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A CHARACTERIZATION OF CERTAIN SEQUENCES OF NORMING CONSTANTS[1]

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1. Introduction and summary. Let $\{Y_j\}_{j=1}^{\infty}$ be a sequence of random variables defined on a probability space (Ω, F, P), which are not necessarily independent, or identically distributed. Let $S_n = Y_1 + \cdots + Y_n$. Assume that there exists sequences of constants $\{A_n\}$, $\{B_n\}$, $B_n > 0$ such that the limit distribution of $(S_n - A_n)/B_n$ exists. For a class of limit distributions which includes the stable distributions, we give a characterization of $\{B_n\}$ in terms of the dispersion of the sequence of partial sums S_n. Such a characterization will be useful in obtaining stable limit theorems for Markov chains since it allows a description of the norming constants which is not dependent on any particular state of the state space of the Markov chain. In addition, using this characterization and that of Tucker in [6], we obtain a partial strengthening of Paul Lévy's Theorem on the Augmentation of the Dispersion.

2. Preliminary definitions and results. Let us recall the following well known:

Definition 1. For $0 \leqq \gamma \leqq 1$, the dispersion of a random variable X for the probability γ, denoted $D(\gamma, X)$ is the infimum of the lengths of the closed intervals to which X belongs with probability $\geqq \gamma$.

The next lemma shows that the infimum in the definition of the dispersion is actually assumed. This lemma corresponds to that given by Tucker in [7] for concentration functions.

Lemma 1. *Let X be a random variable and let $0 \leqq \gamma \leqq 1$ be given. Then there exists an interval $[a, b]$ such that $D(\gamma, X) = b - a$ and $P\{[a, b]\} \geqq \gamma$.*

Proof. Let $L\{I\}$ denote the length of the interval I. Let Γ be the collection of all closed intervals I such that $P\{I\} \geqq \gamma$. Let $k = \inf\{L\{I\} \mid I \varepsilon \Gamma\}$. Then $k = D(\gamma, X)$. Pick a subsequence of intervals $\{I_j\}_{j=1}^{\infty}$ from Γ such that $L\{I_j\} \to k$ and $L\{I_j\} < 2k + 1$ for $j = 1, 2, \cdots$. Let F be the distribution function of X and let a, b be chosen so that $F(a) < \gamma/4$ and $1 - F(b) < \gamma/4$. Then for all j, $I_j \subseteq [a - 2k - 1, b + 2k + 1]$. If $I_j = [a_j, b_j]$, it follows that $\{a_j\}_{j=1}^{\infty}$ has a cluster point p. Let $\{a_{1,j}\}_{j=1}^{\infty}$ be a subsequence of $\{a_j\}$ such that $a_{1,j} \to p$. Then $b_{1,j} \to p + k$. The lemma follows by observing $P\{[p, p + k]\} \geqq \gamma$.

Motivated by the preceding we are led to call the interval $[a, b]$ a γ-interval for the random variable X iff $b - a = D(\gamma, X)$ and $P\{a \leqq X \leqq b\} \geqq \gamma$. We observe that the dispersion of a random variable X for a given probability γ

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depends only on the distribution function F of X. Thus we write $D(\gamma, F)$ and $D(\gamma, X)$ interchangeably. The next lemma is well known and verifies ones expectations.

Lemma 2. *Let X be a random variable and let h, c be real constants. Then for $0 \leqq \gamma \leqq 1$ arbitrary, but fixed, $D(\gamma, cX + h) = |c|D(\gamma, X)$.*

We observe that the absolute value sign is needed in the preceding lemma since dispersion, being a length, must be positive. Finally, in order to show that the principal theorem of this paper is valid for a sufficiently large class of distributions we shall need a certain property of stable distributions which will be given in the lemma following:

Definition 2. Let F be a distribution function, and let $a = \sup \{x \mid F(x) = 0\}$, $b = \inf \{x \mid F(x) = 1\}$. Then $a \leqq b$ and $[a, b] \cap (-\infty, +\infty)$ is termed the interval of variation of F.

The property of stable distributions described in the following lemma is probably known although it does not appear to have been stated before.

Lemma 3. *A stable distribution of index α, $0 < \alpha \leqq 2$, is strictly monotone over its interval of variation. Further, for $1 \leqq \alpha \leqq 2$, the interval of variation is the entire real line.*

Proof. For $1 \leqq \alpha \leqq 2$, the result follows from the fact that a stable distribution of index α is an analytic function [3], page 183, on the entire real line, coupled with the fact that an analytic function cannot be constant over any interval unless it is identically constant. For $0 < \alpha < 1$, the result follows from the series expansion of the density function [2], page 549, since a distribution function having continuous density is constant over an interval iff the density function is identically zero over this interval.

3. The principal result:

Theorem 4. *Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of random variables and let $S_n = X_1 + \cdots + X_n$. Assume there exist sequences of constants $\{A_n\}$, $\{B_n\}$, $B_n > 0$ such that the limit distribution G of $Y_n = (S_n - A_n)/B_n$ exists and is strictly monotone over its interval of variation. Then for $0 < \gamma < 1$, arbitrary but fixed:*

$$\lim D(\gamma, Y_n) = D(\gamma, G), \tag{3.1}$$

and

$$\lim D(\gamma, S_n)/B_n = D(\gamma, G). \tag{3.2}$$

The following corollary follows immediately from a well known lemma [1], page 254.

Corollary 1. *Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of random variables and let $S_n = X_1 + \cdots + X_n$. Assume there exist sequences of constants $\{A_n\}$, $\{B_n\}$, $B_n > 0$ such that the limit distribution G of $Y_n = (S_n - A_n)/B_n$ exists and is strictly monotone over its interval of variation. Then for $0 < \gamma < 1$, arbitrary but fixed, the limit distribution of $(S_n - A_n)/D(\gamma, S_n)$ exists and is $G(xD(\gamma, G))$.*

By restricting the random variables $\{X_n\}$ in the preceding corollary to be independent and identically distributed, we have by Lemma 3:

COROLLARY 2. *Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of independent, identically distributed random variables whose distribution function F belongs to the domain of attraction of a stable law G of index α, $0 < \alpha \leqq 2$. Let $S_n = X_1 + \cdots + X_n$. Then there exists a sequence of constants $\{A_n\}$ such that for $0 < \gamma < 1$, arbitrary but fixed, the limit distribution of $(S_n - A_n)/D(\gamma, S_n)$ exists and is of the same type as G.*

PROOF OF THEOREM 4. First let us observe that (3.2) follows from (3.1) by virtue of Lemma 2. Thus we need only prove (3.1). In order to do this we need the following lemma where $d \lim Z_n = Z$ indicates that the limit in distribution of the sequence of random variables $\{Z_n\}$ is Z.

LEMMA 5. *If $d \lim Z_n = Z$, then for $0 < \gamma < 1$, arbitrary but fixed:*

$$D(\gamma, Z) \leqq \liminf D(\gamma, Z_n) \leqq \limsup D(\gamma, Z_n) \leqq \lim_{\alpha \downarrow \gamma} D(\alpha, Z). \tag{3.3}$$

PROOF. Let us first obtain the left hand inequality. Thus, let $k = \liminf D(\gamma, Z_n)$ and choose a subsequence $\{Z_{1,n}\}$ of $\{Z_n\}$ such that $\lim D(\gamma, Z_{1,n}) = k$ and $D(\gamma, Z_{1,n}) < 2k + 1$ for all n. Let F be the distribution function of Z and let a, b be continuity points of F such that $F(a) < \gamma/4$ and $1 - F(b) < \gamma/4$. Then there exists an integer N_1 such that $n > N_1$ implies $F_n(a) < \gamma/2$ and $1 - F_n(b) < \gamma/2$. For such n, it follows that all γ-intervals for $Z_{1,n}$ are contained in the interval $[a - 2k - 1, b + 2k + 1]$, which we call J. If $I_{1,n} = [a_{1,n}, b_{1,n}]$ is a γ-interval for $Z_{1,n}$ then as in Lemma 1 it follows that $\{a_{1,n}\}$ has a cluster point p in J. Again, as in the proof of Lemma 1, we obtain an interval $[p, p + k]$ having probability $\geqq \gamma$. Thus $D(\gamma, Z) \leqq k$ and the left hand inequality of the lemma follows.

Since the middle inequality of the lemma is obvious, we proceed to verify the right hand inequality. Thus, let $k = \limsup D(\gamma, Z_n)$ and pick a subsequence $\{Z_{1,n}\}$ such that $\lim D(\gamma, Z_{1,n}) = k$.

The inequality is now obtained by showing $k \leqq D(\alpha, Z)$ for each $\alpha > \gamma$. Assume that for some $\alpha_0 > \gamma$, we have $k > D(\alpha_0, Z)$. Let $[a, b]$ be an α_0-interval for Z and pick $\epsilon, r > 0$ so that $L\{[a - \epsilon, b + \epsilon]\} < r$, $r < k$ and $a - \epsilon$, $b + \epsilon$ are both continuity points for the distribution F of Z. Thus there exists $N > 0$ such that for $n > N$, $\gamma + (\alpha_0 - \gamma)/2 < F_{1,n}(b + \epsilon) - F_{1,n}(a - \epsilon)$ where $F_{1,n}$ is the distribution function of $Z_{1,n}$. Hence for such n, $D(\gamma, Z_{1,n}) < b - a + 2\epsilon < r$. Since $r < k$, this contradicts the fact that $\lim D(\gamma, Z_{1,n}) = k$. Thus the third inequality and hence the lemma. We remark that the conclusion of the preceding lemma can be given a more symmetric appearance by observing that $D(\gamma, Z) = \lim_{\alpha \uparrow \gamma} D(\alpha, Z)$.

The next lemma, coupled with the inequalities of the preceding lemma yields the theorem.

LEMMA 6. *Let F be a distribution function which is strictly monotone over its interval of variation I. Then for $0 < \gamma < 1$:*

$$\lim_{\alpha \downarrow \gamma} D(\alpha, F) = D(\gamma, F). \tag{3.4}$$

PROOF. Let $J = [a, b]$ be a γ-interval. Then $J \subseteq I$. Let $\lim_{\alpha \downarrow \gamma} D(\alpha, F) = k$

and assume that $k > b - a$. Choose $\epsilon > 0$ so that $k > b - a + 2\epsilon$. By the strict monotonicity of F over I, $P\{[a - \epsilon, b + \epsilon]\} = \pi > \gamma$. However $L\{[a - \epsilon, b + \epsilon]\} < \lim_{\alpha \downarrow \gamma} D(\alpha, F)$ which is a contradiction and yields the lemma.

4. Application. We obtain a partial strengthening of Lévy's theorem on the augmentation of the dispersion stated in [5], page 155:

THEOREM 7. *Let γ and β be given ($0 < \gamma < 1$, $0 < \beta$). Then there exists $k = k(\gamma, \beta) > 0$ and $N = N(\gamma, \beta)$ having the following property: if $n > N$, and if S is the sum of n independent random variables X_r such that the dispersion of each X_r for the probability γ is $> 2t$, then the dispersion of S for the probability β is $> ktn^{1/2}$.*

Our partial strengthening is:

THEOREM 8. *Let $\{X_j\}$ be independent, identically distributed random variables belonging to the domain of attraction of a stable law of index α, $0 < \alpha < 2$. Let $0 < \gamma < 1$ be arbitrary but fixed, and let $S_n = X_1 + \cdots + X_n$. Then given $\epsilon > 0$, there exists $N(\epsilon)$ such that for $n > N(\epsilon)$:*

$$n^{\alpha^{-1}-\epsilon} < D(\gamma, S_n) < n^{\alpha^{-1}+\epsilon}. \tag{4.1}$$

PROOF. By Corollary 2, we know $\{D(\gamma, S_n)\}$ is a sequence of norming coefficients for the partial sums S_n. Therefore, it follows from [6], page 9, that $D(\gamma, S_n) = n^{\alpha^{-1}}L(n)$ where L is a slowly varying function. However, from [4], page 45, it therefore follows that $n^{\alpha^{-1}}L(n)/n^{\alpha^{-1}-\epsilon} \to +\infty$, $n^{\alpha^{-1}}L(n)/n^{\alpha^{-1}+\epsilon} \to 0$. The theorem follows.

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