The Annals of Mathematical Statistics
1968, Vol. 39, No. 2, 365–371

A REMARK ON HITTING PLACES FOR TRANSIENT STABLE PROCESS[1]

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1. Introduction. In this note we will consider a non-degenerate, drift free, d-dimensional stable process $X(t)$, having exponent α, $0 < \alpha \leqq 2$, and transition density $p(t, x)$ satisfying the scaling property

$$p(rt, r^{1/\alpha}x)r^{d/\alpha} = p(t, x). \tag{1.1}$$

Thus if $\alpha = 1$ the process must be isotropic, while if $\alpha \neq 1$ the process is arbitrary. In addition, we assume that $X(t)$ is a version of the process which is a standard Markov process. (See [1] for a description of a standard process.)

In [3] Taylor states the following:

Theorem 1. *Assume* $p(1, 0) > 0$. *Then* $p(t, x) > 0$ *for all* $t > 0$ *and all* $x \varepsilon R^d$ (*d-dimensional Euclidean space*).

The proof given by Taylor seems incomplete. Our first result will be to present a complete proof of this useful fact. It is a pleasure for me to thank J. Folkman for helpful conversations on this matter. In fact, by using induction on the dimension, and a much more refined version of the arguments used to prove this theorem, Folkman was able to demonstrate the following much more precise fact:

$$\{y : p(t, y) > 0 \quad \text{for some} \quad t\} = \{y : p(t, y) > 0 \quad \text{for all} \quad t > 0\}.$$

Suppose now, in addition to the assumptions made above, that $\alpha < d$ so that $X(t)$ is a transient process.

Let B be a bounded Borel subset of R^d, and let $T_B = \inf\{t > 0 : X(t) \varepsilon B\}$ ($= \infty$ if $X(t) \not\varepsilon B$ for all $t > 0$) denote the first hitting time of B. Recall that B is said to be polar if $P_x(T_B < \infty) \equiv 0$. Let $\hat{T}_B$ denote the quantity T_B for the dual process $-X(t)$. The potential kernel of the process $X(t)$ is the quantity

$$g(x) = \int_0^\infty p(t, x)\, dt,$$

while the kernel for the dual process is $\hat{g}(x) = g(-x)$. For an event A, let $P_x(A)$ denote the probability of A given $X(0) = x$, and for a random variable Z, let $E_x[Z; A] = \int_A Z(w)P_x(dw)$.

The main purpose of this note is to establish the following:

Theorem 2. *Suppose* $\alpha < d$ *and that* $p(1, 0) > 0$. *Let B be a bounded Borel set, and let f be an arbitrary continuous function on the closure* $\bar{B}$ *of B. If the kernel* $g(x) < \infty$, $|x| = 1$, *and is such that for any compact* A,

$$\lim_{|x|\to\infty} \sup_{y \varepsilon A} g(x + y)g(x)^{-1} = 1, \tag{1.2}$$

Received 18 October 1967.
[1] Research supported by NSF GP-8049.

then there is a unique bounded measure π_B, *having support on* $\bar{B}$, *such that*

$$(1.3)\qquad \lim_{|x|\to\infty} E_x[f(X(T_B));\, T_B < \infty]g(-x)^{-1} = \int_{\bar{B}} f(z)\pi_B(dz).$$

Moreover, the measure π_B *is also the unique measure having support on* $\bar{B}$ *such that*

$$(1.4)\qquad P_x(\hat{T}_B < \infty) = \int_{\bar{B}} g(x-y)\pi_B(dy).$$

Equation (1.4) shows that the measure π_B is the dual (co) capacitory measure of B. Applied to the dual process $-X(t)$, Theorem 2 shows that if (1.2) holds, then there is a unique bounded measure μ_B having support on $\bar{B}$ (the capacitory measure of B) such that

$$(1.5)\qquad \lim_{|x|\to\infty} E_x[f(X(\hat{T}_B));\, \hat{T}_B < \infty]g(x)^{-1} = \int_{\bar{B}} f(z)\mu_B(dz),$$

and

$$(1.6)\qquad P_x(T_B < \infty) = \int_{\bar{B}} g(y-x)\mu_B(dy).$$

Applying (1.3) to the function $f \equiv 1$, we see that under condition (1.2),

$$\lim_{|x|\to\infty} g(-x)^{-1}P_x(T_B < \infty) = \pi_B(\bar{B}),$$

while (1.6) shows that

$$\lim_{|x|\to\infty} g(-x)^{-1}P_x(T_B < \infty) = \mu_B(\bar{B}).$$

Thus we derive directly for our processes the known fact that $\mu_B(\bar{B}) = \pi_B(\bar{B})$. The common total mass of these measures, $C(B)$, is called the capacity of B. Clearly B is polar iff $C(B) = 0$. If B is not polar, then in view of (1.6) we may rewrite (1.3) as

$$(1.7)\qquad \lim_{|x|\to\infty} E_x[f(X(T_B)) \mid T_B < \infty] = \int_{\bar{B}} f(z)\pi_B(dz)/C(B).$$

Now it is a fundamental fact of probabilistic potential theory (see chapter 6 of [1]) that representations (1.4) and (1.6) hold for a large variety of Markov processes, including all transient stable processes. The main point of Theorem 2 is that for those transient stable processes satisfying (1.2), the co-capacitory (respectively capacitory) measure of a non-polar set has the intuitively appealing interpretation as the conditional hitting (respectively co-hitting) measure at ∞.

For isotropic processes, $p(1, 0) > 0$, and $g(x)$ is a multiple of $|x|^{\alpha-d}$, and thus (1.2) holds for all such processes. Previously, (Theorem 2 of [2]) we deduced that (1.3) held for compact B in the isotropic case by an argument which used the symmetry in an essential way (so it could not be carried over to processes just satisfying (1.2)), and also used (1.4) and several other key properties of the capacitory potential. The proof we will give here is far more elementary in nature, and we will deduce (1.4) directly during the course of the argument, thereby providing a new proof of that basic result. As will be seen below, (1.2) holds for all one dimensional processes with $\alpha < 1$ which are not one-sided, so there are certainly non-isotropic processes to which the theorem applies.

Since (as pointed out by Taylor [3]) the set

$$K = \{x : p(t, x) > 0 \quad \text{for some} \quad t > 0\}$$

is a convex cone with vertex at 0, it is clear that (1.2) can hold only if $K = R^d$, (i.e. if $p(1, 0) > 0$). It follows from the scaling property that

$$g(x) = |x|^{\alpha-d} g(x/|x|), \tag{1.8}$$

and thus under the assumptions of Theorem 2, $g(x) < \infty$, $x \neq 0$.

In one dimension the unit sphere consists of the two points ± 1, and it easily follows that (1.2) holds for all one dimensional processes such that $p(1, 0) > 0$, (i.e. we exclude only the one-sided processes). In fact, one can explicitly compute $g(x)$ for all one dimensional processes with $\alpha < 1$.

PROPOSITION 1. *Assume $X(t)$ is a one dimensional stable process with exponent $\alpha < 1$ having* log *characteristic function* $-t\,|\theta|^\alpha\,(1 + ih\,\mathrm{sgn}\,(\theta))$, *where* $h = \beta \tan(\frac{1}{2}\pi\alpha)$, $|\beta| \leqq 1$. *Then*

$$g(x) = [1 - \beta\,\mathrm{sgn}\,(x)][2\Gamma(\alpha)\cos(\tfrac{1}{2}\pi\alpha)]^{-1}\,|x|^{\alpha-1}. \tag{1.9}$$

For $d > 1$ matters seem to be more difficult. All that is readily deducible about $g(x)$ (see below) is that it is lower-semi continuous and finite a.e. In [3] Taylor asserts without proof (he comments that the proof is routine) that $g(x)$ is continuous on the unit sphere. However, the author has been unable to show this, and the matter seems entirely non-trivial. Our interest in Taylor's fact is that if $p(1, 0) > 0$ and $g(x)$ is continuous on the unit sphere, then Theorem 1 and a simple argument shows that (1.2) holds. Thus if Taylor's assertion is correct, Theorem 2 holds for all processes with $p(1, 0) > 0$ and $\alpha < d$, $\alpha \neq 1$. As to the the nature of the difficulty here note that (1.1) shows

$$p(t, x) = t^{-d/\alpha} p(1, xt^{-1/\alpha}), \tag{1.10}$$

and as $p(1, x)$ is bounded and continuous, it follows that for any $\delta > 0$, $\int_\delta^\infty p(t, x)\,dt$ is bounded and continuous in x. Thus the continuity properties of $g(x)$ depend on how $p(t, x)$ behaves for small t, and via (1.10) how $p(1, x)$ behaves for large $|x|$. In the isotropic case, we know that $p(1, x)|x|^{\alpha+d} \to C$, $|x| \to \infty$ for some constant C. To establish Taylor's result we need far less. If we could show that for $x = r\xi$, $|\xi| = 1$,

$$\sup_{|\xi|=1} p(1, r\xi) r^d \leqq \text{const},$$

the desired result would follow. Although this is extremely plausible, we have been unable to prove (or disprove) it.

ADDED IN PROOF. Pruitt and Taylor have now found counter examples $g(x)$ on the sphere. In these examples $g(x)$ is in finite at certain points on the sphere.

We conclude by pointing out that Theorem 2 and an obvious modification of the argument used to establish Theorem 3 of [2] shows that the following holds.

THEOREM 3. *Assume the conditions of Theorem 2 are satisfied and let π_B be as in*

Theorem 2. *Let B be a non-polar, bounded Borel set. Then for any continuous function f on $\bar{B}$,*

$$(1.11)\quad \lim_{t\to\infty} t^{(d/\alpha-1)} \int_{\bar{B}} P_x(t < T_B < \infty; X(T_B) \in dy) f(y) = p(1,0)[d/\alpha - 1]^{-1} P_x(T_B = \infty) \int_{\bar{B}} f(y)\pi_B(dy).$$

2. Proof of Theorem 1. The scaling property (1.1) shows that it suffices to prove $p(1, x) > 0$ for all $x \in R^d$. Let

$$\mathcal{S}(t) = \{x \in R^d : p(t, x) > 0\}.$$

If

$$\mathcal{S}(t) + \mathcal{S}(r) = \{x + y : x \in \mathcal{S}(r), y \in \mathcal{S}(t)\},$$

then it follows from the semi-group property of the transition density that

$$(2.1)\qquad \mathcal{S}(t + r) = \mathcal{S}(t) + \mathcal{S}(r),$$

while the scaling property shows that

$$(2.2)\qquad \mathcal{S}(rt) = r^{1/\alpha}\mathcal{S}(t).$$

Set $\mathcal{S}(1) = \mathcal{S}$. Then (2.1) and (2.2) show that

$$r^{1/\alpha}\mathcal{S} + t^{1/\alpha}\mathcal{S} = \mathcal{S}(r) + \mathcal{S}(t) = \mathcal{S}(r + t) = (r + t)^{1/\alpha}\mathcal{S},$$

and, in particular, for $0 \leqq r \leqq 1$ we see that

$$(2.3)\qquad r^{1/\alpha}\mathcal{S} + (1 - r)^{1/\alpha}\mathcal{S} = \mathcal{S}.$$

Suppose $\alpha > 1$. Then (2.3) shows that if $x \in \mathcal{S}$, then so does $r^{1/\alpha}x + (1 - r)^{1/\alpha}x$, $0 \leqq r \leqq 1$. Hence if $x \in \mathcal{S}$, then so does the segment λx where $1 \leqq \lambda \leqq \lambda_0$, where λ_0 is some number >1. Consequently, $x \in \mathcal{S}$ implies $\lambda x \in \mathcal{S}$ for all $\lambda \geqq 1$. But as $p(1, x)$ is continuous at $x = 0$ and $p(1, 0) > 0$, $\mathcal{S}$ must contain a sphere of center 0, and thus $\mathcal{S} = R^d$.

The case when $\alpha < 1$ is a bit more complicated. In this case (2.3) shows that if $x \in \mathcal{S}$, then so does the point λx, where $\lambda_0 \leqq \lambda \leqq 1$ for some $\lambda_0 < 1$, and consequently, the entire segment $\lambda x \in \mathcal{S}$, where $0 < \lambda \leqq 1$. Since $\mathcal{S}$ contains a sphere of center 0, $\mathcal{S}$ is star shaped from 0. From (2.3) we see that if $x, y \in \mathcal{S}$, then so does the arc $r^{1/\alpha}x + (1 - r)^{1/\alpha}y$, $0 \leqq r \leqq 1$. Note that $\mathcal{S} + \mathcal{S} = 2^{1/\alpha}\mathcal{S}$ and $2^{1/\alpha} > 2$. These two facts show that either $\mathcal{S} = R^d$ or $\mathcal{S}$ is contained in a half space. To see that this latter alternative cannot be the case, we proceed by contradiction. If $\mathcal{S}$ is contained in a half space, then let D be a bounding hyperplane of this half space. There are then points of $x \in \mathcal{S}$ arbitrarily near D, and there must also be points $y, z \in \mathcal{S}$ such that $2^{1/\alpha}x = y + z$. Since $2^{1/\alpha} > 2$ we see that this is impossible for x close enough to D. Thus $\mathcal{S} = R^d$.

Finally, if $\alpha = 1$, then the process must be isotropic to satisfy the scaling relation (1.1). But then, if $f(t, u)$ is the density of the stable subordinator of exponent

$\frac{1}{2}$, it is well-known that

$$p(1, x) = \int_0^\infty f(1, u)(4\pi u)^{-d/2} \exp(-|x|^2/4u)\, du,$$

and thus $p(1, x) > 0$ for all $x \varepsilon R^d$. This completes the proof.

3. Proof of Theorem 2. Let B be a non-polar relatively compact Borel set. The basic first passage relation states that for any Borel set A,

$$\int_A [p(t, y - x) - \int_0^t \int_{\bar B} P_x(T_B \varepsilon ds, X(s) \varepsilon dz) p(t - s, y - z)]\, dy = P_x(T_B > t, X(t) \varepsilon A).$$

It easily follows from this that the measure $P_x(T_B > t, X(t) \varepsilon dy)$ has an upper-semi-continuous density $g_B(t, x, y)$ satisfying the relation

$$p(t, y - x) - \int_0^t \int_{\bar B} P_x(T_B \varepsilon ds, X(s) \varepsilon dz) p(t - s, y - z) = g_B(t, x, y) \tag{3.1}$$

Set

$$g(x) = \int_0^\infty p(t, x)\, dt, \qquad g_B(x, y) = \int_0^\infty g_B(t, x, y)\, dt,$$

$$\text{and} \qquad H_B(x, dz) = P_x(X(T_B) \varepsilon dz;\, T_B < \infty).$$

We then obtain from (3.1) and our assumptions on g that for $x \neq y$

$$g(y - x) - \int_{\bar B} H_B(x, dz) g(y - z) = g_B(x, y). \tag{3.2}$$

Since (1.2) holds, we see that

$$\lim_{|y| \to \infty} g_B(x, y)/g(y) = P_x(T_B = \infty). \tag{3.3}$$

The basic duality relation of Hunt (see Section 1 of Chapter 6 of [1]) leads easily to the conclusion

$$g_B(x, y) = \hat g_B(y, x), \tag{3.4}$$

where $\hat g_B$ is the quantity g_B for the dual process $-X(t)$.

Let K be an arbitrary compact set. From (3.2) we see that

$$\int_K g(y - x)[g(-x)]^{-1}\, dy - \int_{\bar B} H_B(x, dz)[g(-x)]^{-1} \int_K g(y - z)\, dy = \int_K g_B(x, y)[g(-x)]^{-1}\, dy\,. \tag{3.5}$$

But

$$\limsup_{|x| \to \infty} \sup_{y \varepsilon K} g_B(x, y)[g(-x)]^{-1} \leqq \lim_{|x| \to \infty} \sup_{y \varepsilon K} g(y - x)[y(-x)]^{-1} = 1,$$

and thus (3.3)–(3.5) and dominated convergence show that

$$\lim_{|x| \to \infty} \int_{\bar B} H_B(x, dz)[g(-x)]^{-1} \int_K g(y - z)\, dy = \int_K P_y(\hat T_B < \infty)\, dy. \tag{3.6}$$

Now it easily follows from the fact that $p(t, x)$ is bounded and continuous in x that $\int_K g(y - x)\, dy$ is also continuous. Since $p(1, 0) > 0$, we see by Theorem 1 that $g(x) > 0$, and thus if K has non-zero Lebesgue measure,

$$\delta = \inf_{z \varepsilon \bar B} \int_K g(y - z)\, dy > 0.$$

Hence by (3.6),

$$\limsup_{|x|\to\infty} H_B(x, \bar{B})/g(-x) \leqq \delta^{-1} \int_K P_y(\hat{T}_B < \infty)\, dy < \infty.$$

Consequently, there is a bounded measure π_B having support on $\bar{B}$, such that for some sequence x_n, $|x_n| \to \infty$,

$$H_B(x_n, dz)/g(-x_n) \to \pi_B(dz)$$

weakly. From (3.6) we then obtain that

$$\int_{\bar{B}} \pi_B(dz) \int_K g(y - z)\, dy = \int_K P_y(\hat{T}_B < \infty)\, dy.$$

Since K is an arbitrary compact, it must be that

$$\int_{\bar{B}} g(y - z)\pi_B(dz) = P_y(\hat{T}_B < \infty) \tag{3.7}$$

holds for a.e. y. But it is clear that

$$P_y(\hat{T}_B < \infty) = \lim_{t\downarrow 0} \int_{R^d} p(t, y - x)P_x(\hat{T}_B < \infty)\, dx$$

and that

$$\int_{\bar{B}} g(y - z)\pi_B(dz) = \lim_{t\downarrow 0} \int_{R^d} p(t, y - x)[\int_{\bar{B}} g(x - z)\pi_B(dz)]\, dx,$$

and thus (3.7) holds for all y. Now it is well-known in general (See chapter 6 of [1]), and quite simple to prove directly for the stable processes in particular, that (3.7) uniquely determines the measure π_B. Consequently, if there is another subsequence of the measures $H_B(x, dz)g(-x)^{-1}$ which converges weakly on $\bar{B}$, then the same argument as above shows that its limit measure must be π_B. Thus (1.3) holds. This completes the proof.

4. Proof of Proposition 1. In view of (1.8) we need only compute $g(\pm 1)$. The computation of $g(1)$ and $g(-1)$ proceed in the same way so we will only consider $g(1)$. Formally $g(1)$ may be computed as follows. Let $h = \beta \tan \frac{1}{2}\pi\alpha$. Then

$$\begin{aligned} 2\pi g(1) &= \int_0^\infty dt \operatorname{Re} \int_{-\infty}^{\infty} e^{-i\theta} \exp(-t|\theta|^\alpha(1 + ih \operatorname{sgn}(\theta))\, d\theta \\ &= \operatorname{Re} \int_{-\infty}^{\infty} d\theta/|\theta|^\alpha(1 + ih \operatorname{sgn}(\theta)), \end{aligned}$$

and further routine computations show that (1.9) holds. The hitch in the above is to justify the interchange in the order of integration, which is not completely trivial. To do this we may proceed as follows:

$$\begin{aligned} 2\pi g(1) = \int_0^\infty dt \int_{-\infty}^{\infty} e^{-t|\theta|^\alpha}[&\cos(\theta) \cos(t|\theta|^\alpha h \operatorname{sgn}(\theta)) \\ &- \sin(\theta) \sin(t|\theta|^\alpha h \operatorname{sgn}(\theta))]\, d\theta. \end{aligned} \tag{4.1}$$

Consider the first term on the right, i.e.,

$$\begin{aligned} I &= \int_0^\infty dt \int_{-\infty}^{\infty} e^{-t|\theta|^\alpha} \cos(\theta) \cos(t|\theta|^\alpha h \operatorname{sgn}(\theta))\, d\theta \\ &= 2 \int_0^\infty dt \int_0^\infty e^{-t\theta^\alpha} \cos(\theta) \cos(t\theta^\alpha h)\, d\theta. \end{aligned}$$

Since for any $\delta > 0$, $e^{-t\theta^\alpha}$ is integrable on $(\delta, \infty) \times [0, \infty)$, we see that

$$I = \lim_{\delta \downarrow 0} 2 \int_0^\infty \cos \theta(\theta)^{-\alpha}(1 + h^2)^{-1}e^{-\delta\theta^\alpha}[\cos (h\delta\theta^\alpha) - h \sin (h\delta\theta^\alpha)]\, d\theta$$
$$= 2(1 + h^2)^{-1}[\int_0^\infty \cos \theta(\theta)^{-\alpha}\, d\theta + \lim_{A\to\infty} \lim_{\delta \downarrow 0} \int_A^\infty \cos \theta(\theta)^{-\alpha}e^{-\delta\theta^\alpha}[\cos (h\delta\theta^\alpha) - h \sin (h\delta\theta^\alpha)]\, d\theta].$$

But

$$e^{-\delta\theta^\alpha}(\theta)^{-\alpha} \downarrow 0, \qquad \theta \downarrow 0,$$

and thus if we can show that

$$\int_A^T \cos (\theta)[\cos (\delta h\theta^\alpha) - h \sin (h\delta\theta^\alpha)]\, d\theta$$

is bounded in T, uniformly in δ, it would follow that

$$\lim_{A \uparrow \infty} \lim_{\delta \downarrow 0} \int_A^T = 0,$$

and the desired result would follow. If $\alpha < (n - 1)/n$, $n \geqq 2$, that may be accomplished by integrating by parts n times and examining the terms.

An analogous argument verifies the interchange for the second term on the right in (4.1). This completes the proof.

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