{X}$ where $p_1, \cdots, p_m$ are certain probabilities ($p_i \geqq 0$; $\sum p_i = 1$).

LEMMA 1. *If these probabilities* $p_1, \cdots, p_m$ *are such that*

$$\gamma_{\varphi,D}(\theta_1) = \gamma_{\varphi,D}(\theta_2) = \cdots = \gamma_{\varphi,D}(\theta_m) \tag{8}$$

holds, then φ *is* MS *size-*α *for problem* $(H, K_{(m)})$.

PROOF. Suppose φ does not minimize the maximum shortcoming over $K_{(m)}$. Then there exists a test $\psi \varepsilon D$ such that $\gamma_{\psi,D}(\theta_i) < \gamma_{\varphi,D}(\theta_i)$ and consequently $\beta_\psi(\theta_i) > \beta_\varphi(\theta_i)$ $(i = 1, \cdots, m)$. But the power of ψ against the above-described simple alternative is $\sum p_i \beta_\psi(\theta_i)$. Consequently ψ would be more powerful than φ. Thus we obtain a contradiction.

A complement of Lemma 1 will be given in the characterization Theorem 2 (Section 4).

Next let $(H, K^{(1)})$ and $(H, K^{(2)})$ be two problems for the variate X over $\mathfrak{X}$, such that $K^{(1)} \subset K^{(2)}$.

LEMMA 2. *If the test* φ *is* MS *size-*α *for Problem* $(H, K^{(1)})$ *and*

$$\sup_{\theta \varepsilon K^{(1)}} \gamma_{\varphi,D}(\theta) = \sup_{\theta \varepsilon K^{(2)}} \gamma_{\varphi,D}(\theta) \tag{9}$$

then φ *is* MS *size-*α *for Problem* $(H, K^{(2)})$.

PROOF. For an arbitrary testing problem $\Pi = (H, K)$ we define

$$b_D(\Pi) = \inf_{\varphi \varepsilon D} \sup_{\theta \varepsilon K} \gamma_{\varphi,D}(\theta), \tag{10}$$

where D is the class of all size-α tests; $b_D(\Pi)$ is the maximum shortcoming on K of the MS size-α test for Problem Π if this test exists. Next let $\Pi^{(i)}$ denote problem $(H, K^{(i)})$ $(i = 1, 2)$. Then it follows from $K^{(1)} \subset K^{(2)}$ that $b_D(\Pi^{(1)}) \leqq b_D(\Pi^{(2)})$ (see Lemma 3 in Section 7).

Moreover the left-hand side of (9) is equal to $b_D(\Pi^{(1)})$, for φ is MS size-α for Problem $\Pi^{(1)}$. Thus it follows from (9) that $b_D(\Pi^{(1)}) \geqq b_D(\Pi^{(2)})$. Hence $b_D(\Pi^{(1)}) = b_D(\Pi^{(2)})$ and (9) shows that φ is MS size-α for Problem $\Pi^{(2)}$.

In order to obtain the MS size-α test φ^* for Problem (H, K_1), we shall try to find m parameterpoints $\theta_1, \cdots, \theta_m$ in K_1, defining pdf's $f_{\theta_i}(x)$ over R^n, with probabilities $p_1, \cdots, p_m$ ($\sum p_i = 1$) such that the MP size-α test φ for H against the simple alternative that X has the pdf $\sum p_i f_{\theta_i}(x)$ satisfies (8), and (9) with $K_{(m)}$ and K_1 substituted for $K^{(1)}$ and $K^{(2)}$. In that case the lemmas show that φ is the MS size-α test φ^* for Problem (H, K_1) and we write θ_i^* and p_i^* instead of θ_i and p_i.

Let $f_1, \cdots, f_n$ be an orthonormal basis of R^n such that $f_{n-s+r+1}, \cdots, f_n$ span R^{s-r} and $f_{n-s+1}, \cdots, f_n$ span R^s; consequently $f_{n-s+1}, \cdots, f_{n-s+r}$ span R^r (see Section 2, [9] Section 3, or [10] Section 2.5). We shall only look for $\theta_1^*, \cdots, \theta_m^*$ among the points of K_1'. Suppose that $\theta_1, \cdots, \theta_m$ are m parameterpoints in K_1'. With respect to the basis $f_1, \cdots, f_n$ the coordinates of θ_i are $\eta_{i1}, \cdots, \eta_{in}$ where $\eta_{ij} = 0$ $(j = 1, \cdots, n - s; n - s + r + 1, \cdots, n)$ and accordingly θ_i defines the pdf

$$f_{\theta_i}(x) = (2\pi)^{-\frac{1}{2}n} \exp\{-\tfrac{1}{2} \textstyle\sum_{j=1}^n (y_j - \eta_{ij})^2\}$$

in the point x of R^n with coordinates $y_1, \cdots, y_n$ with respect to $f_1, \cdots, f_n$.

Next suppose that $p_1, \cdots, p_m$ are m probabilities defining the pdf $\sum p_i f_{\theta_i}(x)$ with respect to $f_1, \cdots, f_n$, over R^n. The MP size-α test φ for the simple hypothesis: $\eta_i = 0$ $(i = 1, \cdots, n)$, against the simple alternative that X has the pdf $\sum p_i f_{\theta_i}(x)$ is obtained by applying the Neyman-Pearson fundamental lemma. This test rejects when

$$(11) \qquad \sum_{i=1}^{m} p_i \exp\{\sum_{j=n-s+1}^{n-s+r} \eta_{ij} Y_j\} \geqq c_\alpha,$$

where c_α is determined such that the probability of (11) is equal to α when $Y_1, \cdots, Y_n$ are independent $N(0, 1)$ variates. Obviously this test φ is of size-α for testing the whole hypothesis H and consequently φ is MP size-α for testing H against the simple alternative that X has the pdf $\sum p_i f_{\theta_i}(x)$.

In the rest of this section and in the Sections 5, 6 and 7 we confine our attention to Problems (H, K_1) where K_1' is a *symmetrical* polyhedral angle (see Sections 6 and 7 for the definition of symmetry in case $r = 3$ or 4); $\bar{e}_i$ will denote the unit vector along the ith edge e_i of K_1' $(i = 1, \cdots, r)$. For these problems we look for $\theta_1^*, \cdots, \theta_m^*$ only among the r-tuples $(\theta_1, \cdots, \theta_r)$ where $\theta_i = \nu\bar{e}_i$ $(i = 1, \cdots, r)$. On account of the symmetry of K_1', the MP size-α test φ_ν for H against the simple alternative that X has the pdf $\sum p_i f_{\theta_i}(x)$ where $\theta_i = \nu\bar{e}_i$, will satisfy (8) if $p_i = r^{-1}$ $(i = 1, \cdots, r)$. So we look for φ^* only among these tests $\varphi_\nu(\nu > 0)$. The shortcoming of φ_ν in all r points $\theta_i = \nu\,\bar{e}_i$ is equal to the following function of ν:

$$(12) \qquad \gamma_{\varphi_\nu, D}\{\theta(\nu)\} = \Phi^{\times}(u_\alpha - \nu) - \beta_{\varphi_\nu}(\nu\bar{e}_i)$$

(the notation $\theta(\nu)$ is introduced in order to indicate that this shortcoming does not depend on i). Next we define ν^* by the requirement that

$$(13) \qquad \gamma_{\varphi_{\nu^*}, D}\{\theta(\nu^*)\} = \sup_{\nu>0} \gamma_{\varphi_\nu, D}\{\theta(\nu)\}$$

stating that φ_{ν^*} is a test of the form φ_ν such that the shortcoming in $\theta(\nu)$ (or equivalently in $\nu\bar{e}_i$) is *maximized.*

The maximum shortcoming of φ_{ν^*} over the half-line l in K_1' is determined by (see Formula (7))

$$(14) \qquad \gamma_{\varphi_{\nu^*}, D}(l) = \sup_{\kappa>0}\{\Phi^{\times}(u_\alpha - \kappa) - \beta_{\varphi_{\nu^*}}(\kappa\bar{l})\}$$

where $\bar{l}$ is the unit vector spanning l.

THEOREM 1. *The test* φ_{ν^*} *is the* MS *size-*α *test* φ^* *for Problem* (H, K_1) *if*

$$(15) \qquad \gamma_{\varphi_{\nu^*}, D}(l) \leqq \gamma_{\varphi_{\nu^*}, D}\{\theta(\nu^*)\}$$

holds for each half-line l *in* K_1'.

PROOF. φ_{ν^*} is of the form φ_ν. Consequently (Lemma 1) φ_{ν^*} is MS size-α for Problem $(H, K_{(r)}(\nu^*))$ where $K_{(r)}(\nu^*)$ consists of the r points $\nu^*\bar{e}_i$ $(i = 1, \cdots, r)$ in K_1'. We have

$$(16) \qquad \sup_{\theta \epsilon K_{(r)}(\nu^*)} \gamma_{\varphi_{\nu^*}, D}(\theta) = \gamma_{\varphi_{\nu^*}, D}\{\theta(\nu^*)\}.$$

On account of (15) we have

$$\gamma_{\varphi_{\nu^*},D}(\theta) \leqq \gamma_{\varphi_{\nu^*},D}\{\theta(\nu^*)\}, \tag{17}$$

for all $\theta \varepsilon K_1'$ with equality for $\theta = \nu^*\bar{e}_i \varepsilon K_1'$. But φ_{ν^*} and consequently $\gamma_{\varphi_{\nu^*},D}(\theta)$ are invariant under all translations parallel to R^{s-r} (see Formula (11)). Hence (17) holds for all $\theta \varepsilon K_1$ with equality for $\theta = \nu^*\bar{e}_i$. This and (16) gives

$$\sup_{\theta \varepsilon K_1} \gamma_{\varphi_{\nu^*},D}(\theta) = \sup_{\theta \varepsilon K_{(r)}(\nu^*)} \gamma_{\varphi_{\nu^*},D}(\theta). \tag{18}$$

By applying Lemma 2 we complete the proof of the theorem.

Remark. *The verification of Condition* (15) is based on the results of the computations. For all problems which will be considered in the Sections 5, 6 and 7 we found

$$\gamma_{\varphi_{\nu^*},D}\{\theta(\nu^*)\} = \gamma_{\varphi_{\nu^*},D}(e_i) \qquad (i = 1, \cdots, r) \tag{19}$$

showing that we indeed *minimize* the *maximum* shortcoming $\gamma_{\varphi_\nu,D}(e_i)$ over e_i $(i = 1, \cdots, r)$ when we *maximize* the shortcoming $\gamma_{\varphi_\nu,D}\{\theta(\nu)\}$ of φ_ν in $\nu\bar{e}_i$ as a function of ν. We shall prove (19) assuming that (i) there exists a ν^* $(0 < \nu^* < \infty)$ satisfying (13) and (ii) the corresponding shortcoming $f(\varkappa) = \gamma_{\varphi_{\nu_*},D}(\varkappa\bar{e}_i)$ over e_i has exactly one maximum whereas $f'(\varkappa) \neq 0$ everywhere else $(\varkappa > 0)$. Let the function $g(\nu)$ denote the left-hand side of (12). Then (i) $g(\nu)$ has a maximum for $\nu = \nu^*$, (ii) $f(\nu^*) = g(\nu^*)$, (iii) $f(\varkappa) \geqq g(\varkappa)$ for all $\varkappa > 0$ (otherwise for a certain $\varkappa$, φ_{ν^*} would be more stringent than $\varphi_\varkappa$ for testing H against $K_{(r)}(\varkappa)$). Hence $f'(\nu^*) = 0$ and on account of the assumptions, $g(\nu^*) = f(\nu^*) = \sup_\varkappa f(\varkappa)$ what proves (19).

Consequently we may verify that

$$\gamma_{\varphi_{\nu^*},D}(l) \leqq \gamma_{\varphi_{\nu^*},D}(e_i) \tag{20}$$

holds for each half-line l in K_1', instead of Condition (15). Of course the right-hand side in (20) does not depend on i $(i = 1, \cdots, r)$. It will turn out in the Sections 5, 6 and 7 that Condition (20) is satisfied provided that Ψ_0 (see Section 2) is *not too large*. If Ψ_0 is so large that (20) is not satisfied, then φ_{ν^*} will be not MS size-α for Problem (H, K_1) (this follows from Theorem 4 under the assumptions of Theorem 3).

4. Back-ground material for Section 3. The Lemmas 1 and 2 and Theorem 1 give *sufficient* conditions for a test to be MS size-α. This section is devoted to some complementary results. It is not necessary to read this section before the Sections 5, 6 and 7 where only Theorem 1 is needed in order to prove that certain tests are MS size-α.

Condition (8) is sufficient for φ to be MS size-α (Lemma 1) for the general problem $(H, K_{(m)})$ described at the beginning of Section 3. In case $m = 2$, Condition (8) is also necessary for φ to be MS size-α. For suppose $\gamma_{\varphi,D}(\theta_1) < \gamma_{\varphi,D}(\theta_2)$ and let φ_2 be the MP size-α test for H against the simple alternative θ_2, then (for sufficiently small $\rho > 0$) $(1 - \rho)\varphi + \rho\varphi_2 \varepsilon D$ will be more stringent than φ for

Problem $(H, K_{(2)})$. Moreover in case $m = 2$ it can be shown easily that there exist probabilities $p_1, p_2 (p_1 + p_2 = 1)$ such that the corresponding MP test φ satisfies (8). In case $m > 2$ there will not always exist probabilities $p_1, \cdots, p_m$ such that (8) holds. For let φ be MS size-α for Problem $(H, K_{(2)})$ where $K_{(2)}$ consists of $\theta_1, \theta_2 \, \varepsilon \, K_1'$ (θ_1 and θ_2 not on the same half-line l). Let $\theta_3, \cdots, \theta_m$ be parameter-points in K_1' such that $\gamma_{\varphi,D}(\theta_i) < \gamma_{\varphi,D}(\theta_1) = \gamma_{\varphi,D}(\theta_2)$ $(i = 3, \cdots, m)$. Then φ is MS size-α for Problem $(H, K_{(m)})$ (Lemma 2) but φ does not satisfy (8).

CHARACTERIZATION THEOREM 2. *If the sample space* $\mathfrak{X}$ *satisfies a regularity condition* ([7] p. 354) *then there exists a number* k $(2 \leqq k \leqq m)$, *indices* i_j $(j = 1, \cdots, k; i_j = 1, \cdots, m)$ *and probabilities* $p(i_j)$ $(\sum p(i_j) = 1)$ *such that the* MP *size-*α *test* φ *for* H *against the simple alternative that* X *has the* pdf $\sum p(i_j) f_{\theta(i_j)}(x)$ *satisfies*

$$\gamma_{\varphi,D}(\theta_{i_1}) = \cdots = \gamma_{\varphi,D}(\theta_{i_k}) = \gamma^* = \max_{i=1,\cdots,m} \gamma_{\varphi,D}(\theta_i)$$

so that φ *is* MS *size-*α *for Problem* $(H, K_{(m)})$ *on account of the Lemmas* 1 *and* 2.

PROOF. This is the situation of a *S-game* ([2] p. 47) where Player I (Nature) has a *finite* number of pure strategies θ_i $(i = 1, \cdots, m)$ while Player II (Statistician) selects his strategy (test function) φ from the class D of all size-α tests. In the R^m of points $g = (g_1, \cdots, g_m)$ we define the subset S of all points g with coordinates $g_i = \gamma_{\varphi,D}(\theta_i)$ $(i = 1, \cdots, m)$ $(\varphi \, \varepsilon \, D)$. Obviously equivalent tests (with the same power over $K_{(m)}$) are represented by the same point in S. Let K denote the unit hypercube of all points g with $0 \leqq g_i \leqq 1$ $(i = 1, \cdots, m)$. S is a *closed, convex* subset of K and S has *a point in common* with each of the m faces $g_i = 0$ $(i = 1, \cdots, m)$ of K. The convexity of S can be proved by remarking that when $\varphi, \varphi' \, \varepsilon \, D$ are characterized by the points $g, g' \, \varepsilon \, S$, then for $0 \leqq \rho \leqq 1$, the test $\rho\varphi + (1 - \rho)\varphi' \, \varepsilon \, D$ and this test is characterized by the point $\rho g + (1 - \rho) g'$. It follows from the Weak Compactness Theorem ([7] p. 354) provided that $\mathfrak{X}$ satisfies a regularity condition, that (i) there exists a MP size-α test for H against the simple alternative θ_i (S has a point in common with $g_i = 0$) and (ii) S is closed.

Next ([2] p. 49) let $\Gamma_\gamma = \{g; g \, \varepsilon \, R^m, g_i < \gamma \; (i = 1, \cdots, m)\}$ denote the "negative orthant" in R^m shifted so that the vertex $(0, \cdots, 0)$ is at the point $\gamma = (\gamma, \cdots, \gamma)$. Γ_γ is obviously open and convex. Let $\gamma^* = \sup_\gamma \{\gamma; \Gamma_\gamma \cap S = \emptyset\}$. Then $\Gamma_{\gamma^*} \cap S = \emptyset$; hence there exists a hyperplane $\sum p_i g_i = c$ separating Γ_{γ^*} and S. Without loss of generality we can assume $p_i \geqq 0$ $(i = 1, \cdots, m)$, $\sum p_i = 1$; moreover the hyperplane contains the vertex γ^* of Γ_{γ^*} ([2] page 43, [5] pages 11 and 12). $[\Gamma_{\gamma^*}] \cap S \neq \emptyset$. Let $g^* \, \varepsilon \, S$ be such that $g^* \, \varepsilon \, [\Gamma_{\gamma^*}]$. S is convex and has a point in common with each of the m faces $g_i = 0$ of K. Consequently at least two coordinates g_i^* of g^* are equal to γ^*. So there exists a number k $(2 \leqq k \leqq m)$ and indices i_j $(j = 1, \cdots, k; i_j = 1, \cdots, m)$ such that the coordinates $g^*(i_j)$ are equal to $\gamma^* (j = 1, \cdots, k)$ whereas the other coordinates of g^* are smaller. The separating hyperplane $\sum p_i g_i = c$ contains g^* and the vertex γ^*. Consequently $p_i = 0$ for $i \neq i_j$ $(j = 1, \cdots, k)$. We complete the proof by remarking that the test φ which is represented by $g^* \, \varepsilon \, S$ is MP size-α for testing H against the simple alternative that X has the pdf $\sum p_i f_{\theta(i)}(x)$; for the power $\sum p_i \beta_\varphi(\theta_i)$ against this simple alternative becomes as large as possible when $\sum p_i \gamma_{\varphi,D}(\theta_i)$ is minimized; the latter expression is $\geqq c$ for all points $g \varepsilon S$ and $=c$ for the point g^*.

Theorem 2 characterizes the parameterpoints $\theta(i_j)$ $(j = 1, \cdots, k)$ as those points where the shortcoming $\gamma_{\varphi,D}(\theta)$ of the MS size-α test φ for Problem $(H, K_{(m)})$ is as large as possible. We remark that each MS size-α test ψ for Problem $(H, K_{(m)})$ is MP size-α for H against the simple alternative that X has the pdf $\sum p(i_j)f_{\theta(i_j)}$ (x) for otherwise $\sum p(i_j)\beta_\psi\{\theta(i_j)\} < \sum p(i_j)\beta_\varphi\{\theta(i_j)\}$ and hence $\sum p(i_j)\gamma_{\psi,D}\{\theta(i_j)\} > \gamma^*$ and $\gamma_{\psi,D}\{\theta(i_j)\} > \gamma^*$ for some index j. This remark will be useful for proving that the MS size-α test for Problem $(H, K_{(m)})$ is *uniquely determined* under certain assumptions (see Theorem 4).

It can be proved that there exists a MS size-α test ψ for Problem (H, K_1) which is "invariant" or in other words, which only depends on the coordinates y_i $(i = n - s + 1, \cdots, n - s + r)$ of the sample point x (see Section 3). For such tests we can restrict our attention to the power over K_1'.

CHARACTERIZATION THEOREM 3. *If there exists an invariant* MS *size-α test ψ for Problem (H, K_1), such that a (finite) number m and parameterpoints $\theta_i \varepsilon K_1'$ $(i = 1, \cdots, m)$ exist with the property that*

$$\gamma_{\psi,D}(\theta_i) = \sup_{\theta \varepsilon K_1'} \gamma_{\psi,D}(\theta) = \gamma^*, \qquad (i = 1, \cdots, m),$$

while for all other $\theta \varepsilon K_1'$ we have $\gamma_{\psi,D}(\theta) < \gamma^*$ (and even $\gamma_{\psi,D}(\theta) < \gamma^* - \epsilon$ for some $\epsilon > 0$, provided that $\|\theta\|$ is sufficiently large) then ψ is MS *size-α for Problem $(H, K_{(m)})$.*

PROOF. Suppose ψ is not MS size-α for Problem $(H, K_{(m)})$. Let φ be a MS size-α test of the form (11), for Problem $(H, K_{(m)})$ (see Theorem 2). Then $\gamma_{\varphi,D}(\theta_i) < \gamma_{\psi,D}(\theta_i)$ $(i = 1, \cdots, m)$ and we can construct ρ $(0 < \rho < 1)$ such that the test $\rho\varphi + (1 - \rho)\psi \varepsilon D$ is more stringent for Problem (H, K_1) than ψ. Thus we obtain a contradiction.

UNIQUENESS THEOREM 4. *If (H, K_1) satisfies the condition of Theorem* 3, *then each* MS *size-α test φ^* for Problem (H, K_1) satisfies $\varphi^* = \psi$ a.e. (μ).*

PROOF. ψ is MS size-α for Problem $(H, K_{(m)})$ and $\gamma_{\psi,D}(\theta_i) = \gamma^*$ $(i = 1, \cdots, m)$ (Theorem 3). Hence φ^* is MS size-α for Problem $(H, K_{(m)})$. Consequently both ψ and φ^* are MP size-α for testing H against the simple alternative that X has a pdf $\sum p(i)f_{\theta(i)}(x)$ where $\theta(i) \varepsilon K_1'$. Hence $\varphi^* = \psi$ a.e. (μ) where μ is the Lebesgue measure over R^n.

In the following sections, the condition of Theorem 3 turns out to hold for certain problems. For problems (H, K_1) satisfying the condition (we conjecture that each problem (H, K_1) will do that) there obviously exist probabilities p_i $(i = 1, \cdots, m)$ such that the MS size-α test φ^* for Problem (H, K_1) can be obtained by applying Lemma 1 (Condition (8) is satisfied) and Lemma 2. This shows that Condition (20) (or (15)) is necessary for φ_ν to be MS size-α for Problem (H, K_1) where K_1' is symmetrical (Remark in Section 3). By introducing the condition of Theorem 3, we avoided the difficulties that emerge when the alternative consists of an infinite number of points

Bartholomew [1], *Kudô* [6] a.o. applied the L.R. (*likelihood-ratio*) criterion to problems of the form (H, K_1). Their L.R. size-α test φ' is not of the form (11). Consequently φ' is not MS size-α (if at least the condition of Theorem 3 is satisfied). Theoretically, the L.R. criterion has the disadvantage that it is not formulated as an optimum property in terms of the power ([7], page 15). From a practical point of view, φ' seems to stand somewhere midway between φ_0 and φ^*: the maximum shortcoming will lie between that of φ_0 and φ^*; the computations involved are more complicated than for φ_0 but much less difficult than for φ^*.

Doornbos and Prins [3], *Kander and Zacks* [4], a.o. considered testing problems which can be regarded as (generalizations of) special cases of Problem (H, S_1) where on the basis of an outcome of the random variable X described in Section 2, the hypothesis H has to be tested against the alternative S_1 corresponding with the subset of R^s which is defined by the existence of an index h $(h = n - s + 1, \cdots, n - s + r)$ such that strict inequality holds in (4) for this index whereas equality holds for all other indices.

Example. Let $X_1, \cdots, X_r$ have independent normal $N(\mu_i, 1)$ distributions $(i = 1, \cdots, r)$. The hypothesis $H: \mu_i = 0$ $(i = 1, \cdots, r)$ has to be tested against the alternative S_1 : for some index h $(h = 1, \cdots, r)$ we have $\mu_h > 0$ whereas $\mu_i = 0$ $(i = 1, \cdots, r; i \neq h)$, or in other words that the vector of means $(\mu_1, \cdots, \mu_r)$ is situated on one of the edges of the positive orthant.

Obviously for the general problem (H, S_1), the intersection S_1' of S_1 and R^r (see Section 2) consists of the r edges e_i $(i = 1, \cdots, r)$ of K_1'. The class of all SMP size-α tests for Problem (H, S_1) consists of the r tests $0X^I \geqq u_\alpha$ where X^I is the projection of X on the R^1 spanned by the half-line l and $l = e_i$ $(i = 1, \cdots, r)$. Consequently the criterion MSSMP is inappropriate for Problem (H, S_1). It is seen easily that the MSSMP size-α test φ_0 for Problem (H, K_1) is MS (C) for Problem (H, S_1) when C is the class of all tests of the form $0X^I \geqq u_\alpha$, which are based on a *linear* combination of the coordinates $X_1, \cdots, X_n$ (Kander and Zacks applied a Bayesian approach with a limiting argument and thus obtained a test of the form $0X^I \geqq u_\alpha$ which is not MS (C)). The restriction to the above-mentioned class C seems to be rather undesirable for problems of the form (H, S_1). The method described after the proof of Lemma 2 where $m = r$ and $\theta_i = \nu_i \bar{e}_i$ $(i = 1, \cdots, r)$ will provide the MS size-α test for Problem (H, S_1). The computations are feasible if we restrict our attention to Problems (H, S_1) where the corresponding polyhedral angle K_1' is *symmetrical* (see Example). For such cases it follows from (19) that φ_{ν^*} *is the* MS *size-α test for Problem* (H, S_1) (Condition (20) needs not to be satisfied).

The above-mentioned Example provides an interesting test-case. The corresponding tests φ_0, φ_ν and the L.R. size-α test φ' respectively reject when $\sum X_i$, $\sum \exp(\nu X_i)$ and $\max X_i$ are sufficiently large. φ_0 and φ' may be regarded as limiting cases ($\nu \to 0$ and $\nu \to \infty$ respectively) of φ_ν; consequently the MS size-α test φ_{ν^*} has an intermediate position between φ_0 and φ'. The method of *Kander and Zacks* inheres in the construction of φ_0 (their limiting argument corresponds with $\nu \to 0$ in φ_ν) whereas the method of *Doornbos and Prins* provides

φ'. The tests φ_0, φ_{ν^*} and φ' respectively have the maximum shortcomings .26, .11, .11 ($r = 2$); .35, .17, .17 ($r = 3$) and .47, .21, .21 ($r = 4$) over the alternative S_1 in case $\alpha = .05$. This shows that for these cases φ' does not differ much from the MS size-α test φ_{ν^*}. The maximum shortcoming of φ_0 is unsatisfactory large; the corresponding advantages of φ_{ν^*} (and φ') over φ_0 are not neutralized by advantages of φ_0 over φ_{ν^*} from other reasonable over-all points of view (see [11] Figure 5.2 where the shortcoming of φ_0 and φ_{ν^*} is plotted for the $r = 2$ case).

5. Comparing φ_0 and φ^* for Problems (H, K_1) with $r = 2$. In the case $r = 2$, K_1' is an angle with vertex 0 in $R^r = R^2$ and K_1' is obviously symmetrical. We introduce a new orthonormal basis $g_1, \cdots, g_n$ for R^n with $g_1 = \|\bar{e}_1 - \bar{e}_2\|^{-1}(\bar{e}_1 - \bar{e}_2)$ and $g_2 = \|\bar{e}_1 + \bar{e}_2\|^{-1}(\bar{e}_1 + \bar{e}_2)$. The half-line l_0 satisfying (6) is spanned by g_2. Further $\Psi_0 = \Psi(g_2, e_1)$. The coordinates of the sample point X with respect to the basis $g_1, \cdots, g_n$ are denoted by $Z_1, \cdots, Z_n$. Obviously φ_0 rejects when $Z_2 \geqq u_\alpha$. Graphs of $\gamma_{\varphi_0,D}(l)$ as a function of $\Psi = \Psi(l, l_0)$ (see Formula (7)) are given in the Figures 3, 4 and 5.

Next we try to obtain φ^* by determining φ_{ν^*} (see Section 3). Using the basis $g_1, \cdots, g_n$ we have $\bar{e}_i = ((-1)^{i-1} \sin \Psi_0, \cos \Psi_0, 0 \cdots)$ $(i = 1, 2)$; $\theta_i = \nu\bar{e}_i$ defines the n-variate normal $N(\theta_i, I)$ distribution with pdf $f_{\theta_i}(x)$ $(i = 1, 2)$. The MP size-α test φ_ν for H against the simple alternative that X has the pdf $\frac{1}{2}f_{\theta_1}(x) + \frac{1}{2}f_{\theta_2}(x)$ rejects when

$$\{\exp(-\tfrac{1}{2}Z_1^2 - \tfrac{1}{2}Z_2^2)\}^{-1} [\exp[-\tfrac{1}{2}\{(Z_1 - \nu \sin \Psi_0)^2 + (Z_2 - \nu \cos \Psi_0)^2\}] + \exp[-\tfrac{1}{2}\{(Z_1 + \nu \sin \Psi_0)^2 + (Z_2 - \nu \cos \Psi_0)^2\}]]$$

is sufficiently large, or equivalently when

$$Z_2 \geqq f_{\nu,\alpha,\Psi_0}(Z_1), \tag{21}$$

where

$$f_{\nu,\alpha,\Psi_0}(z_1) = (\nu \cos \Psi_0)^{-1} \log_e \{c_\alpha(\nu, \Psi_0)/[\exp(z_1 \nu \sin \Psi_0) + \exp(-z_1 \nu \sin \Psi_0)]\} \tag{22}$$

with $c_\alpha(\nu, \Psi_0)$ such that Test (21) is of size-α for testing H or equivalently for testing the hypothesis that Z_1 and Z_2 have independent normal $N(0, 1)$ distributions.

Let l denote the half-line in K_1' spanned by the vector $\bar{l} = (\sin \Psi, \cos \Psi, 0 \cdots)$ where $|\Psi| \leqq \Psi_0$. The *power* of φ_ν in the arbitrary point $\kappa\bar{l}$ of l $(\kappa > 0)$ is determined by

$$\beta_{\varphi_\nu}(\kappa\bar{l}) = P\{Z_2 \geqq f_{\nu,\alpha,\Psi_0}(Z_1)\}, \tag{23}$$

where in the right-hand side Z_1 and Z_2 have independent normal $N(\kappa \sin \Psi, 1)$ and $N(\kappa \cos \Psi, 1)$ distributions respectively.

Next we compute ν^* satisfying (13) (the computations are based on the assumption that the function $g(\nu)$ that is defined by the left-hand side of (12) has exactly one summit for $\nu > 0$, cf [11] Theorem 4.1). Thus we obtain the test φ_{ν^*}

(that is MS size-α for the allied Problem (H, S_1) described in Section 4) and the problem arises whether φ_{ν^*} is the MS size-α test φ^* for Problem (H, K_1) or equivalently whether Condition (20) is satisfied for each half-line l in K_1' (see also (14) and (23)). In the limiting case $\Psi_0 = \frac{1}{2}\pi$ the tests $\varphi_\nu(\nu > 0)$ degenerate into the same test φ' which rejects when $|Z_1| \geqq u_{\frac{1}{2}\alpha}$. We have $\gamma_{\varphi',D}(l_0) = 1 - \alpha$ and (20) is not satisfied. There will exist a critical angle $\Psi_0^{(cr)}(\alpha)$ such that (20) is satisfied and φ_{ν^*} is the MS size-α test φ^* for Problem (H, K_1), if and only if

$$(24) \qquad 0 \leqq \Psi_0 \leqq \Psi_0^{(cr)}(\alpha).$$

In order to determine this critical angle we examined Condition (20) by considering graphs of $\gamma_{\varphi_{\nu^*},D}(l)$ as a function of $\Psi = \Psi(l, l_0)$ (see Figures 3, 4 and 5). Indeed an angle $\Psi_0^{(cr)}(\alpha)$ turned out to exist such that $\varphi_{\nu^*} = \varphi^*$ if (24) holds, whereas otherwise $\gamma_{\varphi_{\nu^*},D}(l)$ obtains an absolute maximum for $l = l_0$. In this connection we refer to [11] where *van Zwet and Oosterhoff* proved analytically for the $\Psi_0 = 45°$ case that $\gamma_{\varphi_{\nu^*},D}(l)$ as function of $\Psi = \Psi(l, l_0)$ can only have maxima for $\Psi = 0$ and $\Psi = \Psi_0$.

The test φ_{ν^*} is of the form (21), (22) and is completely determined if we know ν^* and $c_\alpha(\nu^*, \Psi_0)$. Writing $\nu^* = \nu_\alpha^*(\Psi_0)$ and $c_\alpha(\nu_\alpha^*(\Psi_0), \Psi_0) = c_\alpha^*(\Psi_0)$ we indicate that ν^* and c^* are completely determined by α and Ψ_0. Corresponding graphs for all values Ψ_0 satisfying (24) are given in the Figures 1 and 2. These figures determine φ_{ν^*} completely, exactly in those cases where φ_{ν^*} is the MS size-α test φ^* for Problem (H, K_1). Moreover these figures determine $\Psi_0^{(cr)}(\alpha)$.

We proposed to compare the MSSMP size-α test φ_0 and the MS size-α test φ^* for Problem (H, K_1) with $r = 2$ by means of graphs of $\gamma_{\varphi,D}(l)$ as function of $\Psi = \Psi(l, l_0)$ for $\varphi = \varphi_0$ and $\varphi = \varphi^*$. Such graphs are given in the Figures 3, 4 and 5 ($\alpha = .05$).

By definition, the maximum shortcoming of φ^* is smaller than that of φ_0. But from other points of view φ_0 may be better. In Figure 6 for example we give a graph of (see also [10] Section 2.15)

$$\lambda_\alpha(\Psi_0) = \Psi_0^{-1}\Psi_1(\Psi_0, \alpha),$$

where $\Psi_1(\Psi_0, \alpha)$ is by definition such that the condition $\Psi(l, l_0) \leqq \Psi_1(\Psi_0, \alpha)$ is necessary and sufficient for $\gamma_{\varphi_0,D}(l) \leqq \gamma_{\varphi^*,D}(l)$. So φ_0 is regarded as better than φ^* from this point of view if $\lambda_\alpha(\Psi_0) > \frac{1}{2}$.

We can also compute (cf [10] Sections 2.15 and 4.3)

$$A_\varphi(\Psi_0) = \tfrac{1}{2}\Psi_0^{-1}\int_{-\Psi_0}^{\Psi_0} \gamma_{\varphi,D}(l)\, d\Psi$$

where $\gamma_{\varphi,D}(l)$ is regarded as a function of $\Psi = \Psi(l, l_0)$. Corresponding graphs for φ_0 and φ^* ($0 \leqq \Psi_0 \leqq \Psi_0^{(cr)}(\alpha)$) are given in Figure 7.

Interpreting the Figures 3, $\cdots$, 7 and the theory of [10] Section 2.15 we arrive at the following *rule of thumb*: for Problems (H, K_1) with $r = 2$ and $\Psi_0 < 60°$, φ^* does not provide a worth-while improvement upon φ_0.

Remark. In [9] Section 2 and [10] Section 2.3 we made objections to the criterion MS size-α. Supposing that $\gamma_{\varphi^*,D}(l)$ was almost constant over $|\Psi| \leqq \Psi_0$ for

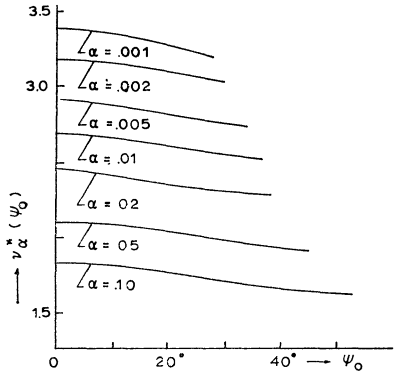

FIG. 1.

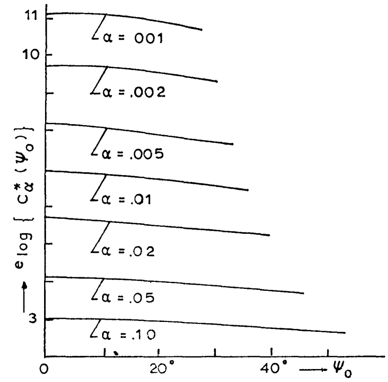

FIG. 2.

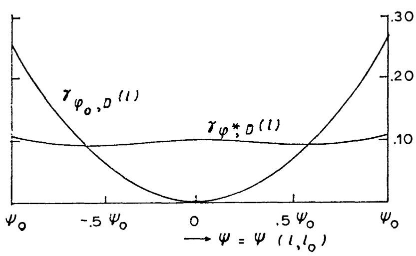

FIG. 3. (α = .05; ψ_0 = 45°)

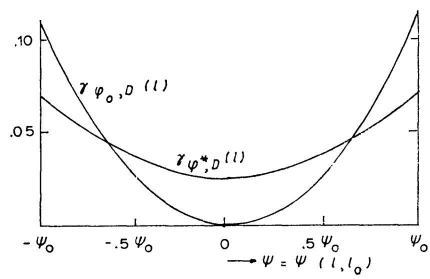

FIG. 4. (α = .05; ψ_0 = 30°)

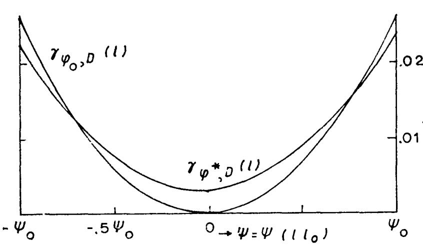

FIG. 5. (α = .05; ψ_0 = 15°)

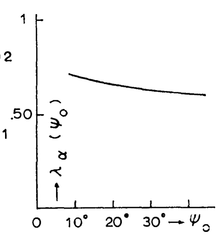

FIG. 6. (α = .05);

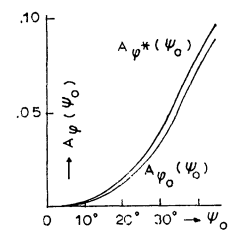

FIG. 7. (α = .05)

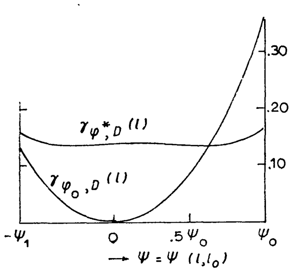

FIG. 8. (α = .05; ψ = 52.5°);

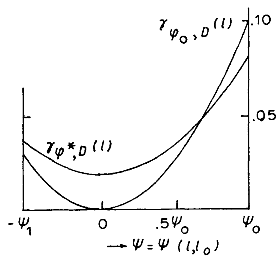

FIG. 9. (α = .05; ψ = 30°)

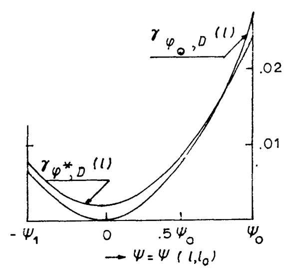

FIG. 10. (α = .05; ψ = 15°)

$\Psi = \Psi(l, l_0)$, we expected that φ_0 would provide a worth-while improvement upon φ^* for Problems (H, K_1) with Ψ_0 small. The Figures 4 and 5 show that these objections do not apply to Problems (H, K_1) in case $r = 2$. For such problems both φ_0 and φ^* have similar power properties if Ψ_0 is small.

6. Comparing φ_0 and φ^* for Problem (H, K_1) with $r = 3$ and K_1' symmetrical. In this case, K_1' is a symmetrical polyhedral angle in $R^r = R^3$. The symmetry may be defined by $\Psi(e_1, e_2) = \Psi(e_1, e_3) = \Psi(e_2, e_3)$. We can introduce an orthonormal basis $g_1, \cdots, g_n$ for R^n such that the unit vectors $\bar{e}_i$ along the edges e_i of K_1' have the coordinates

$$\bar{e}_1 = (\sin \Psi_0, 0, \cos \Psi_0, 0 \cdots)$$
$$\bar{e}_2 = (-\tfrac{1}{2} \sin \Psi_0, \tfrac{1}{2} 3^{\frac{1}{2}} \sin \Psi_0, \cos \Psi_0, 0 \cdots)$$
$$\bar{e}_3 = (-\tfrac{1}{2} \sin \Psi_0, -\tfrac{1}{2} 3^{\frac{1}{2}} \sin \Psi_0, \cos \Psi_0, 0 \cdots).$$

The half-line l_0 satisfying (6) is spanned by g_3; $\Psi(l_0, e_i) = \Psi_0$ $(i = 1, 2, 3)$. The coordinates of the sample point X with respect to the basis $g_1, \cdots, g_n$ are denoted by $Z_1, \cdots, Z_n$. Obviously φ_0 rejects when $Z_3 \geqq u_\alpha$.

The test φ_ν (see Sections 3 and 5) rejects when

$$(25) \qquad Z_3 \geqq f_{\nu,\alpha,\Psi_0}(Z_1, Z_2)$$

where f is defined by

$$(26) \quad (\nu \cos \Psi_0) f_{\nu,\alpha,\Psi_0}(z_1, z_2) = \log_e\{c_\alpha(\nu, \Psi_0)\ [\exp(z_1 \nu \sin \Psi_0)$$
$$+ \exp(-\tfrac{1}{2} z_1 \nu \sin \Psi_0)\ \{\exp(\tfrac{1}{2} 3^{\frac{1}{2}} z_2 \nu \sin \Psi_0) + \exp(-\tfrac{1}{2} 3^{\frac{1}{2}} z_2 \nu \sin \Psi_0)\}]^{-1}\},$$

with $c_\alpha(\nu, \Psi_0)$ such that Test (25) is of size-α for testing H.

The test φ_{ν^*} is determined according to (12) and (13) (the computations are based on the assumption that (12) as function of ν has exactly one summit). Condition (20) is examined by computing $\gamma_{\varphi_{\nu^*},D}(l)$ as a function of $\Psi = \Psi(l, l_0)$ for l varying over the intersection of K_1' and the R^2 spanned by e_1 and g_3. So Ψ varies over $[-\Psi_1, \Psi_0]$ where

$$(27) \quad \Psi_1 = \arccos\{\Psi(\bar{e}_2 + \bar{e}_3, g_3)\} = \arccos\{2 \cos \Psi_0/(1 + 3 \cos^2 \Psi_0)^{\frac{1}{2}}\}.$$

Corresponding graphs of $\gamma_{\varphi,D}(l)$ for $\varphi = \varphi_0$ and $\varphi = \varphi_{\nu^*} = \varphi^*$ are given in the Figures 8, 9 and 10. Here φ_{ν^*} is assumed to be the MS size-α test φ^* for Problem (H, K_1) because the function $\gamma_{\varphi^*,D}(l)$ of $\Psi = \Psi(l, l_0)$ over $[-\Psi_1, \Psi_0]$ has an absolute maximum for $\Psi = \Psi_0$ (see Formula (19)).

Interpreting these figures (and other results mentioned in [8] and in [10] Section 4.3) we obtain the opinion that the *rule of thumb* at the end of Section 5 may be modified for Problems (H, K_1) with $r = 3$, substituting 70° for 60° for example.

From certain over-all points of view, φ_0 provides a slight improvement upon φ^* if $40° < \Psi_0 < 60°$. Further the power properties of φ_0 and φ^* are quite similar for $\Psi_0 < 30°$.

7. Comparing φ_0 and φ^* for Problems (H, K_1) with $r > 3$. The case K_1' symmetrical and $r = 4$ was treated also along the lines of Section 3. In this case we introduce an orthonormal basis $g_1, \cdots, g_n$ for R^n such that

$$\bar{e}_1 = (\sin \Psi_0, 0, 0, \cos \Psi_0, 0 \cdots),$$
$$\bar{e}_2 = (-3^{-1} \sin \Psi_0, 3^{-1} 8^{\frac{1}{2}} \sin \Psi_0, 0, \cos \Psi_0, 0 \cdots),$$
$$\bar{e}_3 = (-3^{-1} \sin \Psi_0, -3^{-1} 2^{\frac{1}{2}} \sin \Psi_0, 3^{-\frac{1}{2}} 2^{\frac{1}{2}} \sin \Psi_0, \cos \Psi_0, 0 \cdots),$$
$$\bar{e}_4 = (-3^{-1} \sin \Psi_0, -3^{-1} 2^{\frac{1}{2}} \sin \Psi_0, -3^{-\frac{1}{2}} 2^{\frac{1}{2}} \sin \Psi_0, \cos \Psi_0, 0 \cdots),$$

and the MSSMP size-α test φ_0 rejects when $Z_4 \geqq u_\alpha$ whereas the test φ_ν (see Sections 3 and 6) rejects when

$$Z_4 \geqq f_{\nu,\alpha,\Psi_0}(Z_1, Z_2, Z_3) \tag{28}$$

where f is defined by a formula similar to (26) but with a more intricate denominator in the right-hand side.

Again φ_{ν^*} can be determined and Condition (20) is examined by computing $\gamma_{\varphi_{\nu^*},D}(l)$ as a function of $\Psi = \Psi(l, l_0) = \Psi(l, g_4)$ for l varying over the intersection of K_1' and the R^2 spanned by e_1 and g_4. So Ψ varies over $[-\Psi_1, \Psi_0]$ where

$$\Psi_1 = \Psi(\bar{e}_2 + \bar{e}_3 + \bar{e}_4, g_4) = \text{arc cos} \{3 \cos \Psi_0/(1 + 8 \cos^2 \Psi_\theta)^{\frac{1}{2}}\}. \tag{29}$$

Interpreting the corresponding graphs of $\gamma_{\varphi_{\nu^*},D}(l)$ we obtain the opinion that the *rule of thumb* at the end of Section 5 may be modified for Problems (H, K_1) with $r = 4$, substituting 75° for 60° for example.

From certain over-all points of view, φ_0 provides a worth-while improvement upon φ^* if $35° < \Psi_0 < 65°$. The power properties of φ_0 and φ^* are quite similar for $\Psi_0 < 30°$.

The rest of this section is devoted to a comparison of φ_0 and φ^* for Problems (H, K_1) where K_1' is an *orthant*: $\Psi(e_i, e_j) = 90°$ for $i, j = 1, \cdots, r$; $i \neq j$ (in [10] Section 4.3 we examined such problems as feasible test-cases for the provisional classification in [10] Section 2.13). For such problems we have $\Psi_0 =$ arc cos $(r^{\frac{1}{2}})$. Let $A_\varphi(\Psi_0)$ denote the averaged maximum shortcoming $\gamma_{\varphi,D}(l)$ when $\bar{l}$ has the uniform distribution over the intersection of K_1' and the surface of the unit sphere in R^r. For $\alpha = .05$ we have

$$A_{\varphi_0}(\Psi_0) \approx (1 - \alpha)\{1 - \pi^{-\frac{1}{2}}[\Gamma\{\tfrac{1}{2}(r + 1)\}]^{-1} r^{\frac{1}{2}} \Gamma(\tfrac{1}{2}r)\} \tag{30}$$

(see [10] Section 4.3, the formula for $A_\alpha(m)$). We shall use $A_\varphi(\Psi_0)$ as a measure for comparing φ_0 and φ^* for Problems (H, K_1) where K_1' is an orthant, $r = 2, 3, 4, \cdots$ and $\alpha = .05$.

Figure 3 applies to *the case* $r = 2$. We compute $A_{\varphi^*}(\Psi_0) = .097$ and $A_{\varphi_0}(\Psi_0) = .084$; further $\Psi_0^{(cr)}(\alpha) \approx 46°$. Neither φ_0 nor φ^* provides a worth-while improvement upon the other.

Condition (20) is not satisfied in *the case* $r = 3$: $\Psi_0^{(cr)}(\alpha) \approx 54° < \Psi_0 = 54.7°$. $\gamma_{\varphi_{\nu^*},D}(l)$ is almost constantly equal to .16 for l varying over the region studied (see Section 6). We shall have $A_{\varphi^*}(\Psi_0) \approx .16$ for the unknown MS size-α test φ^*. From (30) we obtain $A_{\varphi_0}(\Psi_0) \approx .13$. Neither φ_0 nor φ^* provides a worth-while improvement upon the other; φ_0 is slightly better from the $A_\varphi(\Psi_0)$ point of view.

In *the case* $r = 4$ we have $A_{\varphi_0}(\Psi_0) \approx .15$ (see (30)). Condition (20) is not satisfied: $\Psi_0^{(cr)}(\alpha) \approx 59° < \Psi_0 = 60°$. But $\gamma_{\varphi_\nu^*, D}(l)$ is almost constantly equal to about .21 for l varying over the region studied. We shall have $A_{\varphi^*}(\Psi_0) \approx .21$ for the unknown MS size-α test φ^*. So φ_0 provides a worth-while improvement upon φ^* from the $A_\varphi(\Psi_0)$ point of view whereas the maximum shortcoming of φ^* (about .21) is much smaller than the maximum shortcoming (.47) of φ_0.

Next, by applying two simple lemmas, we prove Theorem 5 elucidating *the cases* $r > 4$.

LEMMA 3. *If* $\Pi^{(1)} = (H, K^{(1)})$ *and* $\Pi^{(2)} = (H, K^{(2)})$ *are two testing problems for the variate* X *over* $\mathfrak{X}$ *such that* $K^{(1)} \subset K^{(2)}$, *then* $b_D(\Pi^{(1)}) \leqq b_D(\Pi^{(2)})$ ($b_D(\Pi)$ was defined in (10)).

Next let Π_r and $\bar{\Pi}_r$ be two testing problems of the form (H, K_1) such that for both problems K_1' is an orthant in an R^r (n and s may be different).

LEMMA 4. *For* Π_r *and* $\bar{\Pi}_r$ *we have* $b_D(\Pi_r) = b_D(\bar{\Pi}_r)$.

We shall elucidate the cases $r > 4$ by considering a sequence $\{\Pi_r\}$ ($r = 2, 3, \cdots$) of problems of the form (H, K_1) where K_1' is an orthant in R^r.

THEOREM 5. *For each sequence* $\{\Pi_r\}$ ($r = 2, 3, \cdots$) *we have* (i) $b_D(\Pi_r) \leqq b_D(\Pi_{r+1})$ ($r = 2, 3, \cdots$) *and* (ii) $\lim_{r\to\infty} b_D(\Pi_r) = 1 - \alpha$.

PROOF. (i) may be proved by applying Lemma 3 and Lemma 4. (ii) will be proved by applying Lemma 1 and Lemma 3. First we remark that the trivial size-α test $\varphi_t(x) \equiv \alpha$ belongs to D. Moreover $\sup_{\theta\in K_1} \gamma_{\varphi_t, D}(\theta) = 1 - \alpha$. Hence $b_D(\Pi_r) \leqq 1 - \alpha$ ($r = 2, 3, \cdots$).

We shall construct for $\epsilon > 0$ an integer r_ϵ such that $b_D(\Pi_r) > 1 - \alpha - \epsilon$ for $r > r_\epsilon$. For that purpose we define ν_ϵ such that $\Phi^\times(u_\alpha - \nu_\epsilon) - \alpha = 1 - \alpha - \frac{1}{2}\epsilon$. With respect to Problem Π_r we introduce an orthonormal basis $g_1, \cdots, g_n$ for R^n such that $g_i = \bar{e}_i$ ($i = 1, \cdots, r$). This is possible because K_1' is an orthant. The coordinates of the sample point X are denoted by $Z_1, \cdots, Z_n$. By applying Lemma 1 we obtain the MS size-α test φ_{ν_ϵ} for H against the alternative $K_{(r)}(\nu_\epsilon) \subset K_1$ where $K_{(r)}(\nu_\epsilon)$ consists of the parameterpoints $\nu_\epsilon g_i$ ($i = 1, \cdots, r$). φ_{ν_ϵ} rejects when

$$\textstyle\sum_{i=1}^r \exp(\nu_\epsilon Z) \geqq c_{\alpha,r}(\nu_\epsilon)$$

where $c_{\alpha,r}(\nu_\epsilon)$ is determined such that the test is of size-α. By applying the Central Limit Theorem for $r \to \infty$ (ν_ϵ is fixed) it can be shown that the power of φ_{ν_ϵ} in $\nu_\epsilon g_i$ tends to α as $r \to \infty$. Thus we can choose r_ϵ such that for $r > r_\epsilon$ this power is smaller than $\alpha + \frac{1}{2}\epsilon$. Consequently the corresponding shortcoming is larger than $\Phi^\times(u_\alpha - \nu_\epsilon) - \alpha - \frac{1}{2}\epsilon$. But this shortcoming is equal to $b_D(H, K_{(r)}(\nu_\epsilon))$. By applying Lemma 3 we obtain

$$b_D(\Pi_r) > \Phi^\times(u_\alpha - \nu_\epsilon) - \alpha - \tfrac{1}{2}\epsilon \geqq 1 - \alpha - \epsilon, \quad \text{for} \quad r > r_\epsilon .$$

This completes the proof of the theorem.

By definition the MS size-α test φ^* has the advantage over the MSSMP size-α test φ_0, that the corresponding maximum shortcoming is smaller. Theorem 5 shows that this advantage becomes unimportant when $r \to \infty$, for the maximum shortcoming tends to $1 - \alpha$ as $r \to \infty$ both for φ^* and φ_0.

φ_0 has the important advantage that good power properties for large regions inside K_1 are warranted: the expression in the right-hand side of (30) tends to $(1 - \alpha)(1 - 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}) \approx .19$ as $r \to \infty$ $(\alpha = .05)$.

Acknowledgments. The computations have been performed on the Telefunken TR-4 of Groningen University by means of programs written by Drs. H. J. van Linde and Mr. D. Velvis. The power functions were computed by means of Gauss-Hermite quadrature formulas. Tables of numerical results will be given in [8]. We are indebted to the referee for some comments leading to Section 4.

REFERENCES

[1] Bartholomew, D. J. (1961). A test of homogeneity of means under restricted alternatives (with discussion). *J. Roy. Statist. Soc. Ser. B* **23** 239–281.

[2] Blackwell, D. and Girshick, M. A. (1954). *Theory of Games and Statistical Decisions.* Wiley, New York.

[3] Doornbos, R. and Prins, H. J. (1958). On slippage tests. *Indag. Math.* **20** 38–55 and 438–447.

[4] Kander, K. and Zacks, S. (1966). Test procedures for possible changes in parameters of statistical distributions occurring at unknown time points. *Ann. Math. Statist.* **37** 1196–1210.

[5] Karlin, S. (1959). *Mathematical Methods and Theory in Games, Programming, and Economics* II. Pergamon, London.

[6] Kudô, A. (1963). A multivariate analogue of the one-sided test. *Biometrika* **50** 403–418.

[7] Lehmann, E. L. (1959). *Testing Statistical Hypothesis.* Wiley, New York.

[8] Linde, H. J. van, Velvis, D. and Schaafsma, W. (1967). Tables for the construction of MSSMP and MS size-α tests for problems with a restricted alternative. *Report* T.W. 40, Mathematical Institute. Reitdiepskade 4 Groningen.

[9] Schaafsma, W. and Smid, L. J. (1966). Most stringent somewhere most powerful tests against alternatives restricted by a number of linear inequalities. *Ann. Math. Statist.* **37** 1161–1172.

[10] Schaafsma, W. (1966). *Hypothesis Testing Problems with the Alternative Restricted by a Number of Inequalities.* Noordhoff, Groningen.

[11] Zwet, W. R. van and Oosterhoff, J. (1967). On the combination of independent test statistics. *Ann. Math. Statist.* **38** 659–680 (1966: Preliminary Report S366, VP 27 Mathematisch Centrum Amsterdam).