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TRANSFORMS OF STOCHASTIC PROCESSES[1]

BY P. WARWICK MILLAR[2]

University of Illinois

0. Summary. In this note, the notion of an optimal transform of a (discrete parameter) stochastic process is introduced. Such transforms are shown to exist in certain cases, and a relationship to optimal stopping times is discussed. These ideas lead naturally to the representation of any given stochastic process as the transform of a submartingale. This type of representation theorem is extended to continuous parameter processes, where it is shown that in certain cases a quasi-martingale can be represented as a stochastic integral with respect to a submartingale.

1. Optimal transforms of stochastic processes. Let $(\Omega, \mathfrak{F}, P)$ be a probability space, and $\{\mathfrak{F}_n, n = 0, 1, \cdots\}$ an increasing sequence of sub-sigma-fields of $\mathfrak{F}$. For brevity, we will call $z = \{z_n, \mathfrak{F}_n, n = 1, 2, \cdots\}$ a *stochastic process* if for each n, z_n is a real random variable which is $\mathfrak{F}_n$ measurable; z is *integrable* if $E|z_n| < \infty$ for each n. Define $d_1 = z_1$, $d_n = z_n - z_{n-1}$ for $n \geqq 2$. If $v = \{v_n, \mathfrak{F}_{n-1}, n = 1, 2, \cdots\}$ is a stochastic process (v_n is $\mathfrak{F}_{n-1}$ measurable), define the process $v \cdot z$ by $(v \cdot z)_n = \sum_{k=1}^{n} v_k d_k$; and let $(v \cdot z)_\infty = \lim_{n\to\infty} (v \cdot z)_n$ whenever this limit exists. The process $v \cdot z = \{(v \cdot z)_n, \mathfrak{F}_n\}$ is the v-transform of z, and v is called a multiplier sequence. Such transforms have been studied recently by Burkholder [1], when the process z is a martingale or submartingale.

We consider two special classes of multiplier sequences: v will be said to belong to the class $V(0, 1)$ if $v_1 = 1$ and $0 \leqq v_k \leqq 1$ for $k > 1$; $v \varepsilon V(-1, 1)$ if $-1 \leqq v_k \leqq 1$ for all k. Let $z = \{z_n, \mathfrak{F}_n\}$ be an integrable stochastic process, and V some class of multiplier sequences. An optimal transform of z for the class V is a transform $\gamma \cdot z$, $\gamma \varepsilon V$, with the property that, for each n,

$$E(\gamma \cdot z)_n = \sup_{v \varepsilon V} E(v \cdot z)_n. \tag{1}$$

The reader may obtain an interesting gambling interpretation of (1) by perusing the introduction of [1]. As discussed below, the stopping times belong to the class $V(0, 1)$. The following proposition treats the existence of optimal transforms.

PROPOSITION 1. *Let z be an integrable stochastic process. Then optimal transforms for the classes $V(0, 1)$ and $V(-1, 1)$ exist.*

PROOF. To treat the $V(0, 1)$ case, define the process γ by $\gamma_1 = 1$, $\gamma_k = I\{E(z_k \mid \mathfrak{F}_{k-1}) > z_{k-1}\}, k > 1$. ($I\{A\}$ = indicator of A). Then γ_k is $\mathfrak{F}_{k-1}$ measurable, and $\gamma \varepsilon V(0, 1)$. If $v \varepsilon V(0, 1)$, then $E(\gamma \cdot z)_n \geqq E(v \cdot z)_n$; for, $E[(\gamma \cdot z)_n - (v \cdot z)_n]$

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[2] Now at the University of California, Berkeley.

$= E[(\gamma_1 - v_1)d_1] + \cdots + E[(\gamma_n - v_n)d_n] = E[(\gamma_1 - v_1)E(d_1 \mid \mathfrak{F}_0)] + E[(\gamma_2 - v_2)E(d_2 \mid \mathfrak{F}_1)] + \cdots + E[(\gamma_n - v_n)E(d_n \mid \mathfrak{F}_{n-1})] \geqq 0$ by the definition of γ.

To obtain the result for $V(-1, 1)$, define γ by $\gamma_1 = \operatorname{sign} Ez_1$ and for $k > 1$, $\gamma_k = 1$ on $\{E(z_k \mid \mathfrak{F}_{k-1}) > z_{k-1}\}$, and $\gamma = -1$ on $\{E(z_k \mid \mathfrak{F}_{k-1}) \leqq z_{k-1}\}$.

The following corollary is immediate.

COROLLARY 1. *The optimal* $V(0, 1)$ *and* $V(-1, 1)$ *transforms constructed in Proposition* 1 *are submartingales.*

Proposition 1 has the following relationship to the theory of optimal stopping times. Let $v \varepsilon V(0, 1)$ satisfy:

(a) $v_1 \equiv 1$ and for $k > 1$, v_k assumes only 0 and 1 for values

(b) $v_{k+1}(\omega) = 1$ implies $v_k(\omega) = 1$.

Define the stopping time τ by $\tau = \inf\{n : v_{n+1} = 0\}$, $\tau = \infty$ if there is no such n. Then one has $(v \cdot z)_n = z_{\tau \wedge n}(\tau \wedge n = \min\{\tau, n\})$, and $(v \cdot z)_\infty = z_\tau$ whenever the latter makes sense. On the other hand, if τ is any stopping time, define $v \varepsilon V(0, 1)$ by $v_k = I\{\tau \geqq k\}$; then v satisfies (a) and (b), and $(v \cdot z)_n = z_{\tau \wedge n}$. Stopping times may therefore be regarded as elements of $V(0, 1)$, and one can now inquire when the optimal sequence $\gamma \varepsilon V(0, 1)$ constructed in Proposition 1 corresponds to a stopping time. Denote by B_n the set $\{E(z_{n+1} \mid \mathfrak{F}_n) \leqq z_n\}$. The definition of γ together with the above remarks imply that γ corresponds to a stopping time if and only if $(M_1) B_n \subset B_{n+1}$ for every n. The stopping time σ corresponding to this γ is then $\sigma = \inf\{n : \gamma_{n+1} = 0\} = \inf\{n : E(z_{n+1} \mid \mathfrak{F}_n) \leqq z_n\}$. If one also has $(M_2) \bigcup_n B_n = \Omega$, then $P\{\sigma < \infty\} = 1$. (The conditions M_1 and M_2 are termed the "monotone case," and σ the "conservative" stopping time by Chow and Robbins [2].) If z satisfies M_1 and if $v \varepsilon V(0, 1)$ (in particular, if v corresponds to a stopping time τ) then Proposition 1 yields

$$E(v \cdot z)_n \leqq Ez_{\sigma \wedge n} (Ez_{\tau \wedge n} \leqq Ez_{\sigma \wedge n}) \quad \text{for every} \quad n. \tag{2}$$

In the presence of regularity conditions, one hopes that (2) will yield $E(v \cdot z)_\infty \leqq Ez_\sigma (Ez_\tau \leqq Ez_\sigma)$ whenever the integrands make sense.

As an illustration, assume z is an integrable stochastic process in the monotone case; that $\liminf \int_{\{\sigma > n\}} z_n^+ = 0$; and that τ is a finite valued stopping time such that $\liminf \int_{\{\tau > n\}} z_n^- = 0$. Assume z_σ and z_τ are both integrable; then $Ez_\tau \leqq Ez_\sigma$. This theorem is established by Chow and Robbins [3], under a weaker definition of integrability. To establish the result, let $\epsilon > 0$. Since

$$\begin{aligned} \int z_{\sigma \wedge n} = \sum_{k=1}^{n} \int_{\{\sigma = k\}} z_k + \int_{\{\sigma > n\}} z_n &= \int z_\sigma - \int_{\{\sigma > n\}} z_\sigma + \int_{\{\sigma > n\}} z_n \\ &\leqq \int z_\sigma - \int_{\{\sigma > n\}} z_\sigma + \int_{\{\sigma > n\}} z_n^+ \leqq \int z_\sigma + \epsilon \quad \text{i.o.}, \end{aligned} \tag{3}$$

and since $\int z_{\sigma \wedge n}$ increases with n (Corollary 1), it follows that $\int z_\sigma \geqq \int z_{\sigma \wedge n}$ for every n. With a similar computation one establishes that $\int z_\tau \leqq \int z_{\tau \wedge n} + \epsilon$ i.o., so that $\int z_\tau \leqq \int z_{\tau \wedge n} + \epsilon \leqq \int z_{\sigma \wedge n} + \epsilon \leqq \int z_\sigma + \epsilon$, establishing the result.

We conclude this section with the following simple "representation theorem."

PROPOSITION 2. *Let* z *be any integrable stochastic process. Then there exists a submartingale* $m = \{m_n, \mathfrak{F}_n\}$ *and a multiplier sequence* v *such that* $z = v \cdot m$.

PROOF. Let γ be the multiplier sequence, constructed in Proposition 1, which gives an optimal $V(-1, 1)$ transform for z. Let the submartingale m be given by $m = \gamma \cdot z$ (Corollary 1). It is readily verified that $\gamma \cdot m = \gamma \cdot [\gamma \cdot z] = \gamma^2 \cdot z = z$.

This representation provides a connecting link between the (sub)martingale transform theory of Burkholder and a general theory of stochastic processes. It also provides motivation for the related result on quasi-martingales given in the following section.

2. A representation theorem for quasi-martingales. We will show that, in certain cases of frequent occurrence, a quasi-martingale may be regarded as a stochastic integral with respect to a submartingale. Our terminology will follow as closely as possible that of [4], to which the reader is referred for the meaning of terms not defined below.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, and $\{\mathfrak{F}_t, t \in R_+\}$ an increasing, right continuous family of sub-sigma-fields of $\mathfrak{F}$. We suppose that $\mathfrak{F}_0$ contains all null sets, and that $\{\mathfrak{F}_t\}$ has no time of discontinuity: i.e., if $\{\tau_n, n = 1, 2, \cdots\}$ is an increasing sequence of stopping times, then $\mathfrak{F}_{(\lim \tau_n)} = \vee_n \mathfrak{F}_{\tau_n}$. Let $X = \{X_t, t \in R_+\}$ be a stochastic process adapted to the family $\{\mathfrak{F}_t\}: X_t$ is $\mathfrak{F}_t$ measurable. We will assume in this section that all processes vanish at 0. Let $\mathfrak{I}(\mathscr{I})$ be the σ-field on $R_+ \times \Omega$ generated by the adapted processes having right continuous paths and left limits; $\mathfrak{I}(\mathscr{I}')$ the σ-field on $R_+ \times \Omega$ generated by stochastic intervals $[\sigma, \tau]$ where σ, τ are stopping times and σ is accessible; and $\mathfrak{I}(\mathscr{I}'')$ the σ-field generated by the adapted processes having left continuous paths. Then $\mathfrak{I}(\mathscr{I}) \supset \mathfrak{I}(\mathscr{I}') \supset \mathfrak{I}(\mathscr{I}'')$, and $\mathfrak{I}(\mathscr{I}') = \mathfrak{I}(\mathscr{I}'')$ if $\{\mathfrak{F}_t\}$ is free of times of discontinuity (see Meyer's theorems in ([5], ch. VIII, Section 2 and chapter VII, Theorem 45). If the mapping $(t, \omega) \rightarrow X_t(\omega)$ is measurable with respect to $\mathfrak{I}(\mathscr{I})$, then X is called "well-measurable" (WM); if it is measurable with respect to $\mathfrak{I}(\mathscr{I}'')$, X is called "very well measurable" (VWM).

An adapted, right continuous process $M = \{M_t, t \in R_+\}$ is a local martingale if there exists an increasing sequence of stopping times $\{\tau_n\}$ such that $\lim \tau_n = \infty$ and the processes $M^n = \{M_{t \wedge \tau_n}\}$ are uniformly integrable martingales. Stochastic integrals of the form $Y_t = \int_0^t v_s \, dM_s$ have been studied recently in [6] and in [7] for arbitrary right continuous martingales and local martingales. If $v = \{v_t\}$ is a bounded VWM process, and M is a martingale in L_p for some $p > 1$, then the processes $Y = \{Y_t\}$ is also a martingale in L_p ([7], Section 8). (Actually, Y was shown in [7] to be a martingale for all bounded processes v in the closure of the step functions under the norm n_p (defined in [7]); that the bounded VWM processes belong in this category may be deduced using an argument found, for example, in Theorem 2 of [3]). If v is VWM and M is a local martingale, then Y is also a local martingale ([6], part II). Let now $W = \{W_t, \mathfrak{F}_t, t \in R_+\}$ be a submartingale of the class DL. Then according to the Meyer decomposition theorem ([5], chapter VII, T31) W may be written uniquely in the form $W_t = M_t + A_t$, where $M = \{M_t\}$ is a martingale, and $A = \{A_t\}$ is a natural increasing process. It is then reasonable to define $\int_0^t v_s \, dW_s = Q_t$ by

$$Q_t = \int_0^t v_s \, dM_s + \int_0^t v_s \, dA_s \tag{4}$$

whenever both integrals are defined. If $v = \{v_t\}$ is bounded and VWM, then the first term on the right is a martingale (or local martingale) and the second term is a process whose paths are of bounded variation.

A quasi-martingale (see [4], [6]) is a process $Q = \{Q_t\}$ having a decomposition $Q_t = N_t' + C_t'$, where $N' = \{N_t'\}$ is a martingale (or local martingale) and $C' = \{C_t'\}$ is a process of the form $C_t' = B_t^1 - B_t^2$, where $\{B_t^i\}$ is an increasing, right continuous process, integrable and adapted to the family $\{\mathfrak{F}_t\}$. We assume $N_0' = B_0^i = 0$. From the Meyer decomposition theorem, B^i may be written as the sum of a martingale and a natural increasing process, so that every quasi-martingale has a unique decomposition

$$Q_t = N_t + C_t \tag{5}$$

where $\{N_t\}$ is a martingale and $C_t = A_t^1 - A_t^2$ is the difference of two natural increasing processes. The preceding paragraph shows: Submartingale integrals $\{\int_0^t v_s\, dW_s\}$ are quasi-martingales when $v = \{v_t\}$ is VWM and W is a submartingale of class DL. The remainder of this section will establish Proposition 3; the parallel with Proposition 2 is clear.

PROPOSITION 3. *Let $\{\mathfrak{F}_t\}$ be free of times of discontinuity. Let $Q = \{Q_t\}$ be a quasi-martingale with canonical decomposition* (5); *assume that $N = \{N_t\}$ is a martingale in L_p for some $p > 1$. Then Q may be represented as a submartingale integral: $Q_t = \int_0^t v_s\, dW_s$, where $v = \{v_t\}$ is* VWM *and $W = \{W_t\}$ is a submartingale.*

REMARK. It will be clear from the proof that an analogous theorem can be obtained assuming that N is only locally in L_p. The submartingale W of the proposition will have decomposition $W_t = M_t + A_t$, with $M = \{M_t\}$ a martingale in L_p.

PROOF. The argument of ([6], I, Proposition 1) establishes the existence of a WM process $v = \{v_t\}$ which assumes only ± 1 as values, and such that

$$\int_0^t v_s\, dC_s = \int_0^t |dC_s|. \tag{6}$$

Set $A_t = \int_0^t v_s\, dC_s$. We will prove

(a) v may be assumed VWM.

(b) $A = \{A_t\}$ is a natural increasing process.

Supposing for the moment that (a) and (b) are true, one completes the proof as follows. Define the submartingale $W = \{W_t\}$ by $W_t = M_t + A_t$, where $M_t = \int_0^t v_s\, dN_s$, $A_t = \int_0^t v_s\, dC_s$. Since v is VWM, $M = \{M_t\}$ is a martingale (in L_p); and by (b), A is natural. Thus W is already given in its unique Meyer decomposition. We therefore may compute, according to the definition (4):

$$\begin{aligned} \int_0^t v_s\, dW_s &= \int_0^t v_s\, dM_s + \int_0^t v_s\, dA_s \\ &= \int_0^t v_s^2\, dN_s + \int_0^t v_s^2\, dC_s \\ &= N_t + C_t = Q_t . \end{aligned}$$

To prove (a), recall that ([5], chapter VIII, T20) there exists a VWM process $v' = \{v_t'\}$ such that: (i) for almost all ω $v_t(\omega) = v_t'(\omega)$, except for at most countably many values of t; and, (ii) $v_T = v_T'$ a.s. for every accessible stopping time T. Write $C_t = C_t^c + C_t^d$, where $\{C_t^c\}$, $\{C_t^d\}$ are respectively the continuous and discontinuous parts of $\{C_t\}$ (see [5], chapter VII, 10). Then (6) becomes

$$\int_0^t |dC_s| = \int_0^t v_s\, dC_s^c + \int_0^t v_s\, dC_s^d. \tag{7}$$

Whenever ω is not in an exceptional null set, (i) and the continuity of $\{C_t^c\}$ yield

$$\int_0^t v_s\, dC_s^c = \int_0^t v_s'\, dC_s^c. \tag{8}$$

Let $\{T_n\}$ be a sequence of stopping times which enumerate the jumps of $\{C_s^d\}$. (Such a sequence is easily constructed.) Since the jumps of C^d are the jumps of A^1 and A^2, both of which are natural, it follows that the stopping times $\{T_n\}$ are accessible ([5], VII, T49). Therefore, for any t,

$$\left|\int_0^t v_s\, dC_s^d - \int_0^t v_s'\, dC_s^d\right| \leqq \int_0^\infty |v_s - v_s'|\, |dC_s^d| = \sum_n |v_{T_n} - v_{T_n}'|\, |C_{T_n} - C_{T_n^-}|.$$

Since the T_n are accessible, (ii) implies that the last sum is a.s. zero. Hence, for ω not in some exceptional null set

$$\int_0^t v_s\, dC_s^d = \int_0^t v_s'\, dC_s^d \quad \text{for all} \quad t. \tag{9}$$

Combining (7), (8) and (9), one obtains (a).

Since we assume $\{\mathfrak{F}_t\}$ free of times of discontinuity, (b) is an easy consequence of ([5], VII, T49).

REMARKS. (A) The statement (b) can be derived directly from (a), without using the assumption that $\{\mathfrak{F}_t\}$ is free of times of discontinuity. Hence, should one desire to establish Proposition 3 without this assumption, it would be enough to verify (a).

(B) Let $Q = \{Q_t\}$, $Q_t = N_t + C_t$ be a quasi-martingale such that: both N and C have a.s. continuous paths, N is a local martingale, and C is as in (5). We do not assume $\{\mathfrak{F}_t\}$ free of times of discontinuity. Then $Q_t = \int_0^t v_s\, dW_s$, where v is WM, and $W_t = M_t + A_t$, M a local martingale, and A natural. To prove this, it suffices (by a standard stopping time argument) to consider the case when N is a martingale in L_p for some $p > 1$. If v is the WM process at the beginning of the proof of Proposition 3, then $\int_0^t v_s\, dC_s$ is continuous, hence natural. Since N has a.s. continuous paths, the process $\{S^2(N)_t\}$ has a.s. continuous paths and so v is in the closure of the left continuous step functions under the n_p norm (see [7], Section 8 for explanation of this terminology). Therefore, the integral $\int_0^t v_s\, dN_s$ is defined, and the proof can proceed much as before.

REFERENCES

[1] BURKHOLDER, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1505.

[2] CHOW, Y. S. and ROBBINS, H. (1963). A martingale system theorem and applications *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **1** 93–104.

[3] COURRÈGE, P. (1963). Intégrales stochastiques et martingales de carré intégrable. Seminaire de Théorie du Potentiel, Brelot-Choquet-Deny. Institut Henri Poincaré, Paris, 7e année, 1962–63.

[4] FISK, D. L. (1965). Quasi-martingales. *Trans. Amer. Math. Soc.* **120** 369–405.

[5] MEYER, P. A. (1966). *Probability and Potentials.* Blaisdell, Waltham.

[6] MEYER, P. A. (1967). Intégrales Stochastiques I, II, III, IV. *Séminaire de Probabilités I, Université de Strasbourg,* 72–162. Springer-Verlag, Heidelberg.

[7] MILLAR, P. W. (1967). Martingale integrals. (To appear in the *Transactions of the Amer. Math. Soc.*)