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ON CONVERGENCE IN r-MEAN OF NORMALIZED PARTIAL SUMS

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The question of strengthening Minkowski's inequality when certain conditions are imposed on the summands has been considered for example in [1], [2], [3] and [4], p. 225. In this paper we prove a related result for the case of independent and identically distributed random variables (rv); namely that the finiteness of the rth moment implies that the normalized partial sums converge in r-mean if $0 < r < 2$.

Let $X_1, X_2, \cdots, X_n$ be independent rv's with a common distribution function. Set $S_n = X_1 + \cdots + X_n$.

Theorem. *For each* $r \in (0, 2)$ *the following statements are equivalent*:

(i) $E|X_1|^r < \infty$, (*assume* $EX_1 = 0$ if $r \geqq 1$),

(ii) $n^{-1/r}S_n \rightarrow_{\text{a.s.}} 0$,

(iii) $E|S_n|^r = o(n)$,

(iv) $n^{-1/r}S_n \rightarrow_r 0$.

Proof. It is known that (i) and (ii) are equivalent ([6], p. 243). Also, (iii) and (iv) are equivalent by definition. Since it is easy to see that (iii) implies (i) it remains to prove only that (i) implies (iii).

Set $A_{ni} = [|X_i| > n^{1/r}]$, $B_{ni} = A_{ni}^c$, $p_n = P(A_{ni})$ and $q_n = 1 - p_n$. Write $Y_i = X_i I_{A_{ni}}$, $Z_i = X_i - Y_i$, $S_n' = Y_1 + \cdots + Y_n$, $S_n'' = Z_1 + \cdots + Z_n$. $M = \sum_{i=1}^n I_{A_{ni}}$ and $K = n - M$.

Let $1 \leqq r < 2$. Consider S_n' first and write

$$E|S_n'|^r = E[E(|S_n'|^r \mid M)] = E[E(|S_M{}^*|^r \mid M)] \tag{1}$$

where $S_m{}^*$ is a sum of m independent rv's which are independent of M and have a common df equal to that of the conditional df of X_1 given A_{n1}. By Minkowski's inequality

$$E[|S_m{}^*|^r \mid M] \leqq M^r E[|X_1|^r \mid A_{n1}]. \tag{2}$$

Since M is a $B(n, p_n)$ rv, it follows that $EM^r \leqq EM^2 = np_nq_n + n^2p_n^2$. Since $E|X_1|^r < \infty$ it is known ([6], p. 242) that $\sum p_n < \infty$. Consequently $np_n = o(1)$, so that $EM^r \leqq cnp_n$ for some constant c. From (1) and (2) it therefore follows that

$$E|S_n'|^r \leqq cnp_n E[|X_1|^r \mid A_{n1}] = cnE[|X_1|^r I_{A_{n1}}] = o(n). \tag{3}$$

Similarly, consider S_n'' and write

$$E|S_n''|^r = E[E(|S_n''|^r \mid K)] = E[E(|S_K{}^{**}|^r \mid K)] \tag{4}$$

where $S_k{}^{**}$ is a sum of k independent rv's which are independent of K and have a

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common df equal to that of the conditional df of X_1 given B_{n1}. Since $\{E|X|^r\}^{1/r}$ is non-decreasing in r and $r/2 \leqq 1$, one may write

$$(5) \qquad E[|S_K^{**}|^r \mid K] \leqq \{E[|S_K^{**}|^2 \mid K]\}^{r/2} = \{K\sigma_n^2 + (K\mu_n)^2\}^{r/2}$$
$$\leqq K^{r/2}\sigma_n^r + K^r\mu_n^r \leqq n^{r/2}(\sigma_n^2 + \mu_n^2)^{r/2} + (n\mu_n)^r$$

where μ_n and σ_n^2 denote the conditional mean and variance of X_1 given B_{n1}. Since $B_{n1} = [|X_1| \leqq n^{1/r}]$ we have

$$(6) \qquad \sigma_n^2 + \mu_n^2 \equiv q_n^{-1}E[X_1^2 I_{B_{n1}}] \leqq q_n^{-1}n^{2/r-1}E[|X_1|^r|n^{-1/r}X_1|^{2-r}I_{B_{n1}}].$$

Also, since $EX_1 = 0$

$$\mu_n \equiv q_n^{-1}E[X_1 I_{B_{n1}}] = -q_n^{-1}E[X_1 I_{A_{n1}}]$$

so that

$$(7) \qquad |\mu_n| \leqq q_n^{-1}E[|X_1|^r/|X_1|^{r-1}I_{A_{n1}}] \leqq q_n^{-1}n^{1/r-1}E[|X_1|^r I_{A_{n1}}].$$

From (7) it follows that $(n\mu_n)^r = o(n)$. From (6), it follows by the dominated convergence theorem that $n^{r/2}(\sigma_n^2 + \mu_n^2)^{r/2} = o(n)$ since $n^{-1/r}X_1 \rightarrow_{\text{a.s.}} 0$. These results, together with (4) and (5), imply that $E|S_n''|^r = o(n)$.

For $0 < r < 1$ the proof is simpler. First of all, by the c_r-inequality ([6], p. 155)

$$E|S_n'|^r \leqq nE[|X_1|^r I_{A_{n1}}] = o(n).$$

Secondly, since $\{E|X|^r\}^{1/r}$ is non-decreasing and since $n^{-1/r}X_1 \rightarrow_{\text{a.s.}} 0$,

$$E|S_n''|^r \leqq \{E|S_n''|\}^r \leqq n^r\{E[|X_1|I_{B_{n1}}]\}^r$$
$$= n\{E[|X_1|^r|n^{-1/r}X_1|^{1-r}I_{B_{n1}}]\}^r = o(n)$$

by the dominated convergence theorem.

It should be remarked that a slightly different approach would be to assume that X_1 is a symmetric rv. This assumption can be made without loss of generality since if X_1 is not symmetric it may be symmetrized in the usual way. Recall that symmetrization can only increase $E|S_n|^r$. If one uses this approach the only change would be the omission of the computation which yields (7); since $\mu_n \equiv 0$.

A consequence of the above theorem, together with Markov's inequality, is the result of Brillinger [1] that whenever $E|X_1|^r < \infty$ and $1 \leqq r < 2$, then $P[|n^{-1}S_n| > \epsilon] = o(n^{1-r})$ for all $\epsilon > 0$. (The authors are indebted to Vernon Johns for bringing reference [1] to their attention.) Another related result is that of Katz [5] who proves for example that $E|X_1|^r < \infty$ $(r \geqq 1, EX_1 = 0)$ is equivalent to the convergence of the series $\sum \{n^{r-2}P[|n^{-1}S_n| > \epsilon]\}$ for all $\epsilon > 0$. Although this result is equivalent to ours for $1 \leqq r \leqq 2$ its form gives much more explicit information about the asymptotic behavior of $P[|n^{-1}S_n| > \epsilon]$. A similar comparison holds also for $r < 1$.

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