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ABSTRACTS OF PAPERS

(*Abstracts of papers presented at the Central Regional meeting, East Lansing, Michigan, March 18–20, 1968.*)

2. Distribution of linear combination of order statistics from the rectangular population. M. M. Ali, University of Western Ontario.

Let $X_1 < X_2 < \cdots < X_n$ be the order statistics of a random sample of size n from the Rectangular (0, 1) population. We prove that $F_n(z) = \Pr(\sum_1^n 1_i X_i \leqq z)$ is given by $F_n(z) = \sum_{\nu=0}^n H(z, a_\nu, n)/P_\nu(a_\nu)$ for all z where $a_\nu = \sum_{i=\nu}^n 1_i$, $\nu = 1, \cdots, n$, and $a_0 = 0$, $P_\nu(x) = (a_0 - x)(a_1 - x) \cdots (a_n - x)/(a_\nu - x)$ and $H(x, a_\nu, n) = \{\frac{1}{2}(z - a_\nu) + \frac{1}{2}|z - a_\nu|\}^n$. When some of the a_ν's coincide $F_n(z)$ is still well defined by taking appropriate limit of the expression for $F_n(z)$. (Received 15 January 1968.)

3. Distribution-free tests for multivariate independence, symmetry and k-sample problems. C. B. Bell and Paul Smith, Case Western Reserve University.

Let H_0': $F(x_1, \cdots, x_s) = F(t(x_1, \cdots, x_s))$ for all permutations t; and H_0'': There exist G_i such that $F(x_1, \cdots, x_s) = \prod_1^m G_i(x_{r_i}, \cdots, x_{q_i})$, where $r_i = 1 + q_{i-1}$; and $z = (x_{11}, \cdots, x_{1s}; \cdots; x_{n1}, \cdots, x_{ns})$ be the generic data point. The permutation groups under which the likelihood functions are invariant are, respectively, S' of order $(n!)(s!)^n$ and S'' of order $n1(n!)^m$. Theorem 1. *A statistic is* DF wrt $H_0'[H_0'']$ iff *it is a measurable function of some permutation statistic based on* S' $[S'']$. *In either case, its null distribution is discrete with probabilities integral multiples of the reciprocal of the order of the permutation group.* Theorem 2. *Against a simple alternative the most powerful* DF *test is a permutation test based on the alternative likelihood function.* This test is also most powerful DF for a Koopman-Pitman class generated by the alternative. For the k-sample problem $z = (x_{111}, \cdots, x_{11s}; \cdots, x_{kn_ks})$ and H_0''': $F_1(x_1, \cdots, x_s) = \cdots = F_k(x_1, \cdots, x_s)$. The permutation group S''' is of order $(n_1 + \cdots + n_k)!$. Theorems analagous to Theorems 1 and 2 are valid here. (Received 2 February 1968.)

4. On the monotonicity of $E_p(S_t/t)$. Y. S. Chow and W. J. Studden, Purdue University.

Let $S_n = X_1 + \cdots + X_n$ be the sums of independent, identically distributed random variables X_n with $P[X_n = 1] = p$ and $P[X_n = 0] = q = 1 - p$. Let t be a stopping time relative to the sequence X_n. The following theorem was conjectured by H. Robbins: Theorem. *If* $P_p[t < \infty] = 1$ *for every* $0 < p < 1$, *then* $E_p(S_t/t) \leqq E_{p'}(S_t/t)$ *for* $0 < p \leqq p' < 1$. In the proof, the Wald's equation $E_pS_t = pE_pt$ for a bounded stopping time t has been utilized. The result holds when the X_i are iid with an exponential density $C(p)e^{Q(p)x}$ with respect to some measure $d\mu$ where $Q(p)$ is increasing in p. This includes the normal and poisson distributions. (Received 29 January 1968.)

5. Some remarks on Scheffé's solution to the Behrens-Fisher problem. Morris L. Eaton, University of Chicago.

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ $(m \leqq n)$ be two independent random samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ populations respectively. In this paper, it is shown that Scheffé's proposed solution (Scheffé, H. (1943), *Ann. Math. Statist.* **13** 371-388) to the problem of testing that $\mu_1 = \mu_2$ is equivalent to the following procedure: (i) form $\bar{X}$, $\bar{Y}$ and the joint sample covariance matrix V, based on the X's and the first m Y's, (ii) construct Hotelling's T^2 statistic for a bivariate normal distribution for testing $\mu_1 = \mu_2$. It is also shown that the extended Scheffé solution for k-multivariate normal populations has a similar equivalent procedure in terms of Hotelling's T^2. This interpretation of Scheffé's solution is used to generate possible statistical procedures for testing and simultaneous inference problems in the Behrens-Fisher situation. Also some alternatives to the Scheffé solution are discussed. (Received 2 February 1968.)

6. Inadmissibility of the best invariant test when the moment is infinite under one of the hypotheses.

MARTIN FOX and S. K. PERNG, Michigan State University.

We observe (X, Y) where X is real and $Y \varepsilon \mathcal{Y}$. Under H_i $(i = 1, 2)$ assume Y is distributed according to probability measure ν_i and, given $Y = y$, the distribution function of X is $F_i(\cdot, -\theta, y)$. Lehmann and Stein (*Ann. Math. Statist.* **24** 473–479) showed that the best invariant test (BIT) is admissible when $E_{i\theta}|X| < \infty$ $(i = 1, 2)$ with an additional assumption to guarantee the BIT is unique. We present an example satisfying all these conditions except that $E_{i\theta}|X| = \infty$ for which the BIT is inadmissible. For this example we may fix $\delta > 0$ and have $E_{i\theta}|X|^{1-\delta} < \infty$. Furthermore, $\mathcal{Y}$ is the reals and the $F_i(\cdot, y)$ are all absolutely continuous with respect to Lebesgue measure. This example is an improvement over Perng's (Ph.D. thesis). (Received 9 January 1968.)

7. Finite splittings of the states of a stationary process to produce a Markov process.

MARTIN FOX and HERMAN RUBIN, Michigan State University and Mathematics Research Center, U. S. Army; Michigan State University and Purdue University.

Let $\{Y_k\}$ be a stochastic process with state space U_k at time k $(k = 1, 2, \cdots$ or $k = 0$ $\pm 1, \pm 2, \cdots)$. Let $p_n(S, \circ, T)$ be the Radon-Nikodym derivative of

$$P_n(S, \circ, T) = P((\cdots, X_{n-2}, X_{n-1}) \epsilon S, X_n \epsilon \circ, (X_{n+1}, X_{n+2}, \cdots) \epsilon T)$$

with respect to $Q_n = P_n(\cdots U_{n-2} \times U_{n-1}, \circ, U_{n+1} \times U_{n+2} \times \cdots)$. Assume the σ-field on each U_k is separable and that p_n is a regular conditional probability. Let $C_{nq} \subset U_n$ be the set on which $p_n(S, \epsilon, T) = \sum_{i=1}^q g_i(S, \epsilon) h_i(T, \epsilon)$ for measurable, essentially linearly independent g_i and h_i. We say that C_{nq} is the *set of states of rank q at time n*. This is an extension of the definition of $n(\epsilon)$ by Gilbert (*Ann. Math. Statist.* **30** 688–697). Assume also that the Radon-Nikodym derivative of $P_n(S \times \circ, \circ, T)$ with respect to $Q_{n-1} \times Q_n$ is a regular measure. The condition of Dharmadhikari (*Ann. Math. Statist.* **34** 1033–1041) under which stationary $\{Y_k\}$ can be expressed as a function of a stationary Markov process and well as Gilbert's result extend to this case. Proofs are direct extensions of those in the cited papers. (Received 7 February 1968.)

8. The behavior of some robust estimators on dependent data (preliminary report).

JOSEPH L. GASTWIRTH and HERMAN RUBIN, The Johns Hopkins University and Purdue University.

Most of the recent research on robust estimators has been devoted to developing estimators which are insensitive to outliers or "wild observations". In this paper the authors study the effect of serial dependence in the data on the efficiency of some robust estimators. When the observations are from a stationary Gaussian process obeying a uniform mixing condition, the empiric cdf is shown to approach a Gaussian process on any finite interval. From the result the asymptotic distributions of the median and the trimmed mean are seen to be normal. The asymptotic variance of the Hodges-Lehmann estimator is also obtained. On first order autoregressive normal data, the Hodges-Lehmann estimator is the most robust estimator studied. Finally, some non-normal first order autoregressive processes are examined. (Received 31 January 1968.)

9. On blocking 2^n factorials (preliminary report). DENNIS C. GILLILAND, Michigan State University.

There is a standard procedure for blocking a 2^n factorial design into 2^q blocks. The experimenter chooses a subgroup H of order 2^q of the group of factorial effects and then blocks to completely confound the elements of the subgroup. The factorial effects group is denoted by $G = \{I, A, B, AB, C, \cdots\}$ and the isomorphic group of treatments is given by $\{1, a, b, ab, c, \cdots\}$. The standard procedure is to block the treatments into the principal block (the set of all treatments having an even number of letters in common with all $X \varepsilon H$) and its cosets. This procedure completely confounds H and leaves each $X \varepsilon G - H$ estimable. We show that this standard procedure is uniquely optimal in the following sense.

PROPOSITION. *Let the treatments be blocked into 2^q non-empty sets and let $S = \{X \varepsilon G \mid X$ is non-estimable$\}$. Then S has order at least 2^q and exactly 2^q if and only if S is a subgroup of G and the blocking is into the corresponding principal block and its cosets.* (Received 19 January 1968.)

10. The resultant of a sample of points on a hypersphere. A. LAWRENCE GOULD, Research Triangle Institute.

Let an n-dimensional hypersphere have a von Mises distribution on its surface. A sample of points on the surface of this hypersphere may be regarded as a sample of vectors of constant length. The standard derivation of the resultant of the sample, and its length, is based on an argument involving discontinuous integrals first put forth by Kluyver in 1906, and subsequently by Rayleigh in 1919. By starting with the characteristic functions, the present treatment obtains in a unified way the distributions of the sample resultant, its length and, in addition, the distribution of the inner product of the sample resultant with an arbitrary known vector. A useful approximate form for one of the distributions is also given. (Received 25 January 1968.)

11. Certain optimal nonparametric tests for slippage. Z. GOVINARAJULU and H. SMITH HALLER, Case Western Reserve University and B. F. Goodrich Co.

Let population π_i have an unknown continuous cumulative distribution function $F_i(x)$, $(i = 1, 2, \cdots, c)$. The nonparametric slippage problem considered is to test the null hypothesis, $H_0 : F_1(x) = F_2(x) = \cdots = F_c(x)$ for all x against the c alternative hypotheses, H_i^c: $F_i(x) < F(x) = F_j(x)$ (or $F_j(x) = F(x) < F_i(x)$) for all $j \neq i$ and all $x, i = 1, 2, \cdots, c$. Karlin and Truax (*Ann. Math. Statist.* **31** (1960) 296–324) obtained an invariant Bayes solution to the parametric slippage problem when the joint probability density satisfies a monotonicity property. The results of Karlin and Truax (1960) for the parametric case are shown to be valid for a less restrictive class of loss functions introduced by Eaton (*Ann. Math. Statist.* **38** (1967) 124–137). Locally, an invariant Bayes solution to the nonparametric slippage problem is obtained for the class of Lehmann alternatives. Unlike the approach used by Karlin and Truax (1960) we show that the joint distribution of statistics due to Savage (*Ann. Math. Statist.* **27** (1956) 590–615) has locally the required monotonicity property. Asymptotically, when a control population is available, an invariant Bayes solution to the nonparametric slippage problem is obtained for "near" alternatives using statistics of the type introduced by Savage (1956) and Deshpande (*J. Indian Statist. Assoc.* **3** (1965) 10–29). (Received 31 January 1968.)

12. c-sample tests of homogeneity against ordered alternatives. Z. GOVINDARAJULU and H. SMITH HALLER, Case Western Reserve University and B. F. Goodrich Co.

Let population π_i have an unknown continuous cumulative distribution function $F_i(x)$, $(i = 1, 2, \cdots c)$. The problem considered is to test the null hypothesis, H_0, that the F_i's are equal against the alternative: $F_1(x) \geqq F_2(x) \geqq \cdots \geqq F_c(x)$ (or $F_1(x) \leqq F_2(x) \leqq \cdots \leqq F_c(x)$) for all x with at least one inequality strict. Nonparametric tests for this problem were proposed by Jonckheere (*Biometrika* **41** (1945) 135–145), Chacko (*Ann. Math. Statist.* **34** (1963) 945–956), and Puri (*Comm. Pure Appl. Math.* **18** (1965) 51–63). These tests are for location shift alternatives and are based on the asymptotic distribution of the statistics. For tests of H_0 against two classes of ordered alternatives of the Lehmann type, we obtain a locally most powerful rank order test. Also proposed are a class of test statistics for location shift and change of scale alternatives which are weighted sums of rank order statistics of the type introduced by Chernoff-Savage (*Ann. Math. Statist.* **29** (1958) 972–994), Bhapkar (*Ann. Math. Statist.* **32** (1961) 1108–1117), and Deshpande (*J. Indian Statist. Assoc.* **3** (1965) 20–29). These Chernoff-Savage statistics are easier to compute than and asymptotically equivalent to those proposed by Puri (1965) for equally spaced location shift alternatives. (Received 31 January 1968.)

13. On exchangeability, coherence and order statistics. BRUCE M. HILL, University of Michigan. (Invited).

A Bayesian approach to inference about the percentiles and other characteristics of a finite population is proposed. The approach need not depend upon the use of parametric models. Some related questions concerning the existence of exchangeable distributions are considered. It is shown that there are no countably additive exchangeable distributions on the space of observations which give ties probability 0 and are such that a next observation is conditionally equally likely to fall in any of the open intervals between successive order statistics of a given sample. (Received 29 January 1968.)

14. Properties of the compound multinomial distribution useful in a Bayesian analysis of categorical data from finite populations. A. BRUCE HOADLEY, Bell Telephone Labs. (Invited)

A Bayesian analysis of the parameter vector, **W** (W_j = number of elements in category j), of a multivariate hypergeometric distribution is considered. It is shown that if, a priori, **W** is compound multinomial (CMtn), then, *a posteriori*, **W** is a translated CMtn. Many properties of the CMtn distribution are derived. These include joint moments of all orders; a characterization in terms of independent compound Poisson variables, regression properties; and joint distributions of disjoint and overlapping sums of the components. It is shown that in applications related to analysis of variance and contingency tables, the parameters of interest are linear and nonlinear functions of **W**. A method is proposed for approximating the posterior distributions of such parameters. (Received 30 January 1968.)

15. Cyclic designs with the folded cubic association scheme. PETER W. M. JOHN, The University of Texas.

The folded cubic association scheme is a partially balanced association scheme for $v = 64$ with four associate classes. (John, *Ann. Math. Statist.* **38** (1967) 1311–1312). Cyclic designs are obtained for the following sets of parameters (b, d; λ_1, λ_2, λ_3, λ_4): (96, 4; 1, 0, 0, 0), (128, 4; 0, 1, 0, 0), (96, 4; 0, 0, 1, 0); (16, 4; 0, 0, 0, 1); (128, 6; 0, 2, 0, 4), (96, 6; 1, 0, 0, 3), (24, 8; 1, 0, 0, 1), (96, 8; 2, 0, 2, 4), (72, 8; 1, 0, 2, 3). (Received 26 January 1968.)

16. Cross classification models with random effects (preliminary report). LEONE Y. LOW, ARL, Wright State University.

In variance component and mixed model analysis of variance, relative efficiency of estimators may depend on the ratios of the variance components. This is the case with the estimators that are considered for the unequal numbers cross classification case. A treatment adjusted for block approach is compared with one that used subclass totals. They are the same when the incidence matrix of the design consists of zeros and ones. Although the general case is considered, designs with all cells filled are the most tractible. Incidence matrices which allow unbiased tests and contrasts of fixed effects with equal variances under permutations of treatments are investigated. (Received 25 January 1968.)

17. A useful device for continuity theorems. HERMAN RUBIN, Purdue University.

Let $\{X_n\}$ be a sequence of stochastic processes on [0, 1]. Then by the well-known Prokhorov theorem, for $\{X_n\}$ to converge in law to a process with continuous sample functions, it is necessary and sufficient that the finite dimensional distributions converge and that $\lim_{\delta\to 0} \limsup_{n\to\infty} P(\sup_{|t-u|<\delta} |X_n(t) - X_n(u)| > \epsilon) = 0$ for all $\epsilon > 0$. We show that, if for $t > u$, $X_n(t) - X_n(u) \geqq -h_n(t - u)$, h_n an increasing function for each n, that the continuity condition is satisfied if $E(|X_n(t) - X_n(u)|^\alpha) \leqq A|t - u|^{1+\beta} + B_n|t - u|$, $\alpha > 0$, $\beta > 0$, and $B_n(\log h_n^{-1}(\epsilon))^{2+\delta} \to 0$ for some $\delta > 0$. This result can be applied in many problems where otherwise more involved methods would be needed. (Received 23 January 1968.)

18. Distribution of non-central positive definite quadratic functions from a multivariate normal distribution. B. K. SHAH, Yale University.

Let X: $p \times n$ be distributed as a multivariate normal distribution with mean matrix μ: $p \times n$ and variance covariance matrix Σ. Then the density function of $S = XLX'$, L being a $n \times n$ positive definite matrix has been treated by Khatri C. G. (*Ann. Math. Statist.* **37** 468–479) for the central case. In this paper we derive the distribution of S in the central and non-central cases as a mixture of Wishart distributions. We also show that the distribution of S in the non-central case has a similar representation to that of the central case. Using these results we find that the distribution of $F = Y'(XLX')^{-1}Y$, where Y: $p \times m$ matrix is independently distributed as multivariate normal, is also the same as that of the central case (see Theorem 3, Khatri, *Ann. Math. Statist.* **37**). (Received 29 January 1968.)

19. On the distribution of rank orders from several samples and asymptotically locally most powerful rank order test statistics. CHIA KUEI TSAO, Wayne State University.

Let $\{X_{ij}\}$, $j = 1, \cdots, n_i$; $i = 1, \cdots, k$, be k independent random samples drawn from k absolutely continuous cdf's $F(y; \theta_1), \cdots, F(y; \theta_k)$ having positive pdf's $f(y; \theta_1), \cdots, f(y; \theta_k)$. Let $N = n_1 + \cdots + n_k$ and $V_1, \cdots, V_N$ be the order statistics of the combined sample. Let $Z_{ij} = 1$ if $V_j = X_{ij'}$ for some $j' = 1, \cdots, n_i$, and 0 otherwise, $i = 1, \cdots, k$. Then, the joint pdf of the $k \times N$ rank matrix $Z = \|Z_{ij}\|$ and the order-statistic vector $V = (V_1, \cdots, V_N)$ can be written as $\Pr(Z = m, V = v) = [N!/c(N; n)] \exp [L_f(v_1; \theta)C_1 +$

$\cdots + L_f(v_N\,;\,\theta)C_N]$, where $c(N; n) = N!/\pi n_i$, $m = \|C_1 \cdots C_N\|$ is a matrix with columns $C_1, \cdots, C_N$, $v = (v_1, \cdots, v_N)$ is an N-vector and $L_f(w; \theta) = (\ln f(w; \theta_1), \cdots, \ln f(w; \theta_k))$. Derived in this paper are a few explicit expressions of the marginal pdf $p(m; \theta)$ of z for a few special cases, including a generalization of formula (4.5) of Lehmann (*Ann. Math. Statist.* **24** (1952) 23) and Theorem 7.a.1 and Corollary 7.a.1 of Savage (*Ann. Math. Statist.* **27** (1956) 590). It is also shown that, under certain regularity conditions, a class of locally most powerful rank order test statistics is asymptotically equivalent to a subclass of the L-statistics proposed by Puri (*Ann. Math. Statist.* **35** (1964) 102). It is done by estimating the parameters in the linear terms of the analytic expansion of $p(m; \theta)$. (Received 18 January 1968.)

20. On the computation of normal probabilities and percentiles. JOHN S. WHITE, General Motors Corporation.

The problem of computing normal probabilities and normal percentile points to many decimal places is discussed in this paper. Specifically we wish to compute $p = F(x) = \int_{-\infty}^{x} \exp(-t^2/2)\, dt/(2\pi)^{\frac{1}{2}}$ and its inverse $x = x(p) = F^{-1}(p)$ for given x and p respectively. Several approximations for $F(x)$ and $x(p)$ are derived. A table of $x(p)$ to 20 decimal places is also given. (Received 23 January 1968.)

(Abstracts of papers presented at the Eastern Regional meeting, Blacksburg, Virginia, April 8–10, 1968.)

3. Berry-type bounds on test size. ROBERT BOHRER, Research Triangle Institute.

Based on results of Agnew ((1959), *Ann. Math. Statist.* 30 721–37) bounds are derived for the true size of normal-approximation tests based on sample means. For some examples with binomial random variables, the bounds obtained are shown to sharpen the related ones of Berry ((1941), *Trans. Am. Math. Soc.* 49 122–36) and Zolatorev ((1967), *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **8** 332–42). (Received 29 February 1968.)

4. ϵ-topothetical procedures. T. CACOULLOS, New York University.

On the basis of an observation x for a p-variate normal population $\Pi_0 : N(\mu_0, \Sigma)$ with unknown mean μ_0, we wish to decide which of the k known normal populations $\Pi_i : N(\mu_i, \Sigma)$ is closest to Π_0. As distance between Π_i and Π_j we use the Mahalanobis distance. Let m be the dimension of the linear space spanned by the points $\mu_1, \cdots, \mu_k (1 \leqq m \leqq k - 1)$. The cases $m = 1$ and $m = p - 1$ have been treated by the author (Comparing Mahalanobis distances, *Sankhya-Ser. A* (1965) 1–32). Topothetical procedures are now given for the case $1 < m < p - 1$. These within-ϵ-admissible procedures are constructed by "raising" $k - m - 1$ of the points $\mu_1, \cdots, \mu_k$ by an appropriate amount $\delta = \delta(\epsilon)$ in different dimensional directions thus creating a set of k points spanning a linear space of maximum dimensionality $m = p - 1$. (Received 19 February 1968.)

5. A contribution to a detection problem (preliminary report). KLAUS H. DANIEL, University of Maryland.

A particle is located in one of a countable number of possible states. If the particle is in state i it will be overlooked with probability a_i. A procedure is required which, for a given finite N, maximizes the probability of detecting the particle in at most N attempts. THEOREM 1. *An optimal procedure is obtained by selecting the maximal* a posteriori *probability at the kth stage that the particle is in state i given that the previous $k - 1$ attempts failed to detect the particle, $1 \leqq k \leqq N$.* THEOREM 2. *All optimal procedures are permutations* *of procedures as characterized in Theorem* 1. THEOREM 3. *For* $N \to \infty$ *an optimal procedure will stop with probability one.* (Received 16 February 1968.)

6. Some infinitely divisible discrete distributions (preliminary report). S. K. KATTI and WILLIAM WARDE, The Florida State University.

A number of discrete distributions have been developed and analyzed for their infinite divisibility using the necessary and sufficient conditions developed in Katti (*Ann. Math Statist.* **38** (1968) 1306–1308). The list of distributions includes many distributions for which characteristic functions cannot be obtained in any compact form. A set of sufficient conditions is derived which is much simpler than the necessary and sufficient conditions mentioned above. For the common distributions such as the logarithmic distribution and others, it is shown that these conditions are satisfied and hence the more involved proofs, using the necessary and sufficient conditions, are not needed. These sufficient conditions also permit construction of other discrete distributions which are infinitely divisible. We wish to note that some of the proofs are very involved and we were guided by the numerical schemes that can be used for discrete distributions. Counter examples were established when parameter values were discovered for which the necessary and sufficient conditions were not satisfied. It has been further shown that the monotone increasing function under the integral sign in the Levy-Khintchine form becomes discrete when the original distribution is discrete and that the jumps of this function are proportional to the determinants used in the necessary and sufficient conditions. (Received 12 February 1968.)

7. The non-central distribution of range and studentized range in normal samples. P. A. LACHENBRUCH and H. A. DAVID, University of North Carolina.

Let X_i ($i = 1, 2, \cdots, n$) be independent normal $N(\alpha_i, \sigma^2)$ variates and let S^2 be an independent mean square estimator of σ^2 based on ν df. Then $W' = X_{\max} - X_{\min}$ and $Q' = W'/S$ are respectively the non-central range and studentized range in normal samples. The distributions of W' and Q' for given n are not determined by $\phi = \sum \alpha_i^2/n$ alone but depend on the $n - 1$ differences $\alpha_h - \alpha_1$ ($h = 2, 3, \cdots, n$). Tables have been constructed giving the maxima and minima of the power function $\Pr\{Q' > q_\alpha \mid \phi\}$, where q_α is the upper α significance point of the (central) studentized range, for $\alpha = 0.05, 0.01$; $n = 4(2)10, 20$; $\nu = 4(2)10, 15, 20, \infty$; and various ϕ. These tables are useful in assessing the performance of the studentized range test which is used, for example, as the first step in the Newman-Keuls and Tukey multiple range procedures for the analysis of variance in the fixed effects case. (Received 8 February 1968.)

8. A bivariate Weibull distribution. LAKSHMI U. TATIKONDA, Louisiana State University.

A bivariate distribution is not determined by the knowledge of the marginal distributions. A bivariate distribution with Weibull marginals is discussed. Various properties of the regression curves are discussed. The conditional expectation of one of the variables decreases to zero with increasing values of the other variable. The conditional variance of one of the variables also decreases to zero with increasing values of the other variable. The correlation coefficient is never positive. (Received 16 February 1968.)

9. Surveillance of a Poisson production process under monotone failure rate average breakdown (preliminary report). JAGDISH K. PATEL, Louisiana State University.

Consider a Poisson production process $x(t)$ which produces output in a continuous stream when it is not in the repair state. The production process is such that at any instant of

time it may go to the next state or may breakdown independently. The breakdown is assumed to be a non-negative continuous random variable having monotone failure rate average distribution. It is also assumed that the process can be observed without cost at all times of production and the errorless observations become available immediately. When $x(t) = x$, the income per unit of time is $i(x)$. The cost of repair depends on how the process comes to stop. The basic block of time is a cycle, the time from beginning production after repair until the recurrence of that event. The problem is to formulate a strategy which specifies when to stop production and start repairs. An expression for the long run average income per unit of time is obtained and it is shown that under certain conditions there exists an optimal strategy. Some special cases and comparisons are considered. (Received 16 February 1968.)

10. A characterization of the ϵ, δ-topology in random normed spaces. B. J. PROCHASKA, Clemson University.

Serstnev [*Dokl. Akad. Nauk. SSSR* **149** (1963) 280–283] introduces the concept of a random normed space and its "natural" topology, called the ϵ, δ-topology. Let (L, f, μ) be a random norm space and B be the collection of real valued left-continuous non-increasing function of a real variable such that $F(x) = 1$ whenever $x \leqq 0$ and $\lim_{x\to\infty} F(x) = 0$. By constructing a metric d on B which is analogous to the Lévy metric on the set of cumulative distribution functions the following characterization of the ϵ,δ-topology is given. THEOREM. *The ϵ,δ-topology for (L, f, μ) is identical to the smallest linear topology for L such that f on L into (B, d) is continuous at θ, the origin in L.* (Received 5 February 1968.)

11. Asymptotic expansions of the power functions of the likelihood ratio tests for multivariate linear hypothesis and independence. NARIAKI SUGIURA, University of North Carolina.

Asymptotic non-null distribution of the likelihood ratio criterion for testing the linear hypothesis in multivariate analysis is obtained up to the order N^{-1}, where N means the sample size, by using the characteristic function expressed by the hypergeometric function of matrix argument due to Constantine (*Ann. Math. Statist.* **34** (1963) 1270–1285). It is expressed by the sum of noncentral χ^2 distribution with some degrees of freedom and coefficients depending on noncentrality matrix. This result holds without any assumption on the rank of noncentrality matrix and coincides with that of Posten and Bargmann (*Biometrika* **51** (1964) 467–480) in case of rank two. Asymptotic expansion of the likelihood ratio criterion for testing the independence between two sets of variates is also obtained up to the order $N^{-\frac{1}{2}}$ by the same way. After some normalization of the test statistic it is expressed by the standard normal distribution and its derivatives with coefficients depending on canonical correlations. (Received 5 February 1968.)

(*Abstracts of Papers to be presented at the European meeting, Amsterdam, Netherlands, September 2–7, 1968. An additional abstract appeared in the April issue.*)

2. Asymptotic properties of conditional maximum likelihood estimators. ERLING BERNHARD ANDERSEN, Copenhagen School of Economics and Business Administration.

Let $X_1, X_2, \cdots$ be an infinite sequence of independent random variables and Θ and Ω two subsets of the real numbers. The distribution of X_i depends on $\theta \in \Theta$ and $\tau_i \in \Omega$, such that the value of θ is the same for all i, while the value of τ_i changes. We consider the problem of estimating θ. In case there exists a minimal sufficient statistic T_i for each τ_i we consider the conditional distribution of X_i given T_i, which is independent of τ_i. The conditional maximum likelihood estimator (cmle) $\hat{\theta}_n$ is defined as the function of

$X_1, \cdots, X_n$ that maximizes the conditional density $f(x_1, \cdots, x_n \mid \theta, t_1, \cdots, t_n)$ of $X_1, \cdots, X_n$ given $T_1 = t_1, \cdots, T_n = t_n$. Under usual regularity assumptions and the additional assumption that the density of X_i is a continuous function of τ_i and that there exists a closed interval I such that $\tau_i \varepsilon I$ for all i we prove that $\hat{\theta}_n$ is a consistent and asymptotically normal estimator for θ. The efficiency of the cmle measured as the ratio of the asymptotic variance of $\hat{\theta}_n$ and a Cramér-Rao-type lower bound to the asymptotic variance of asymptotically normally distributed estimators for θ is considered. For exponential type distributions it is proved that the cmle is fully efficient if and only if the distribution of T_i has a certain ancillarity property with respect to θ. (Received 8 February 1968.)

3. Raised conditional level of significance for the 2×2 table when testing the equality of two probabilities. R. D. Boschloo, Mathematical Centre, Amsterdam.

The usual test for the hypothesis $H_0 : p_1 = p_2$ given the results of two independent sequences of experiments, is the exact test of R. A. Fisher in the 2×2-table. If α is the level of significance used for this conditional test (further to be called the "conditional level of significance") the true level of significance depends on the common value p of p_1 and $p_2 : \alpha(p) \leqq \alpha$. Numerical calculations show that for small samples $\alpha(p)$ is often smaller than $\frac{1}{2}\alpha$. This leads to a regrettable loss of power of the test. The suggested method (a variation on an idea of G. A. Barnard, 1945) to overcome this disadvantage is to compute for a number of sample sizes a raised conditional level of significance γ such that the inequality $\alpha(p) \leqq \alpha$ (where α is the level of significance chosen) holds for all p. The test can then be applied in the usual way, with γ instead of α as conditional level of significance. Tables of γ for several values of α have been prepared up to sample sizes of 50; a simple footrule which can be used if the table γ is not available, has been developed using a relation between the method of raised conditional levels of significance and the randomised test. (Received 22 February 1968.)

4. On a distribution of MacMahon and Fisher. Pierre Thionet, Université de Poitiers.

This is a sequel of our papers: Sur certains tests non-parametriques bien connus, *Rev. Int. Statist. Inst.* **34** (1966) 13–26, and: Emploi d'une suite markovienne pour retrouver quelques tests nonparametriques connus, Session 1967 I. S. I., Sydney. We study sign runs up and down whose combinatorial distribution is basic for Moore and Wallis' randomization tests. We clarify our symbolic derivation rules, by use of generating functions. We give practical rules to obtain transition matrices of the random point $(t_1, t_2 \cdots)$ in R^{n-1}, t_j being the number of runs whose length is j in sign sequence of any permutation of: 1, 2, $\cdots$ n; n is increasing and so T is markovian. $t_1 + t_2 + \cdots = t$ is also markovian. But t_j individually is not. The longer run (Olmstead, *Amer. Math. Soc.* **17** (1946) 24) is not; Olmstead's recurrence equation is analogous to our matrix equation. We give also rules to follow the random point T on its lattice, forward and backward, useful for computer program. (Received 25 January 1968.)

(Abstracts of papers to be presented at the Annual meeting, Madison, Wisconsin, August 26–30, 1968. Additional abstracts will appear in future issues.)

1. Some results for bulk-arrival queues with state-dependent service times. Carl M. Harris, Research Analysis Corporation.

The Poisson bulk-arrival, single channel, first-come, first-served queuing system is generalized so that the service times are conditioned on the length of the queue at the moment service is begun. Some general theory is developed for the model and three specific cases are explored. For each of the examples results are obtained that characterize queue behavior using the imbedded Markov chain approach. Sufficient conditions for ergodicity are derived for the general model and steady-state probabilities are found for the imbedded chains of each illustration. (Received 30 March 1968.)

2. Distribution theory of positive definite quadratic forms in multivariate normal samples. B. K. SHAH, Yale University.

Let X: $p \times n$ be a multivariate normal with mean matrix 0 and variance covariance matrix Σ. Khatri, C. G. {*Ann. Math. Statist.* **37** (1966)} has given a representation of the density function of quadratic form (multivariate) $S = XLX'$, L: $n \times n$ is a positive definite matrix, in terms of zonal polynomials which is similar to Ruben (*Ann. Math. Statist.* **33** (1962)). In this paper we are giving a different representation of the density function of S in terms of Laguerre polynomials in matrix argument. The convergent rate will be faster here because these polynomials are orthogonal to the weight function of wishart density (see Constantine, A. G., *Ann. Math. Statist.* **37** (1966)). (Received 23 February 1968.)

3. Multiple testing versus multiple estimation. Improper confidence sets. Estimation of directions and ratios. HENRY SCHEFFÉ, University of California, Berkeley. (Invited)

The "S-method" of multiple comparison (*Analysis of Variance*, Wiley, 1959) was intended for multiple estimation, possibly combined with multiple testing. It is shown that if only multiple testing is desired a certain "modified S-method" is more powerful. An asymptotic calculation of the gain in the number of observations saved by the new method leads to the question of whether this is offset by the loss of the possibility of a certain follow-up procedure. The multiple testing problems are related to problems of estimating the direction of a vector or its unoriented direction. Natural approaches to the latter problems lead to improper confidence sets, i.e., sets for which the probability that they give a trivially true statement is positive. The problems are reformulated to permit solutions by proper confidence sets. In the case of the unoriented direction of a q-dimensional vector the confidence sets yield solutions of the problem of joint estimation of $q - 1$ ratios and the problem of multiple estimation of the ratios of all pairs of estimable functions in a q-dimensional space of estimable functions. Specializing to the case $q = 2$ yields a proper confidence set as a substitute for Fieller's improper confidence set for a ratio. (Received 22 February 1968.)

(*Abstracts of papers not connected with any meeting of the Institute.*)

1. A Hájek-Rényi extension of Lévy's inequality and some applications. P. J. BICKEL, University of California, Berkeley.

THEOREM. *Let* $X_1, \cdots, X_n$ *be independent, symmetric about* 0. *Let* g *be a convex nonnegative function, and* $c_1 \geqq c_2 \cdots \geqq c_n > 0$ *be a sequence of constants. Define* $S_n = \sum_{i=1}^{n} X_i$. *Then,*

$$P[\max_{1 \leqq k \leqq n} c_k g(S_k) \geqq 1] \leqq 2P[\{g(S_1)(c_1 - c_2) + \cdots + g(S_{n-1})(c_{n-1} - c_n) + g(S_n)c_n\} \geqq 1].$$

This inequality generalizes Lévy's inequality in the same way that Hájek and Rényi (*Acta Math. Acad. Sci. Hungar.* (1955)) generalized Kolmogorov's inequality. Using this theorem we are able to strengthen and generalize the results of Teicher (*Z. Warscheinlichkeitstheorie und Verw. Gebiete* (1967)), and to give simple proofs of almost all the classical results of Marcinkiewicz and Zygmund "Sur les fonctions indépendantes" (*Fund. Math.* (1937) and *Studia Math.* (1938)). (Received 26 February 1968.)

2. The resultant of a sample of points on a hypersphere. A. Lawrence Gould, Research Triangle Institute.

Let an n-dimensional hypersphere have a von Mises distribution on its surface. A sample of points on the surface of this hypersphere may be regarded as a sample of vectors of constant length. The standard derivation of the resultant of the sample, and its length, is based on an argument involving discontinuous integrals first put forth by Kluyver in 1906, and subsequently by Rayleigh in 1919. By starting with the characteristic functions, the present treatment obtains in a unified way the distributions of the sample resultant, its length and, in addition, the distribution of the inner product of the sample resultant with an arbitrary known vector. A useful approximate form for one of the distributions is also given. (Received 25 January 1968.)

3. Weak convergence of probability measures on product spaces with applications to sums of random vectors. Donald L. Iglehart, Stanford University.

Let C^k be the product of k copies of $C[0, 1]$, the space of continuous functions on $[0, 1]$ with the uniform metric, and D^k be the product of k copies of $D[0, 1]$, the space of right-continuous functions on $[0, 1]$ having left limits with the Skorohod metric. If $\{P_n\}$ is a sequence of probability measures on the Borel sets of C^k, then we let $\{P_n^i\}$ $(i = 1, \cdots, k)$ be the corresponding sequences of marginal measures. Theorem. *Necessary and sufficient conditions for* $\{P_n\}$ *to converge weakly to* P *are that the finite-dimensional distributions of* P_n *converge weakly to those of* P *and that the families of marginal measures* $\{P_n^i\}$ $(i = 1, \cdots, k)$ *be tight.* A similar result holds for sequences of probability measures on the Borel sets of D^k. These results can easily be applied to obtain functional central limit theorems (invariance principles) for sums of random vectors. The random vectors considered are either independent and identically distributed or stationary ϕ-mixing. Extensions to the case of sums of a random number of random variables are also treated. (Received 7 March 1968.)

4. Asymptotic expansions associated with posterior distributions. Richard A. Johnson, University of Wisconsin.

An extension of the investigation of Johnson (1967) *Ann. Math. Statist.* is made by giving a larger class of posterior distributions which possess asymptotic expansions having a normal distribution as a leading term. Asymptotic expansions for the related normalizing transformation and percentiles are also presented. The conditions imposed are sufficient to make the maximum likelihood estimate strongly consistent and asymptotically normal. They also include higher order derivative assumptions on the log of the likelihood. In Section 2, we show that with probability one, the centered and scaled posterior distribution possesses an asymptotic expansion in powers of $n^{-\frac{1}{2}}$ having the standard normal as a leading term. The number of terms in the expansion is two less than the number of continuous derivatives of the log likelihood. All terms beyond the first consist of a polynomial multiplied by the standard normal density. The coefficients of the polynomial depend on the prior density ρ and the likelihood. The moments of the posterior distribution are shown to possess an expansion in Section 3. The following two sections present the normalizing transformation and percentile expansions. These last three expansions also exist for the case considered in the reference above. These results are proved for independent identically distributed random variables but the extension to certain stationary ergodic Markov processes is immediate. The details of the necessary modifications are presented in Section 6. (Received 9 February 1968.)

5. The relative asymptotic efficiency of Wilcoxon to normal scores test in gross-error models. S. R. KULKARNI, Karnatak University.

Let $e_{WN}(F)$ denote the relative asymptotic efficiency of Wilcoxon to normal scores test, where F is a continuous cumulative distribution function. In this note we have studied the properties of $e_{WN}(F_{\epsilon,\tau})$ for large τ, where $F_{\epsilon,\tau}(x) = (1-\epsilon)F(x) + \epsilon F(x \mid \tau)$, $0 < \epsilon < 1$, $\tau > 0$. It is proved that $\lim_{\tau\to\infty} e_{WN}(F_{\epsilon,\tau})$ is an increasing function of ϵ and $\lim_{\tau\to\infty} e_{WN}(\Phi_{\epsilon,\tau}) > 1$ for all $\epsilon > .0067$, where Φ is the standard normal distribution function. (Received 28 February 1968.)

6. The Smirnov two-sample tests as rank tests. G. P. STECK, Sandia Corporation.

Let $X_1 \leqq X_2 \leqq X_m$ be the order statistics from a sample of m independent identically distributed random variables with a continuous distribution function F and let $Y_1 \leqq Y_2 \leqq \cdots \leqq Y_n$ be the order statistics from a sample of n independent identically distributed random variables with a continuous distribution function G. Let $F_m(z)$ and $G_n(z)$ denote the corresponding empirical distribution functions. Finally, let R_i denote the rank of X_i in the combined ordered sample. The Smirnov statistics are: $D^+(m, n) = \sup_z (F_m(z) - G_n(z))$, $D^-(m, n) = \sup_z (G_n(z) - F_m(z))$, $D(m, n) = \max (D^+(m, n), D^-(m, n))$. These statistics are expressible in terms of the R_i as follows: $mnD^+ = \sup_{1\leqq k\leqq m} ((m+n)k - mR_k)$, $mnD^- = \sup_{1\leqq k\leqq m} (mR_k - (m+n)k + n)$, $mnD = \frac{1}{2}n + \sup_{1\leqq k\leqq m} |mR_k - (m+n)k + \frac{1}{2}n|$. Also

$$P(mnD^+ \leqq r \mid F, G) = P(R_k \geqq (k(m+n) - r)/m, \text{ all } k \mid F, G),$$

$$P(mnD^- \leqq r \mid F, G) = P(R_k \leqq (k(m+n) - n + r)/m, \text{ all } k \mid F, G) \text{ and}$$

$$P(mnD \leqq r \mid F, G) = P((k(m+n) - r)/m \leqq R_k \leqq (k(m+n) - n + r)/m, \text{ all } k \mid F, G).$$

Recursion relations are proved for computing the distribution of D^+ under the assumption $G = F^k$, $k > 1$, and it is shown that this distribution can be expressed as a determinant. Finally, if $F = G$, it is shown that a formula of Korolyuk's (*Izvestia Akad. Nauk SSSR Ser. Mat.* **19** (1955) 81–96.) derived under the assumptions $n = mp$, p a positive integer is valid for any m, n, and r whenever the numbers $b_h, b_{h+1}, \cdots, b_m$ form an arithmetic progression where b_i is the integer part of $[k(m+n) - r]/m$ and h is the smallest index such that $b_j \geqq j$. (Received 15 March 1968.)

7. The non-central multivariate beta type 2 distribution. D. J. DE WAAL, University of the Orange Free State.

Let the real random matrices $A(p \times p)$ and $B(p \times p)$ be independently distributed according to $W(\Sigma_1, m)$ and $W(\Sigma_2, n, \theta)$ respectively. If $\Sigma_1 = \Sigma_2 = I$, it is proved that $V = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ is distributed according to a noncentral multivariate beta type 2 distribution with $m/2$, $n/2$ degrees of freedom and non-centrality parameter θ of rank $r \leqq p$. If θ is of rank $r < p$, then the distribution function of V can be written as the product of independent one-dimensional beta type 2 distribution functions and a non-central multivariate beta type 2 distribution function of full rank. The distribution function of (det V) can be written in a related form. The distribution of (tr V) is also derived if $\Sigma_1 \neq \Sigma_2$ and θ of rank $r \leqq p$, from which the distribution derived by Khatri (*Ann. Math. Statist.* **38** (1967) 944) follows as a special case by letting $\theta = 0$. The same is done if the matrices A and B are considered as complex variates. (Received 11 March 1968.)